Control Theory from the Geometric Viewpoint

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Preface

This book presents some facts and methods of the Mathematical Control Theory treated from the geometric point of view. The book is mainly based on graduate courses given by the first coauthor in the years 2000–2001 at the International School for Advanced Studies, Trieste, Italy. Mathematical prerequisites are reduced to standard courses of Analysis and Linear Algebra plus some basic Real and Functional Analysis. No preliminary knowledge of Control Theory or Differential Geometry is required.

What this book is about? The classical deterministic physical world is described by smooth dynamical systems: the future in such a system is completely determined by the initial conditions. Moreover, the near future changes smoothly with the initial data. If we leave room for "free will" in this fatalistic world, then we come to control systems. We do so by allowing certain parameters of the dynamical system to change freely at every instant of time. That is what we routinely do in real life with our body, car, cooker, as well as with aircraft, technological processes etc. We try to *control* all these dynamical systems!

Smooth dynamical systems are governed by differential equations. In this book we deal only with finite dimensional control systems: they are governed by ordinary differential equations on finite dimensional smooth manifolds. A control system for us is thus a family of ordinary differential equations. The family is parametrized by *control parameters.* All differential equations of

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the family are defined on one and the same manifold which is called the *state space* of the control system. We may select any admissible values of the control parameters (i.e. select any dynamical system from the family) and we are free to change these values at every time instant. The way of selection, which is a function of time, is called a *control* or a *control function*.

As soon as a control is fixed, the control system turns into a nonautonomous ordinary differential equation. A solution of such an equation is uniquely determined by the initial condition and is called an admissible trajectory of the control system (associated with a given control). Thus, an admissible trajectory is a curve in the state space. The initial condition (initial state) is just a starting point of the trajectory; different controls provide, generally speaking, different admissible trajectories started from a fixed state. All these trajectories fill the *attainable (reachable) set* of the given initial state.

To characterize the states reachable from a given initial one is the first natural problem to study in Control Theory: the Controllability Problem. As soon as the possibility to reach a certain state is established, we try to do it in the best way. Namely, we try to steer the initial state to the final one as fast as possible, or try to find the shortest admissible trajectory connecting the initial and the final states, or to minimize some other cost. This is the Optimal Control Problem. These two problems are our leading lights throughout the book.

Why Geometry? The right-hand side of the ordinary differential equation is a vector field and the dynamical system governed by the equation is the flow generated by this vector field. Hence a control system is a family of vector fields. The features of control systems we study do not change under transformations induced by smooth transformations of the state space. Moreover, our systems admit a wide class of reparametrizations of the family of vector fields, which are called feedback transformations in Control Theory and gauge transformations in Geometry and Mathematical Physics. This is a formal reason why the intrinsic geometric language and geometric methods are relevant to Control Theory.

There is another more fundamental reason. As we mentioned, a dynamical system is a flow (a one-parametric group of transformations of the state space) generated by a vector field. An admissible trajectory of the control system associated to a constant control is a trajectory of the correspondent flow. Admissible trajectories associated with a piecewise constant control are realized by the composition of elements of the flows corresponding to the values of the control function. The arbitrary control case is realized via an approximation by piecewise constant controls. We see that the structure of admissible trajectories and attainable sets is intimately related to the group of transformations generated by the dynamical systems involved. In turn, groups of transformations form the heart of Geometry.

Now, what could be the position of Control techniques and the Control way of thinking in Geometry and, more generally, in the study of basic structures of the world around us? It is usual in Control Theory for the dimension of attainable sets to be essentially greater than the dimension of the sets of admissible velocities. In particular, a generic pair of vector fields on an n-dimensional manifold provides n-dimensional attainable sets, where n is as big as we want. In other words, constraints on velocities do not imply state constraints. Such a situation is traditionally indicated by saying that constraint are "nonholonomic". Control theory is a discipline that systematically studies various types of behavior under nonholonomic constraints and provides adequate methods for the investigation of variational problems with nonholonomic constraints.

The first chapter of the book is of introductory nature: we recall what smooth manifolds and ordinary differential equations on manifolds are, and define control systems. Chapter 2 is devoted to an operator calculus that creates great flexibility in handling of nonlinear control systems. In Chapters 3 and 4 we introduce a simple and extremely popular in applications class of linear systems and give an effective characterization of systems that can be made linear by a smooth transformation of the state space. Chapters 5–7 are devoted to the fundamental Orbit Theorem of Nagano and Sussmann and its applications. The Orbit Theorem states that any orbit of the group generated by a family of flows is an immersed submanifold. Chapter 8 contains general results on the structure of attainable sets starting from a simple test to guarantee that these sets are full dimensional. In Chapter 9 we introduce feedback transformations, give a feedback classification of linear systems, and effectively characterize systems that can be made linear by feedback and state transformations.

The rest of the book is mainly devoted to the Optimal Control. In Chapter 10 we state the optimal control problem, give its geometric interpretation, and discuss the existence of solutions. Chapter 11 contains basic facts on differential forms and Hamiltonian systems; we need this information to investigate optimal control problems. Chapter 12 is devoted to the intrinsic formulation and detailed proof of the Pontryagin Maximum Principle, a key result in the Optimal Control Theory. Chapters 13–16 contain numerous applications of the Pontryagin Maximum Principle including a curious property of Hamiltonian systems with convex Hamiltonians and more or less complete theories of linear time-optimal problems and linear–quadratic problems with finite horizonts. In Chapter 17 we discuss a Hamiltonian version of the theory of fields of extremals, which is suitable for applications in the Optimal Control, and introduce the Hamilton–Jacobi equation. Chapters 18 and 19 are devoted to the moving frames technique for optimal control problems and to problems on Lie groups. The definition and basic facts on Lie groups are given in the same chapter: they are simple corollaries of the general geometric control techniques developed in previous chapters. Chapters 20 and 21 contain the theory of the Second Variation with second order necessary and sufficient optimality conditions for regular and singular extremals. The short Chapter 22 presents an instructive reduction procedure, which establishes a connection between singular and regular extremals. In Chapter 23 we introduce and compute (in simplest low dimensional cases) the curvature, a remarkable feedback invariant of optimal control problems. Finally in Chapter 24 we discuss the control of a classical nonholonomic system: two bodies rolling one on another without slipping or twisting. The appendix contains proofs of some results formulated in Chapter 2.

This is a very brief overview of the contents of the book. In each chapter we try to stay at textbook level, i.e. to present just the first basic results with some applications. The topic of practically every chapter has an extensive development, sometimes rather impressive. In order to study these topics deeper the reader refereed to research papers.

Geometric Control Theory is a broad subject and many important topics are not even mentioned in the book. In particular, we do not study the feedback stabilization problem and the huge theory of control systems with outputs including fundamental concepts of Observability and Realization. For this and other material see books on Control listed in the Bibliography.