# Approximate controllability on the group of volume-preserving diffeomorphisms

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**Abstract**

We study controlability issues for the group of volume-preserving diffeomorphisms of the torus $\mathbb{T}^d$ for system $\dot{x} = f(x) + u(t)$, where $f$ is a fixed divergence free vector field on $\mathbb{T}^d$ and $u(t)$ are constant vector fields which generate translations of the torus. Main results concern $d$ equals two or three.

## Contents



## 1 Introduction

Let $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ be the $d$-dimensional torus. Translations of the torus are generated by constant vector fields. Let $f$ be a non constant divergence free vector field on $\mathbb{T}^d$: it generates a one-parametric group of volume-preserving diffeomorphisms. In this paper, we try to understand which transformations of $\mathbb{T}^d$ can be reached if we perturb $f$ by constant fields (with the constant depending on time), mainly for $d = 2$ and $d = 3$.

The answer is surprisingly simple in many cases and we hope that it might be useful in the mathematical fluid dynamics, in particular, for the study of turbulent flows. In the next paper, we are going to treat random perturbations.

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We start with a general linear in control system $\dot{x} = \sum_{j=1}^{s} u_j F_j(x)$ on a compact manifold $M$. Theorem 1 characterizes the flows on $M$ which can be uniformly approximated by the flows generated by a time-varying ordinary differential equation of the form $\dot{x} = \sum_{j=1}^{s} u_j(t) F_j(x)$. These are exactly the flows generated by equations $\dot{x} = V_t(x)$, where $V_t \in \overline{\mathrm{Lie}\{F_1, \ldots, F_s\}}$.

In Theorem 2 we deal with the flows which preserve a fixed volume form on $M$. Such flows are generated by the divergence free vector fields. Let $\mathrm{Vec}_0 M$ be the space of divergence free vector fields. Theorem 2 states that the condition $\mathrm{Vec}_0 M \subset \overline{\mathrm{Lie}\{F_1, \ldots, F_s\}}$ is sufficient for the possibility to transfer any finite ensemble of mutually distinct points $x_i \in M,\ i = 1, \ldots, N$, to any other sequence $y_i \in M,\ i = 1, \ldots, N$, by the flow generated by the equation of the form $\dot{x} = \sum_{j=1}^{s} u_j(t) F_j(x)$, where the control $(u_1(\cdot), \ldots, u_s(\cdot))$ and the transfer time are the same for all $x_i,\ i = 1, \ldots, N$.

General theorems 1–2 are rather simple corollaries of earlier results. We need them to study the affine in control system $\dot{x} = f(x) + u$ that is our main objective. Here $u$ is an arbitrary constant vector field on the torus $\mathbb{T}^d$.

Let $f(x) = \sum_{m \in \mathbb{Z}^d} p_m e^{i\langle m, x\rangle}$ and $\mathcal{M}_f = \{m \in \mathbb{Z}^d \mid p_m \neq 0\}$. Theorem 3 and Theorem 4 concern the case where $\#\mathcal{M}_f < \infty$ and $\mathrm{span}\{f(x) \mid x \in \mathbb{T}^d\} = \mathrm{span}\mathcal{M}_f = \mathbb{R}^d$.

In Theorem 3 we describe the closure of the Lie subalgebra of $\mathrm{Vec}_0\mathbb{T}^d$ generated by the fields $f + u$ for $d$ equals two or three. This closure is equal to the space of fields $g \in \mathrm{Vec}_0\mathbb{T}^d$ such that $\mathcal{M}_g$ is contained in the subgroup of $\mathbb{Z}^d$ generated by $\mathcal{M}_f$.

In Theorem 4 we describe the closure of the attainable set of the system $\dot{x} = f(x) + u$ in the group of volume-preserving diffeomorphisms of $\mathbb{T}^d$. It appears that the closure of the attainable set depends only on the Lie algebra computed in Theorem 3 in spite of the fact that our system is not linear in control, it has a nontrivial drift $f$ while its linear part $u$ is commutative.

Finally, Theorem 5 concerns the controllability for finite ensembles of points on $\mathbb{T}^d$. Here we do not need the set $\mathcal{M}_f$ to be finite. The theorem states that for $d$ equals two or three there exists a residual subset of $\mathrm{Vec}_0\mathbb{T}^d$ such that the system $\dot{x} = f(x) + u$ can transfer any finite ensemble of mutually distinct points to any other ensemble with the same number of points. On the other hand, according to the same theorem, whatever $f$ we take, the transfer of at least two points ensemble may require arbitrary long time. No a priori time bound is possible.

These are our main results. In Section 6, the Lie algebra generated by $f + u,\ u \in \mathbb{R}^d$, is computed also in the case where $\mathrm{span}\mathcal{M}_f \neq \mathbb{R}^d$ (see Theorem 6 and Theorem 7).

**Remark 1.** *Admissible controls in this paper are measurable locally bounded vector functions but all results remain valid with much smaller classes of admissible controls. Indeed, we say that a sequence of locally integrable vector functions $u_n(\cdot)$ converges to $u(\cdot)$ on the segment $[0, T]$ in the sliding topology if $\|u_n\|_{L^1}$ are uniformly bounded and $\left|\int_0^t u_n(t) - u(t)\, dt\right| \to 0$ uniformly for $t \in [0, T]$ as $n \to \infty$.*

*The map which sends the control $u(\cdot)$ to the flow $P_\cdot^{u(\cdot)}$ is continuous in the sliding topology (see, for instance, Lemma 8.10 in [5]) and all our results survive if we use only controls from an everywhere dense subset in the sliding topology. For example, it is sufficient to use only controls from the space of vector polynomials or trigonometric polynomials, or piecewise constant vector functions, or even from the set of piecewise constant vector functions with only one nonzero coordinate in every time moment.*

To conclude the introduction, we have to mention that various aspects of the controllability on the group of diffeomorphisms were studied in [2, 6, 7, 8, 9, 10, 11, 12, 13, 14] and many other papers.

## 2 Motion planning in the group of diffeomorphisms and controllability of finite ensembles of points

Let $M$ be a compact Riemannian manifold of class $C^\infty$ and dimension $n \in \mathbb{N}$. Let us consider the linearly controlled equation

$$\dot{x} = \sum_{j=1}^{s} u_j(t) F_j(x), \qquad u(t) = (u_1(t), \dots, u_s(t)) \in \mathbb{R}^s, x \in M, \tag{1}$$

where the measurable map $t \mapsto u(t)$ is locally bounded. The flow of (1) at time $t$ is denoted by $P_t^{u(\cdot)}$. Considering the family of smooth vector fields $\mathcal{F} := \{F_1, \dots, F_s\}$, we would like to understand which trajectories in the group of diffeomorphisms of $M$ could be approximated by the flows $t \mapsto P_t^{u(\cdot)}$, uniformly on any time segment. This is a problem of motion planning in the group of diffeomorphisms.

Let us consider $N \in \mathbb{N}^*$ distinct points of $M$, $\gamma = (\gamma_1, \dots, \gamma_N) \in \hat{M}^N$, where $\hat{M}^N = M^N \setminus \Delta^N$ and $\Delta^N := \left\{\gamma \in M^N \mid \exists k \neq \ell, \gamma_k = \gamma_\ell\right\}$. If we apply the dynamic (1) to the initial positions $(\gamma_1, \dots, \gamma_N)$, the configuration of these points at time $t \geq 0$ is determined by $(P_t^{u(\cdot)}(\gamma_1), \dots, P_t^{u(\cdot)}(\gamma_N))$. We would like to study properties about controllability of finite ensembles of points depending on $\mathcal{F}$.

### 2.1 Motion planning in the group of diffeomorphisms

In what follows, $M$ is a compact Riemannian manifold of class $C^\infty$. Given a smooth tensor field $q \mapsto \omega_q,\ q \in M$, where $\omega_q \in (T_qM)^{\otimes k} \otimes (T_q^*M)^{\otimes \ell}$, the norms $\|\omega\|_m,\ m = 0, 1, 2, \dots$, are defined as follows:

$$\|\omega\|_m = \sup_{q \in M, i \leq m} |\nabla_q^i \omega|,$$

where $\nabla$ is the covariant derivative. Here $k = 1, 2, \dots,\ \ell = 0, 1, 2, \dots$, and $(T^*M)^{\otimes 0} = C^\infty(M)$. The seminorms $\|\cdot\|_m$ define standard $C^\infty$-topology on the space of smooth tensor fields of the prescribed degree. Standard $C^\infty$-topology on the group of diffeomorphisms is induced by the topology on $C^\infty(M)$ if we treat a diffeomorphism $P : M \to M$ as a linear operator $P^*$ on $C^\infty(M)$, where $P^* : a \mapsto a \circ P$, $\|P\|_m = \sup\limits_{\|a\|_m \leq 1} \|P^* a\|_m$. More details about this formalism and chronological calculus can be found in [1, Chap. 6].

Let $\mathrm{Vec}M$ be the Lie algebra of smooth vector fields on $M$. Given $\mathcal{F} \subset \mathrm{Vec}M$, $\mathrm{Lie}\mathcal{F} \subset \mathrm{Vec}M$ is the Lie subalgebra generated by $\mathcal{F}$ and $\overline{\mathrm{Lie}\mathcal{F}}$ is the closure of $\mathrm{Lie}\mathcal{F}$ in the standard topology.

A measurable map $t \mapsto V_t$, where $V_t \in \mathrm{Vec}M,\ t \in \mathbb{R}$, is called a time-varying vector field if $\|v_t\|_m$ is a locally bounded function of $t$ for any $m \geq 0$. Any time-varying vector field defines a flow $P_t \in \mathrm{Diff}M,\ t \in \mathbb{R}$, where $P_0 = \mathrm{Id}$, $\frac{\partial}{\partial t} P_t(x) = V_t(P_t(x))$. We use the standard chronological notation for this flow, $P_t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$. The following result is a corollary of [3, Th. 3].

**Theorem 1.** *Let $\mathcal{F} = \{F_1, \dots, F_s\} \subset \mathrm{Vec}M$. Let $t \mapsto V_t \in \overline{\mathrm{Lie}\mathcal{F}},\ t \in [0, T]$, be a time-varying vector field. Then for every $m \in \mathbb{N}$ and $\varepsilon > 0$ there exists a control $t \mapsto u(t) = (u_1(t), \dots, u_s(t)) \in L^\infty([0, T], \mathbb{R}^s)$, such that the flow $P_t^{u(\cdot)}$, generated by control system (1) and control $u(\cdot)$, satisfies*

$$\|\overrightarrow{\exp} \int_0^t V_\tau \, d\tau - P_t^{u(\cdot)}\|_m \leq \varepsilon, \qquad t \in [0, T].$$

*Proof.* If $V_t = \sum\limits_{i=1}^{k} v_i(t) X_i$, where $X_i \in \mathrm{Lie}\mathcal{F}$, then the desired result is just the statement of Theorem 3 from [3]. On the other hand, we can uniformly and arbitrarily well approximate $V_t$ by such linear combination in any norm $\|\cdot\|_m$. Indeed, $\{V_t \mid\ 0 \leq t \leq T\}$ is a precompact set in the topology $\|\cdot\|_m$ since this set is bounded in the norm $\|\cdot\|_{m+1}$. In other words, for every $\delta > 0$ there exists a finite set

$\{X_1, \dots, X_k\} \subset \mathrm{Lie}\mathcal{F}$ such that the union of the radius $\delta$ balls in norm $\|\cdot\|_m$ centered at $X_i$ covers the set $\{V_t \mid 0 \le t \le T\}$. We present $[0,T]$ as the disjoint union of subsets $S_i$, $[0,T] = \bigcup_i S_i$, such that $\|X_i - V_t\|_m \le \delta$, for every $t \in S_i$, and we set

$$v_i(t) = \begin{cases} 1 & t \in S_i \\ 0 & t \notin S_i, \end{cases}$$

then $\|V_t - \sum_{i=1}^{k} v_i(t) X_i\|_m \le \delta$, for every $t \in [0,T]$. □

Let us consider control system (1). The connected component of the identity in Diff$M$ is denoted by $\mathrm{Diff}^0 M$. For every measurable and locally bounded control $u(\cdot)$, the vector field $t \mapsto \sum_{j=1}^{s} u_j(t) F_j$ is a time-varying vector field according to the previous definition.

**Definition 1.** *An element $P \in \mathrm{Diff}^0 M$ is said to be approximately reachable for system* (1) *in time $t \ge 0$ if for every $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists a measurable and locally bounded control $u(\cdot)$ such that the flow $P_t^{u(\cdot)}$, generated by system* (1) *and control $u(\cdot)$, satisfies*

$$\|P - P_t^{u(\cdot)}\|_m \le \varepsilon.$$

*The set of approximately reachable diffeomorphisms in time $t \ge 0$ is denoted by $\overline{\mathcal{A}}_t$ and $\overline{\mathcal{A}} := \cup_{t \ge 0} \overline{\mathcal{A}}_t$. For every subgroup $D \subset \mathrm{Diff}^0 M$, system* (1) *is said to be approximately controllable in $D$ if $\overline{\mathcal{A}} = D$.*

Let $\omega$ be a fixed volume form on $M$ and $\mathrm{Vec}_0 M$ be the Lie algebra of divergence free vector fields of $M$,

$$\mathrm{Vec}_0 M = \{f \in \mathrm{Vec}M \mid \mathrm{div}_\omega f = 0\}.$$

Any volume-preserving *flow* $P_t$ has a form $P_t = \overrightarrow{\exp} \int_0^t f_\tau d\tau$, where $t \mapsto f_\tau \in \mathrm{Vec}_0 M$. Then $P_t \in \mathrm{Diff}^0 M$. The set of volume-preserving flows of $M$ is denoted by $\mathrm{Diff}_0 M$. This is a subgroup of the connected component of the identity $\mathrm{Diff}^0 M$, and $\mathrm{Diff}_0 M \subset \mathrm{Diff}^0 M$.

**Proposition 1.** *Let $\mathcal{F} = \{F_1, \dots, F_s\}$ be the family of admissible vector fields for system* (1).

— *If $\overline{\mathrm{Lie}\mathcal{F}} = \mathrm{Vec}M$, system* (1) *is approximately controllable in* $\mathrm{Diff}^0 M$.

— *If $\overline{\mathrm{Lie}\mathcal{F}} \supset \mathrm{Vec}_0 M$, system* (1) *is approximately controllable in* $\mathrm{Diff}_0 M$.

*Proof.* This is a corollary of Theorem 1. □

## 2.2 Controllability of finite ensembles of points

In what follows, we study the controllability of finite ensembles of points in $\hat{M}^N$. Indeed system (1) can be lifted to a linear in control system defined on $\hat{M}^N$ by the controlled equations

$$\dot{\gamma}_\ell = \sum_{j=1}^{s} u_j(t) F_j(\gamma_\ell), \qquad u(t) \in \mathbb{R}^s, \quad \ell \in \{1, \dots, N\}, \tag{2}$$

where $(\gamma_1, \dots, \gamma_N) \in \hat{M}^N$ and the map $t \mapsto u(t)$ is measurable and locally bounded. The attainable set at time $t \ge 0$ from $\gamma = (\gamma_1, \dots, \gamma_N) \in \hat{M}^N$ of system (2) is defined by

$$A_\gamma(t) := \left\{ (P_t^{u(\cdot)}(\gamma_1), \dots, P_t^{u(\cdot)}(\gamma_N)) \mid u(\cdot) \in L^\infty([0,t], \mathbb{R}^s) \right\} \subset \hat{M}^N.$$

**Definition 2.** *For a general system of control we also define the attainable set from $\gamma$ by $A_\gamma = \cup_{t \ge 0} A_\gamma(t)$ (which coincides with $A_\gamma(t)$ for every $t \ge 0$ in the case of system* (2)*). Then a system is said globally controllable (respectively globally controllable in time $T \ge 0$) if $A_\gamma = \hat{M}^N$ (respectively if $\cup_{0 \le t \le T} A_\gamma(t) = \hat{M}^N$) for every $\gamma \in \hat{M}^N$.*

**Definition 3.** *Let $N \in \mathbb{N}^*$. System* (1) *is said to be globally controllable (respectively globally controllable in time $T \geq 0$) in the space of $N$-ensembles if system* (2) *is globally controllable (respectively globally controllable in time $T$) in $\hat{M}^N$.*

The space $\hat{M}^N$ has a structure of smooth manifold. For each $\gamma \in \hat{M}^N$, the tangent space $T_\gamma \hat{M}^N$ is isomorphic to $T_{\gamma_1} M \times \cdots \times T_{\gamma_N} M$. The $N$-fold of a vector field $X \in \mathrm{Vec}M$ is defined on $\hat{M}^N$ by $X^N(\gamma_1, \ldots, \gamma_N) = (X(\gamma_1), \ldots, X(\gamma_N))$. If $X$ is complete on $M$ then $X^N$ is also complete on $\hat{M}^N$. The Lie bracket of $N$-folds $X^N, Y^N$ verifies the formula $[X^N, Y^N] = [X, Y]^N$ and the same holds for the iterated Lie brackets.

**Definition 4.** *Let $\mathcal{F}^N = \{F_1^N, \ldots, F_s^N\}$. System* (2) *is said to be Lie bracket generating at $\gamma$ if $\{F(\gamma) \mid F \in \mathrm{Lie}\mathcal{F}^N\} = T_\gamma \hat{M}^N$. It is Lie bracket generating if it is Lie bracket generating at every $\gamma \in \hat{M}^N$.*

As a consequence of Rashevsky-Chow theorem, if system (2) is Lie bracket generating then it is globally controllable, see e.g. [5, Th. 5.2 and Cor 5.2].

**Theorem 2.** *Let $\mathcal{F} = \{F_1, \ldots, F_s\}$. If $\mathrm{Vec}_0 M \subset \overline{\mathrm{Lie}\mathcal{F}}$, then the family $\mathcal{F}^N$ is Lie bracket generating in $\hat{M}^N$ for every $N \in \mathbb{N}^*$, and system* (1) *is globally controllable in the space of $N$-ensembles.*

*Proof.* Let $\gamma \in M^N$, we consider the linear map

$$\varphi_\gamma : \begin{cases} \mathrm{Lie}\mathcal{F} & \to T_\gamma \hat{M}^N \\ X & \mapsto X^N(\gamma), \end{cases}$$

if it is surjective for every $\gamma \in \hat{M}^N$, then system (2) is Lie bracket generating and so it is globally controllable. By assumption, if $\mathrm{Im}\varphi_\gamma$ denotes the image of $\varphi_\gamma$, then $\{X^N(\gamma) \mid X \in \mathrm{Vec}_0 M\} \subset \overline{\mathrm{Im}\varphi_\gamma}$.

Recall that $X \in \mathrm{Vec}_0 M$ if and only if $\mathrm{div}X = 0$. Let us prove that for every $(a_1, \ldots, a_N) \in T_\gamma \hat{M}^N$, there exists $X \in \mathrm{Vec}_0 M$ such that $X^N(\gamma) = (a_1, \ldots, a_N)$. Let $\mathcal{V}_1, \ldots, \mathcal{V}_N$ be open neighborhoods in $M$ such that

$$\gamma_\ell \in \mathcal{V}_\ell, \quad \mathcal{V}_k \cap \mathcal{V}_\ell = \emptyset, \qquad k \neq \ell \in \{1, \ldots, N\},$$

and such that $\mathcal{V}_\ell$ is diffeomorphic to some open neighborhood $\mathcal{O}_\ell \subset \mathbb{R}^n, 0 \in \mathcal{O}_\ell$. Locally, the vector field $X$ can be expressed in coordinates. The charts $\phi_\ell : \mathcal{V}_\ell \to \mathcal{O}_\ell$ are chosen such that the expression of the volume form $\omega$ in coordinates is equal to $dx_1 \wedge \ldots \wedge dx_n$. Given $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and two neighborhoods $\mathcal{O} \subset \mathcal{O}'$ of 0 in $\mathbb{R}^n$, we construct $\bar{X} \in \mathrm{Vec}\mathbb{R}^n$ such that $\bar{X} = a$ on $\mathcal{O}$, $\mathrm{supp}\bar{X} \subset \mathcal{O}'$ and $\mathrm{div}\bar{X} = 0$. Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a smooth cut-off function such that $\chi(x) = \frac{1}{n-1}$ on $\mathcal{O}$ and $\mathrm{supp}\chi \subset \mathcal{O}'$. We consider a $(n-2)$-differential form on $\mathbb{R}^n$,

$$\alpha = \chi \sum_{1 \leq \ell < m \leq n} ((-1)^{m-2} a_\ell x_m + (-1)^{\ell-1} a_m x_\ell) dx_1 \wedge \ldots \widehat{dx_\ell} \ldots \widehat{dx_m} \ldots \wedge dx_n.$$

Then we compute $d\alpha = \sum_{m=1}^n \psi_m dx_1 \ldots \widehat{dx_m} \ldots dx_n$ and we check that $\psi_m = a_m$ on $\mathcal{O}$ and $\mathrm{supp}\psi_m \subset \mathcal{O}'$. Let us consider the vector field $\bar{X} = \sum_{m=1}^n \psi_m \partial_{x_m}$, then $\bar{X} = a$ on $\mathcal{O}$ and $\mathrm{supp}\bar{X} \subset \mathcal{O}'$. Moreover, $dd\alpha = 0 = (\mathrm{div}\bar{X}) dx_1 \wedge \ldots \wedge dx_n$, so $\mathrm{div}\bar{X}$=0. The image of $\varphi_\gamma$ is dense in $T_\gamma \hat{M}^N$, so the map is surjective. □

# 3 Volume-preserving diffeomorphisms on $\mathbb{T}^d$

In the following, we consider the torus $\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$. Vector fields on $\mathbb{T}^d$ are naturally identified with $2\pi$-periodic $d$-vector functions on $\mathbb{R}^d$, i.e. the vector function $f(x) = (f^1(x), \ldots, f^d(x)), x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, corresponds to the field $\sum_{i=1}^d f^i(x) \frac{\partial}{\partial x_i}$. In the following, we study an affine in control system of the form

$$\dot{x} = f(x) + u(t), \qquad u(t) \in \mathbb{R}^d, \tag{3}$$

where $f \in \mathrm{Vec}_0\mathbb{T}^d$ is any divergence free vector field and $t \mapsto u(t)$ is measurable and locally bounded.

**Remark 2.** *By replacing $f$ and $u$ by $f+c$ and $u-c$ where $c \in \mathbb{R}^d$ is a constant, we can suppose that $\int_{\mathbb{T}^d} f = 0$ without changing the set $\overline{\mathcal{A}}$.*

The flow at time $t$ of system (3) is denoted by $P_t^{u(\cdot)}$. The set of approximately reachable elements in the group of diffeomorphisms is denoted by $\overline{\mathcal{A}}$, see Definition 1. We would like to understand which volume-preserving diffeomorphisms could be approximated by the flows of the previous equation, depending on the modes of the Fourier decomposition of $f$. In the following, we study a classification of the approximately reachable set in the group of diffeomorphisms depending on $f$.

## 3.1 Subgroups of volume-preserving flows on $\mathbb{T}^d$

Recall that $\mathbb{Z}^d$ is an additive subgroup of $\mathbb{R}^d$. Let $\Gamma \subset \mathbb{Z}^d$ be a subgroup of $\mathbb{Z}^d$ such that $\mathrm{span}\Gamma = \mathbb{R}^d$. Let $f(x) = \sum_{m \in \mathbb{Z}^d} p_m e^{i\langle m,x\rangle}$ be the Fourier expansion of $f$, where $p_m \in \mathbb{C}^d,\ p_{-m} = \bar{p}_m$. We set

$$\mathcal{M}_f = \left\{ m \in \mathbb{Z}^d \mid p_m \neq 0 \right\},$$

$$\mathrm{Vec}_0(\mathbb{T}^d)_\Gamma = \left\{ f \in \mathrm{Vec}\mathbb{T}^d \mid \mathrm{div} f = 0,\ \mathcal{M}_f \subset \Gamma \right\}.$$

We omit the index $\Gamma$ if $\Gamma = \mathbb{Z}^d$. It is easy to check that $\mathrm{Vec}_0(\mathbb{T}^d)_\Gamma$ is a closed Lie subalgebra of $\mathrm{Vec}\mathbb{T}^d$.

Now we consider the subgroup $\Gamma^* \subset \mathbb{R}^d$ dual to $\Gamma$,

$$\Gamma^* = \left\{ x \in \mathbb{R}^d \mid \langle x, y\rangle \in \mathbb{Z},\ \forall\, y \in \Gamma \right\}.$$

We see that $\mathbb{Z}^d \subset \Gamma^*$ and moreover, $\Gamma^*/\mathbb{Z}^d$ is a finite group. Indeed, $\Gamma = A\mathbb{Z}^d$, where $A$ is a nondegenerate matrix with integral entries. Then $\Gamma^* = A^{*-1}\mathbb{Z}^d$, where $A^*$ is the adjoint matrix of $A$, and $A^{*-1}$ has rational entries.

Moreover, the group $2\pi\Gamma^*/2\pi\mathbb{Z}^d$ acts freely and properly on $\mathbb{T}^d$ by the translations and a divergence free vector field $f$ belongs to $\mathrm{Vec}_0(\mathbb{T}^d)_\Gamma$ if and only if $f$ commutes with this action. The same property can be described in other way if we use the covering $\mathfrak{p}_{\Gamma^*} : \mathbb{T}^d \to \mathbb{T}^d/2\pi\Gamma^*$. Here $\mathbb{T}^d/2\pi\Gamma^*$ is another torus. We see that $f \in \mathrm{Vec}_0(\mathbb{T}^d)_\Gamma$ if and only if $f = \mathfrak{p}^*_{\Gamma^*} g$ where $g \in \mathrm{Vec}_0(\mathbb{T}^d/2\pi\Gamma^*)$.

Any volume-preserving flow $P_t \in \mathrm{Diff}\mathbb{T}^d,\ P_0 = \mathrm{Id}$, has a form $P_t = \overrightarrow{\exp}\int_0^t f_\tau\, d\tau$, where $f_\tau \in \mathrm{Vec}_0\mathbb{T}^d$. This is true for any torus, in particular for the torus $\mathbb{T}^d/2\pi\Gamma^*$. We obtain that the flows generated by the time varying vector fields from $\mathrm{Vec}_0(\mathbb{T}^d)_\Gamma$ are exactly the lifts to $\mathbb{T}^d$ of the volume-preserving flows on $\mathbb{T}^d/2\pi\Gamma^*$.

We denote by $\mathrm{Diff}_0\mathbb{T}^d_\Gamma$ the connected component of the identity in the group of volume-preserving diffeomorphisms of $\mathbb{T}^d$ commuting with the action of $\Gamma^*/\mathbb{Z}^d$. Then

$$\mathrm{Diff}_0\mathbb{T}^d_\Gamma = \left\{ \overrightarrow{\exp}\int_0^t f_\tau\, d\tau \mid f_\tau \in \mathrm{Vec}_0(\mathbb{T}^d)_\Gamma \right\}.$$

## 3.2 Approximation of volume-preserving diffeomorphisms by an affine in control system

Recall that $\mathrm{Vec}_0\mathbb{T}^d$ is the set of divergence free vector fields of $\mathbb{T}^d$. We define the subset $\mathfrak{V}^d \subset \mathrm{Vec}_0\mathbb{T}^d$ as follows. A vector field $f \in \mathrm{Vec}_0\mathbb{T}^d$ belongs to $\mathfrak{V}^d$ if

(i) $\#\mathcal{M}_f < \infty$,

(ii) $\mathrm{span}\mathcal{M}_f = \mathbb{R}^d$,

(iii) $\mathrm{span}\left\{ f(x) \mid x \in \mathbb{T}^d \right\} = \mathbb{R}^d$.

Clearly, $\overline{\mathfrak{V}^d} = \mathrm{Vec}_0\mathbb{T}^d$. Moreover, if $d = 2$, then property (ii) implies (iii). Indeed if $f(x) = \sum_{m\in\mathcal{M}_f} a_m \cos\langle m, x\rangle + b_m \sin\langle m, x\rangle, x \in \mathbb{T}^d$, then $\mathrm{span}\left\{f(x) \mid x \in \mathbb{T}^d\right\} = \mathrm{span}\left\{a_m, b_m \mid m \in \mathcal{M}_f\right\}$. If $d = 2$ and $\mathrm{span}\mathcal{M}_f = \mathbb{R}^2$, there exist $m, n \in \mathcal{M}_f$ such that $a_m, a_n \in \mathbb{R}^2\setminus\{0\}$ and $\langle m, a_m\rangle = \langle n, a_n\rangle = 0$. Then necessarily $\mathrm{span}\left\{a_m, a_n\right\} = \mathbb{R}^2$.

Here we present the main results for system (3) with $f \in \mathfrak{V}^d$. The proofs are given in the following sections.

**Theorem 3.** *Let $d = 2$ or $d = 3$ and $f \in \mathfrak{V}^d$. Let $\Gamma \subset \mathbb{Z}^d$ be the subgroup generated by $\mathcal{M}_f$. Then*

$$\overline{\mathrm{Lie}\{f + u \mid u \in \mathbb{R}^d\}} = \mathrm{Vec}_0(\mathbb{T}^d)_\Gamma.$$

**Theorem 4.** *Under the conditions of Theorem 3, the subgroup $\mathrm{Diff}_0\mathbb{T}^d_\Gamma \subset \mathrm{Diff}_0\mathbb{T}^d$ is invariant for system (3) and moreover the system is approximately controllable in $\mathrm{Diff}_0\mathbb{T}^d_\Gamma$.*

**Theorem 5.** *(i) Let $d = 2$ or $d = 3$. There exists a residual subset $\mathcal{R} \subset \mathrm{Vec}_0\mathbb{T}^d$ such that, for every $f \in \mathcal{R}$ and $N \in \mathbb{N}^*$, system (3) is globally controllable in the space of $N$-ensembles in $\mathbb{T}^d$.*

*(ii) For every $d \geq 2$, $f \in \mathrm{Vec}_0\mathbb{T}^d$, $N \geq 2$, and $T > 0$, system (3) is not globally controllable for time smaller or equal than $T$ in the space of $N$-ensembles in $\mathbb{T}^d$.*

# 4 Proof of Theorem 4

In what follows, $\mathrm{cone}S$ is the convex cone generated by the subset $S$ of a real vector space,

$$\mathrm{cone}S = \left\{\sum_i \alpha_i a_i \mid a_i \in S,\ \alpha_i \geq 0\right\},$$

and $dw$ is the standard volume form on the torus. Let $f \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ be a smooth vector function and $\theta \in \mathbb{T}^d$. We define a vector function $f_\theta$ by the formula $f_\theta(x) = f(x + \theta),\ x \in \mathbb{T}^d$.

Let $t \geq 0$. By applying the variation formula to system (3), see e.g. [5, Section 2.7], we obtain a decomposition of the flow

$$\begin{aligned} P_t^{u(\cdot)} &= \overrightarrow{\exp} \int_0^t f + u(\tau)d\tau \\ &= \left(\overrightarrow{\exp} \int_0^t \left(\mathrm{Ad}e^{\int_0^\tau u(s)ds}\right) f d\tau\right) \circ e^{\int_0^t u(s)ds} \end{aligned}$$

where $(\mathrm{Ad}P^{-1})f = P_* f$ for any $P \in \mathrm{Diff}\mathbb{T}^d$. Let $\theta(t) = \int_0^t u(s)ds$. Notice that $e^{\theta(t)} \in \mathrm{Diff}\mathbb{T}^d$ is the translation by $\theta(t)$. So

$$\left(\mathrm{Ad}e^{\int_0^\tau u(s)ds}\right) f = (\mathrm{Ad}e^{\theta(\tau)})f = f_{\theta(\tau)}.$$

Therefore,

$$P_t^{u(\cdot)} = \left(\overrightarrow{\exp} \int_0^t f_{\theta(\tau)} d\tau\right) \circ e^{\theta(t)}.$$

The map

$$(\theta(\cdot), v) \mapsto \left(\overrightarrow{\exp} \int_0^t f_{\theta(\tau)} d\tau\right) \circ e^v$$

is continuous from $L^1([0, t], \mathbb{R}^d) \times \mathbb{R}^d$ to $\mathrm{Diff}M$, and moreover the map

$$u(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^d) \mapsto (\theta(\cdot), \theta(t)) \in L^1([0, t], \mathbb{R}^d) \times \mathbb{R}^d \tag{4}$$

has dense image in $L^1([0,t],\mathbb{R}^d)\times\mathbb{R}^d$, so the closure of the attainable set verifies

$$\left\{\overrightarrow{\exp}\int_0^t f_{\theta(\tau)d\tau} \mid \theta(\cdot)\in L^1([0,t],\mathbb{R}^d)\right\}\circ\left\{e^v \mid v\in\mathbb{R}^d\right\}\subset\overline{\mathcal{A}}.$$

So the study of $\overline{\mathcal{A}}$ is reduced to the study of the no more linear in the control system

$$\dot{x}=f_{\theta(t)}(x),\qquad \theta(t)\in\mathbb{R}^d, \tag{5}$$

where $\int_{\mathbb{T}^d} f(\tau)d\tau=0$, see Remark 2, and $t\mapsto\theta(t)\in L^1_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^s)$. By standard convexification, see [5, Th. 8.7], the flow of system (5) can approximate the flow of any convex combination of vector fields $f_\theta,\theta\in\mathbb{R}^d$. By re-scaling of the time, the flow of system (5) can approximate the flow of any convex combination of vector fields $f_\theta,\theta\in\mathbb{R}^d$ up to a positive multiplicative constant, that is, the flow of any vector field in the convex subset cone $\left\{f_\theta \mid \theta\in\mathbb{R}^d\right\}$.

**Lemma 1.** *Let $f\in C^\infty(\mathbb{T}^d,\mathbb{R}^n)$. If $\int\limits_{\mathbb{T}^d} f(x)\,dw(x)=0$, then*

$$\overline{\text{cone}\{f_\theta \mid \theta\in\mathbb{T}^d\}}=\overline{\text{span}\{f_\theta \mid \theta\in\mathbb{T}^d\}},$$

*where the closure is taken in the $C^\infty$-topology.*

*Proof.* Assume that $\overline{\text{cone}\{f_\theta\in\mathbb{T}^d\}}$ is not a vector space. Then, according to the standard separation theorem for locally convex topological vector spaces, there exists $\varphi\in C^\infty(\mathbb{T}^d,\mathbb{R}^n)^*$ such that $\varphi$ restricted to $\overline{\text{span}\{f_\theta \mid \theta\in\mathbb{T}^d\}}$ is not identically 0 and $\langle\varphi,f_\theta\rangle\le 0,\ \forall\,\theta\in\mathbb{T}^d$.

Note that $\theta\mapsto\langle\varphi,f_\theta\rangle$ is a continuous function on $\mathbb{T}^d$, hence it is strictly negative on an open subset of $\mathbb{T}^d$. We have

$$0>\int_{\mathbb{T}^d}\langle\varphi,f_\theta\rangle\,dw(\theta)=\langle\varphi,\int_{\mathbb{T}^d}f_\theta\,dw(\theta)\rangle.$$

On the other hand,

$$\left(\int_{\mathbb{T}^d}f_\theta\,dw(\theta)\right)(x)=\int_{\mathbb{T}^d}f(x+\theta)\,dw(\theta)=\int_{\mathbb{T}^d}f(\theta)\,dw(\theta)=0,\qquad x\in\mathbb{T}^d.$$

In other words, $\int\limits_{\mathbb{T}^d}f_\theta\,dw(\theta)=0$ and we obtain a contradiction which proofs the lemma. □

So to summarize, the flow of system (3) can approximate the flow of any vector field of the form $\alpha f_\theta+u$, with $\alpha\in\mathbb{R}$ and $\theta,u\in\mathbb{R}^d$. In particular, according to Theorem 1, the flow of every vector field in $\overline{\text{Lie}\left\{f_\theta+u \mid \theta,u\in\mathbb{R}^d\right\}}$ belongs to $\overline{\mathcal{A}}$. According to Theorem 3, if $f\in\mathfrak{V}^d$ and if $\Gamma$ denotes the subgroup of $\mathbb{Z}^d$ generated by $\mathcal{M}_f$, then $\overline{\text{Lie}\left\{f+u \mid u\in\mathbb{R}^d\right\}}=\text{Vec}_0(\mathbb{T}^d)_\Gamma$ and $\overline{\mathcal{A}}=\text{Diff}_0\mathbb{T}^d_\Gamma$.

# 5 Proof of Theorem 5

**(i)** Let us prove the first statement of Theorem 5. Let $d=2$ or $d=3$. We recall that

$$(\hat{\mathbb{T}}^d)^N=(\mathbb{T}^d)^N\setminus\left\{(y_1,\dots,y_N)\in(\mathbb{T}^d)^N \mid \exists k\neq\ell, y_k=y_\ell\right\}.$$

For every $f\in\text{Vec}_0\mathbb{T}^d$, we consider the lift of control system (3) in the space of $N$-ensembles,

$$\dot{x}_j=f(x_j)+u(t),\qquad u(t)\in\mathbb{R}^d, j\in\{1,\dots,N\}, \tag{6}$$

where $x(t)=(x_1(t),\dots,x_N(t))\in(\hat{\mathbb{T}}^d)^N$ and $u(\cdot)$ is measurable and locally bounded. As explained in the proof of Theorem 4, the attainable set of system (6) has the same closure of the attainable set of the following system,

$$\dot{x}_j=\alpha(t)f(x_j)+u(t),\qquad \alpha(t)\in\mathbb{R}, u(t)\in\mathbb{R}^d, j\in\{1,\dots,N\}, \tag{7}$$

where $\alpha(\cdot)$ and $u(\cdot)$ are measurable and locally bounded. System (7) is linear with respect to the control. According to the Rashevski – Chow theorem, such a system is globally controllable if it is Lie bracket generating. Let us prove that there exists a residual set $\mathcal{R} \subset \mathrm{Vec}_0\mathbb{T}^d$ such that for every $f \in \mathcal{R}$, for every $N \in \mathbb{N}^*$, system (6) is Lie bracket generating in $(\hat{\mathbb{T}}^d)^N$.

Let us fix $N \in \mathbb{N}^*$. We consider $f \in \mathfrak{V}^d$ such that $\Gamma = \mathbb{Z}^d$, then according to Theorem 3, $\overline{\mathrm{Lie}\left\{f + u \mid u \in \mathbb{R}^d\right\}} = \mathrm{Vec}_0\mathbb{T}^d$. In this case, as explained in the proof of Theorem 2, system (6) is Lie bracket generating at every point of $(\hat{\mathbb{T}}^d)^N$.

**Remark 3.** *Although the set* $\left\{f \in \mathfrak{V}^d \mid \Gamma = \mathbb{Z}^d\right\}$ *is dense in* $\mathrm{Vec}_0\mathbb{T}^d$*, it is not residual.*

The manifold $(\hat{\mathbb{T}}^d)^N$ is the union of a countable number of compacts, $(\hat{\mathbb{T}}^d)^N = \bigcup_i K_{Ni}$, where $K_{Ni} \Subset (\hat{\mathbb{T}}^d)^N$, $i = 1, 2, \ldots$. The set of vector fields $f \in \mathrm{Vec}_0\mathbb{T}^d$ such that system (6) is Lie bracket generating at every point of $K_{Ni}$ is open. Moreover, we know that it is dense, hence it is open dense. The desired residual set is just the intersection of these open dense subsets for all $Ni$.

**(ii)** Let us prove the second statement of Theorem 5. Let $d \geq 2$, $f \in \mathrm{Vec}_0\mathbb{T}^d$, $N \geq 2$ and $T > 0$. Let $t \mapsto x(t) = (x_1(t), \ldots, x_N(t))$ be the solution of (6). For every $t \in [0, T]$, $x_1(t) \neq x_2(t)$ because $x_1(0) \neq x_2(0)$, and so $t \mapsto \xi(t) = \ln|x_1(t) - x_2(t)|$, $t \in [0, T]$, is well defined. Moreover, for every $t \in [0, T]$,

$$\dot{\xi}(t) = \frac{\langle f(x_1(t)) - f(x_2(t)), x_1(t) - x_2(t)\rangle}{|x_1(t) - x_2(t)|} \geq -\|f\|_1,$$

and so $|\xi(t)| \leq \xi(0) - T\|f\|_1$. Then $|x_1(t) - x_2(t)| \geq e^{-T\|f\|_1}|x_1(0) - x_2(0)|$ for every $t \in [0, T]$, and so the configurations where $x_1(t)$ and $x_2(t)$ are very close are not reachable in any time $t \in [0, T]$.

# 6 Proof of Theorem 3

The proof of Theorem 3 requires several steps and the study depends on the dimension of the considered torus. For the bi-dimensional torus, the statement of Theorem 3 is proved by Theorem 6, and for the tri-dimensional torus by Theorem 8.

## 6.1 Bi-dimensional torus

On $\mathbb{T}^2$, the volume form $dx \wedge dy$ coincides with the symplectic form, and every divergence free vector field can be written as the sum of a Hamiltonian vector field $\overrightarrow{h} \in \mathrm{Ham}\mathbb{T}^2$ and a constant vector field. Indeed, if we denote $\omega = dx \wedge dy$, according to Cartan's formula, the Lie derivative of $\omega$ along any vector field $V \in \mathrm{Vec}\mathbb{T}^2$ verifies

$$\mathcal{L}_V\omega = (i_V \circ d + d \circ i_V)\omega = d \circ i_V\omega.$$

If $\mathrm{div}V = 0$, then $i_V\omega$ is closed, so there exists a constant vector field $u = u_1\partial_{x_1} + u_2\partial_{x_2}$ such that $di_{V+u}\omega = 0$, so $V + u$ verifies $\mathcal{L}_{V+u}\omega = 0$ and $V + u$ is Hamiltonian.

For this reason we can assume that there exists a smooth function $h \in C^\infty(\mathbb{T}^2, \mathbb{R})$, associated to the Hamiltonian vector field

$$\overrightarrow{h}(x, y) = -\frac{\partial h}{\partial y}(x, y)\partial_x + \frac{\partial h}{\partial x}(x, y)\partial_y, \qquad (x, y) \in \mathbb{T}^2,$$

such that $f = \overrightarrow{h}$.

The non-zero modes that appear in the Fourier decomposition of the function $h$ are exactly those that appear in the decomposition of the vector field $\overrightarrow{h}$. The set of modes in the decomposition of $h$ is denoted by $\mathcal{M}_h$, and the subgroup of $\mathbb{Z}^2$ generated by $\mathcal{M}_h$ is denoted by $\Gamma$. Note that the subgroups of $\mathbb{Z}^2$ generated by $\mathcal{M}_h$ and $\mathcal{M}_f$ are the same. We recall that for $a, b \in C^\infty(\mathbb{T}^2, \mathbb{R})$, their Poisson bracket is defined by

$$\{a, b\} = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x},$$

and the arrow map $C^\infty(\mathbb{T}^2, \mathbb{R}) \mapsto \mathrm{Ham}\mathbb{T}^2$ preserves the Lie algebra structure owing to the relation

$$\overrightarrow{\{a, b\}} = \left[\overrightarrow{a}, \overrightarrow{b}\right].$$

**Theorem 6.** *Let $\#\mathcal{M}_h < \infty$.*

— *If* $\mathrm{span}\Gamma = \mathbb{R}^2$*, then*

$$\mathrm{Lie}\left\{\overrightarrow{h} + u \mid u \in \mathbb{R}^d\right\} = \mathrm{span}\left\{\overrightarrow{\cos}\langle m, \cdot\rangle, \overrightarrow{\sin}\langle m, \cdot\rangle, \partial_x, \partial_y \mid m \in \Gamma\right\}.$$

— *If* $\mathrm{span}\Gamma$ *is of dimension 1, then*

$$\mathrm{Lie}\left\{\overrightarrow{h} + u \mid u \in \mathbb{R}^d\right\} = \mathrm{Lie}\left\{\overrightarrow{\cos}\langle m, \cdot\rangle, \overrightarrow{\sin}\langle m, \cdot\rangle, \partial_x, \partial_y \mid m \in \mathcal{M}_h\right\}.$$

Throughout the proof of Theorem 6, we will use the notation $\mathfrak{L}_h = \mathrm{Lie}\left\{\overrightarrow{h} + u \mid u \in \mathbb{R}^d\right\}$. We will also make use of the following identity:

$$\mathrm{ad}^k_{\partial_x}\mathrm{ad}^\ell_{\partial_y}\overrightarrow{h} = \overrightarrow{\frac{\partial^{k+\ell} h}{\partial x^k \partial y^\ell}} \in \mathfrak{L}_h, \qquad k, \ell \in \mathbb{N}. \tag{8}$$

**Lemma 2.** *If $\#\mathcal{M}_h < \infty$, then*

$$\mathfrak{L}_h = \mathrm{Lie}\left\{\overrightarrow{\cos}\langle m, \cdot\rangle, \partial_x, \partial_y \mid m \in \mathcal{M}_h\right\}.$$

*Proof.* Let $h = \sum_{m \in \mathcal{M}_h} h_m e^{i\langle m, \cdot\rangle}$ be the finite Fourier decomposition of $h$, where the coefficients $h_m$ are complex. The function $f$ is real-valued so $h_{-m} = \overline{h_m}$ for every $m \in \mathcal{M}_h$. Let us prove that $\overrightarrow{\cos}\langle m_0, \cdot\rangle \in \mathfrak{L}_h$ for every $m_0 \in \mathcal{M}_h$. By a straightforward computation,

$$\mathrm{ad}^2_{\partial_x}\overrightarrow{h} = -\sum_{m \in \mathcal{M}_h} m_x^2 h_m \overrightarrow{e^{i\langle m, \cdot\rangle}} \in \mathfrak{L}_h,$$

and then for every $\alpha, \beta \in \mathbb{R}$,

$$(\alpha - \mathrm{ad}^2_{\partial_x})(\beta - \mathrm{ad}^2_{\partial_y})\overrightarrow{h} = \sum_{m \in \mathcal{M}_h} (\alpha - m_x^2)(\beta - m_y^2) h_m \overrightarrow{e^{i\langle m, \cdot\rangle}} \in \mathfrak{L}_h.$$

Let $m_0 = (m_{0x}, m_{0y}) \in \mathcal{M}_h$. For any $m \in \mathbb{Z}^2$ we denote $|m| = |m_0|$ if $|m_x| = |m_{0x}|$ and $|m_y| = |m_{0y}|$. By iteration and thanks to a specific choice of $\alpha, \beta \in \mathbb{R}$, we obtain that

$$\prod_{\substack{m_1, m_2 \in \mathcal{M}_h \\ m_{1x} \neq m_{0x}, m_{2y} \neq m_{0y}}} (m_{1x}^2 - \mathrm{ad}^2_{\partial_x})(m_{2y}^2 - \mathrm{ad}^2_{\partial_y})\overrightarrow{h} = \gamma \sum_{|m| = |m_0|} h_m \overrightarrow{e^{i\langle m, \cdot\rangle}} \in \mathfrak{L}_h,$$

where

$$\gamma = \prod_{\substack{m_1, m_2 \in \mathcal{M}_h \\ m_{1x} \neq m_{0x}, m_{2y} \neq m_{0y}}} (m_{1x}^2 - m_{0x}^2)(m_{2y}^2 - m_{0y}^2) \neq 0,$$

and so $\sum_{|m|=|m_0|} h_m \overrightarrow{e^{i\langle m,\cdot\rangle}} \in \mathfrak{L}_h$. The function $h$ is real-valued, so $\overline{h_m} = h_{-m}$ for every $m \in \mathcal{M}_h$. If $m_0 = (m_{0x}, m_{0y})$, we denote $m_0' = (m_{0x}, -m_{0y})$, and then

$$\sum_{|m|=|m_0|} h_m \overrightarrow{e^{i\langle m,\cdot\rangle}} = 2\mathfrak{Re}(h_{m_0}\overrightarrow{e^{i\langle m_0,\cdot\rangle}} + h_{m_0'}\overrightarrow{e^{i\langle m_0',\cdot\rangle}}) \in \mathfrak{L}_h.$$

Let us consider the case where $m_{0x}, m_{0y} \neq 0$. According to formula (8),

$$-\sum_{|m|=|m_0|} h_m \overrightarrow{\frac{\partial^2}{\partial x \partial y} e^{i\langle m,\cdot\rangle}} = 2m_{0x}m_{0y}\mathfrak{Re}(h_{m_0}\overrightarrow{e^{i\langle m_0,\cdot\rangle}} - h_{m_0'}\overrightarrow{e^{i\langle m_0',\cdot\rangle}}) \in \mathfrak{L}_h,$$

so by linear combination $\mathfrak{Re}(h_{m_0}\overrightarrow{e^{i\langle m_0,\cdot\rangle}}) = \mathfrak{Re}(h_{m_0})\overrightarrow{\cos}\langle m_0,\cdot\rangle - \mathfrak{Im}(h_{m_0})\overrightarrow{\sin}\langle m_0,\cdot\rangle \in \mathfrak{L}_h$. Taking the derivative with respect to one variable we obtain that

$$-\mathfrak{Re}(h_{m_0})\overrightarrow{\sin}\langle m_0,\cdot\rangle - \mathfrak{Im}(h_{m_0})\overrightarrow{\cos}\langle m_0,\cdot\rangle \in \mathfrak{L}_h,$$

and so by linear combination, $(\mathfrak{Re}(h_{m_0})^2 + \mathfrak{Im}(h_{m_0})^2)\overrightarrow{\cos}\langle m_0,\cdot\rangle \in \mathfrak{L}_h$, and so $\overrightarrow{\cos}\langle m_0,\cdot\rangle \in \mathfrak{L}_h$. The other cases can be easily derived from the previous one. □

**Lemma 3.** *Let $m = (m_1, m_2)$ and $n = (n_1, n_2)$. Let $m \wedge n = m_1 n_2 - m_2 n_1$. If $m, n \in \mathcal{M}_h$ and if $m \wedge n \neq 0$, then $\overrightarrow{\cos}\langle m+n,\cdot\rangle \in \mathfrak{L}_h$.*

*Proof.* According to Lemma 2,

$$\{\sin\langle n,\cdot\rangle, \cos\langle m,\cdot\rangle\} = (m \wedge n)\sin\langle m,\cdot\rangle\cos\langle n,\cdot\rangle \in \mathfrak{L}_h,$$
$$\{\cos\langle n,\cdot\rangle, \sin\langle m,\cdot\rangle\} = (m \wedge n)\cos\langle m,\cdot\rangle\sin\langle n,\cdot\rangle \in \mathfrak{L}_h.$$

So by linear combination $\overrightarrow{\sin}\langle m+n,\cdot\rangle \in \mathfrak{L}_h$. By (8), $\overrightarrow{\cos}\langle m+n,\cdot\rangle \in \mathfrak{L}_h$. □

*Proof of Theorem 6.* If span$\mathcal{M}_h$ is of dimension 1, we can assume up to an orthonormal change of variables that $\frac{\partial}{\partial y}h = 0$. The Poisson Bracket of two functions that only depend on $x$ is zero, so

$$\text{Lie}\{\overrightarrow{\cos}\langle m,\cdot\rangle, \partial_x, \partial_y \mid m \in \mathcal{M}_h\} = \text{span}\left\{\overrightarrow{\cos}\langle m,\cdot\rangle, \overrightarrow{\sin}\langle m,\cdot\rangle, \partial_x, \partial_y \mid m \in \mathfrak{L}_h\right\}.$$

If span$\mathcal{M}_h = \mathbb{R}^2$, let us introduce the sets $\mathcal{I}_k(h), k \in \mathbb{N}^*$, defined by $\mathcal{I}_0(h) = \mathcal{M}_h$ and

$$\mathcal{I}_{k+1}(h) = \mathcal{I}_k(h) \cup \{m+n \mid m, n \in \mathcal{I}_k(h), m \wedge n \neq 0\}.$$

According to Lemma 3, $\overrightarrow{\cos}\langle m,\cdot\rangle \in \mathfrak{L}_h$ for every $m \in \cup_{k\in\mathbb{N}}\mathcal{I}_k(h)$. But if span$\mathcal{M}_h = \mathbb{R}^2$, it is easy to see that $\cup_{k\in\mathbb{N}}\mathcal{I}_k(h) = \Gamma$. Indeed every element $m \in \Gamma$ can be written as a sum $m = m_1 \pm \cdots \pm m_p$, with $m_1, \ldots, m_p \in \mathcal{M}_h$. Note that if $m \in \mathcal{M}_h$ then it is also verified that $\overrightarrow{\cos}\langle -m,\cdot\rangle, \overrightarrow{\sin}\langle -m,\cdot\rangle \in \mathfrak{L}_h$. If $m = m_1 + m_2$ and if $m_1 \wedge m_2 = 0$, necessarily there exists $m_3 \in \mathcal{M}_h$ such that $m_1 \wedge m_3 \neq 0$. Then $m_1 + m_3 \in \cup_{k\in\mathbb{N}}\mathcal{I}_k(h)$ and $(m_1+m_3)\wedge m_2 \neq 0$, so $m_1+m_2+m_3 \in \cup_{k\in\mathbb{N}}\mathcal{I}_k(h)$, and $(m_1+m_2+m_3) \neq -m_3$, so finally $m_1 + m_2 \in \cup_{k\in\mathbb{N}}\mathcal{I}_k(h)$ and

$$\mathfrak{L}_h = \text{Lie}\{\overrightarrow{\cos}\langle m,\cdot\rangle, \partial_x, \partial_y \mid m \in \Gamma\} = \text{span}\left\{\overrightarrow{\cos}\langle m,\cdot\rangle, \overrightarrow{\sin}\langle m,\cdot\rangle, \partial_x, \partial_y \mid m \in \Gamma\right\}.$$

Indeed the Lie algebra is composed of linear combinations and derivatives of the modes present in $\Gamma$, which is closed. □

In order to check that $\Gamma = \mathbb{Z}^2$, and so that Lie $\left\{\overrightarrow{h} + u \mid u \in \mathbb{R}^d\right\}$ is dense in Vec$_0M$, we can apply the following criterion from [4, Lem. 1].

**Lemma 4.** *The subgroup generated by $\mathcal{M}_h$ is equal to $\mathbb{Z}^2$ if and only if the greatest common divisor (g.c.d) of the numbers $\{m \wedge n \mid m, n \in \mathcal{M}_h\}$ equals 1.*

## 6.2 Tri-dimensional torus

On $\mathbb{T}^3$, we use the Fourier decomposition of a divergence free vector field,

$$f = \sum_{m \in \mathcal{M}_f} p_m e^{i\langle m, \cdot \rangle} = \sum_{m \in \mathcal{M}_f} a_m \cos\langle m, \cdot \rangle + b_m \sin\langle m, \cdot \rangle,$$

where $p_m$ are linear combinations of $\partial_x, \partial_y, \partial_z$. We identify the constant vector fields $p_m$ with vectors in $\mathbb{C}^3$, whose coordinates correspond to the coefficients in $\partial_x, \partial_y, \partial_z$. In particular $p_m = \frac{a_m - ib_m}{2}$. The components of the vector $\mathfrak{Re}(p_m)$ (respectively $\mathfrak{Im}(p_m)$) correspond to the real parts (respectively to the imaginary parts) of the components of $p_m$. With these notations, and because $\operatorname{div} f = 0$, $\langle m, p_m \rangle = \langle m, a_m \rangle = \langle m, b_m \rangle = 0$ for every $m \in \mathcal{M}_f$. The set of directions that are orthogonal to $m \in \mathbb{Z}^3$ is denoted by $m^\perp := \left\{ v \in \mathbb{R}^3 \mid \langle m, v \rangle = 0 \right\}$. For two vectors $a, b \in \mathbb{R}^3$, their cross product is denoted by $a \wedge b$. The subgroup of $\mathbb{Z}^3$ generated by $\mathcal{M}_f$ is denoted by $\Gamma$. The aim of this section is to characterize the Lie algebra $\operatorname{Lie}\left\{ f + u \mid u \in \mathbb{R}^3 \right\}$. We will use the notation $\mathfrak{L}_f = \operatorname{Lie}\left\{ f + u \mid u \in \mathbb{R}^3 \right\}$.

In the following, we will make use of the following formulas. For every $m \neq 0$,

$$\begin{aligned}
\operatorname{ad}_{\partial_x} a_m \cos\langle m, \cdot \rangle &= -m_x a_m \sin\langle m, \cdot \rangle, \quad & \operatorname{ad}_{\partial_x} b_m \sin\langle m, \cdot \rangle &= m_x b_m \cos\langle m, \cdot \rangle, \\
\operatorname{ad}_{\partial_y} a_m \cos\langle m, \cdot \rangle &= -m_y a_m \sin\langle m, \cdot \rangle, \quad & \operatorname{ad}_{\partial_y} b_m \sin\langle m, \cdot \rangle &= m_y b_m \cos\langle m, \cdot \rangle, \\
\operatorname{ad}_{\partial_z} a_m \cos\langle m, \cdot \rangle &= -m_z a_m \sin\langle m, \cdot \rangle, \quad & \operatorname{ad}_{\partial_z} b_m \sin\langle m, \cdot \rangle &= m_z b_m \cos\langle m, \cdot \rangle.
\end{aligned}$$

**Lemma 5.** *Let $\#\mathcal{M}_f < \infty$. Then*

$$\mathfrak{L}_f = \operatorname{Lie}\left\{ a_m \cos\langle m, \cdot \rangle + b_m \sin\langle m, \cdot \rangle, \partial_x, \partial_y, \partial_z \mid m \in \mathcal{M}_f \right\}.$$

*Proof.* As for the bi-dimensional case, we explain how the isolated frequencies also belong to the Lie algebra. Indeed, let $m_0 \in \mathcal{M}_f$, let us prove that

$$a_{m_0} \cos\langle m_0, \cdot \rangle + b_{m_0} \sin\langle m_0, \cdot \rangle \in \mathfrak{L}_f.$$

By a straightforward computation,

$$\operatorname{ad}^2_{\partial_x} f = - \sum_{m \in \mathcal{M}_f} m_x^2 p_m e^{i\langle m, \cdot \rangle} \in \mathfrak{L}_f,$$

and so for every $\alpha \in \mathbb{R}$,

$$\alpha f - \operatorname{ad}^2_{\partial_x} f = \sum_{m \in \mathcal{M}_f} (\alpha - m_x^2) p_m e^{i\langle m, \cdot \rangle} \in \mathfrak{L}_f.$$

If there exists $m_1 \in \mathcal{M}_f$ such that $|m_{1x}| \neq |m_{0x}|$, then

$$(m_{1x}^2 - \operatorname{ad}^2_{\partial_x}) f = \sum_{m \in \mathcal{M}_f} (m_{1x}^2 - m_x^2) p_m e^{i\langle m, \cdot \rangle} \in \mathfrak{L}_f.$$

By iteration of such operation for every $m_1 \in \mathcal{M}_f$ that verifies $|m_{1x}| \neq |m_{0x}|$, we obtain that

$$\prod_{\substack{m_1 \in \mathcal{M}_f \\ |m_{1x}| \neq |m_{0x}|}} (m_{1x}^2 - \operatorname{ad}^2_{\partial_x}) f = \beta \sum_{\substack{m \in \mathcal{M}_f \\ |m_x| = |m_{0x}|}} p_m e^{i\langle m, \cdot \rangle} \in \mathfrak{L}_f,$$

where

$$\beta = \prod_{\substack{m_1 \in \mathcal{M}_f \\ |m_{1x}| \neq |m_{0x}|}} (m_{1x}^2 - m_{0x}^2) \neq 0.$$

For any $m \in \mathbb{Z}^3$, $|m| = |m_0|$ means that $|m_x| = |m_{0x}|, |m_y| = |m_{0y}|$ and $|m_z| = |m_{0z}|$. By iteration and thanks to an adapted choice of $\alpha, \beta$, we obtain that

$$\prod_{\substack{m_1, m_2, m_3 \in \mathcal{M}_f \\ |m_{1x}| \neq |m_{0x}|, |m_{2y}| \neq |m_{0y}|, |m_{3z}| \neq |m_{0z}|}} (m_{1x}^2 - \mathrm{ad}^2_{\partial_x})(m_{2y}^2 - \mathrm{ad}^2_{\partial_y})(m_{3z}^2 - \mathrm{ad}^2_{\partial_z}) f = \gamma \sum_{\substack{m \in \mathcal{M}_f \\ |m| = |m_0|}} p_m e^{i\langle m, \cdot \rangle} \in \mathfrak{L}_f,$$

where

$$\gamma = \prod_{\substack{m_1, m_2, m_3 \in \mathcal{M}_f \\ |m_{1x}| \neq |m_{0x}|, |m_{2y}| \neq |m_{0y}|, |m_{3z}| \neq |m_{0z}|}} (m_{1x}^2 - m_{0x}^2)(m_{2y}^2 - m_{0y}^2)(m_{3z}^2 - m_{0z}^2) \neq 0,$$

and so

$$\sum_{|m| = |m_0|} p_m e^{i\langle m, \cdot \rangle} \in \mathfrak{L}_f.$$

Let us consider the case where $m_{0x}, m_{0y}, m_{0z} \neq 0$. There are $2^3 = 8$ modes $m \in \mathbb{Z}^3$ that verify $|m| = |m_0|$. The vector field $f$ is real-valued, so $\overline{p_m} = p_{-m}$ for every $m \in \mathcal{M}_f$. There are 4 couples of opposite modes $m \in \mathbb{Z}^3$ that verify $|m| = |m_0|$, so

$$\sum_{|m| = |m_0|} p_m e^{i\langle m, \cdot \rangle} = 2(\mathfrak{Re}(p_{m_0} e^{i\langle m_0, \cdot \rangle}) + \sum_{k=1}^{3} \mathfrak{Re}(p_{m_{0,k}} e^{i\langle m_{0,k}, \cdot \rangle})) \in \mathfrak{L}_f,$$

where

$$m_{0,1} = (m_{0x}, m_{0y}, -m_{0z}), \quad m_{0,2} = (m_{0x}, -m_{0y}, m_{0z}), \quad m_{0,3} = (-m_{0x}, m_{0y}, m_{0z}).$$

Then

$$\frac{1}{m_{0x}}(m_{0x} + \mathrm{ad}_{\partial_x}) \sum_{|m| = |m_0|} p_m e^{i\langle m, \cdot \rangle} = 4(\mathfrak{Re}(p_{m_0} e^{i\langle m_0, \cdot \rangle}) + \sum_{k=1}^{2} \mathfrak{Re}(p_{m_{0,k}} e^{i\langle m_{0,k}, \cdot \rangle})) \in \mathfrak{L}_f.$$

Then we can apply $\frac{1}{m_{0y}}(m_{0y} + \mathrm{ad}_{\partial_y})$ and $\frac{1}{m_{0z}}(m_{0z} + \mathrm{ad}_{\partial_z})$ to the previous vector field and we obtain that

$$\mathfrak{Re}(p_{m_0} e^{i\langle m_0, \cdot \rangle}) = a_{m_0} \cos\langle m_0, \cdot \rangle + b_{m_0} \sin\langle m_0, \cdot \rangle \in \mathfrak{L}_f.$$

The other cases can be easily derived from the previous one. □

The following formulas can be obtained by straightforward computations and will be useful for the remaining proofs.

**Proposition 2.**

1)
$$[p_m \sin\langle m, \cdot \rangle, p_n \cos\langle n, \cdot \rangle] = \langle m, p_n \rangle p_m \cos\langle m, \cdot \rangle \cos\langle n, \cdot \rangle + \langle n, p_m \rangle p_n \sin\langle m, \cdot \rangle \sin\langle n, \cdot \rangle,$$
$$[p_m \cos\langle m, \cdot \rangle, p_n \sin\langle m, \cdot \rangle] = -\langle m, p_n \rangle p_m \sin\langle m, \cdot \rangle \sin\langle n, \cdot \rangle - \langle n, p_m \rangle p_n \cos\langle m, \cdot \rangle \cos\langle n, \cdot \rangle,$$
$$[p_m \cos\langle m, \cdot \rangle, p_n \cos\langle m, \cdot \rangle] = -\langle m, p_n \rangle p_m \sin\langle m, \cdot \rangle \cos\langle n, \cdot \rangle + \langle n, p_m \rangle p_n \cos\langle m, \cdot \rangle \sin\langle n, \cdot \rangle,$$
$$[p_m \sin\langle m, \cdot \rangle, p_n \sin\langle m, \cdot \rangle] = -\langle m, p_n \rangle p_m \cos\langle m, \cdot \rangle \sin\langle n, \cdot \rangle - \langle n, p_m \rangle p_n \sin\langle m, \cdot \rangle \cos\langle n, \cdot \rangle.$$

2)
$$[p_m \sin\langle m, \cdot \rangle, p_n \sin\langle n, \cdot \rangle] - [p_m \cos\langle m, \cdot \rangle, p_n \cos\langle n, \cdot \rangle] = (\langle m, p_n \rangle p_m - \langle n, p_m \rangle p_n) \sin\langle m + n, \cdot \rangle,$$
$$[p_m \cos\langle m, \cdot \rangle, p_n \sin\langle n, \cdot \rangle] + [p_m \sin\langle m, \cdot \rangle, p_n \cos\langle n, \cdot \rangle] = (\langle m, p_n \rangle p_m - \langle n, p_m \rangle p_n) \cos\langle m + n, \cdot \rangle.$$

3)
$$\begin{aligned}&[a_m \cos\langle m, \cdot \rangle + b_m \sin\langle m, \cdot \rangle, c_n \cos\langle n, \cdot \rangle + d_n \sin\langle n, \cdot \rangle] \\ &-[-a_m \sin\langle m, \cdot \rangle + b_m \cos\langle m, \cdot \rangle, -c_n \sin\langle n, \cdot \rangle + d_n \cos\langle n, \cdot \rangle] \\ &= (\langle m, c_n \rangle b_m + \langle m, d_n \rangle a_m - \langle n, b_m \rangle c_n - \langle n, a_m \rangle d_n) \cos\langle m + n, \cdot \rangle \\ &+(\langle m, d_n \rangle b_m - \langle m, c_n \rangle a_m - \langle n, b_m \rangle d_n + \langle n, a_m \rangle c_n) \sin\langle m + n, \cdot \rangle.\end{aligned}$$

**Theorem 7.** *Let $\#\mathcal{M}_f < \infty$. If* $\mathrm{span}\mathcal{M}_f$ *is of dimension 1 or if* ($\mathrm{span}\mathcal{M}_f$ *is of dimension 2 and* $\mathrm{span}\,\{a_m, b_m \mid m \in \mathcal{M}_f\}$ *is of dimension 1), then*

$$\mathrm{Lie}\left\{f + u \mid u \in \mathbb{R}^3\right\} = \mathrm{span}\,\{(\alpha a_m + \beta b_m)\cos\langle m, \cdot\rangle + (-\beta a_m + \alpha b_m)\sin\langle m, \cdot\rangle, \partial_x, \partial_y, \partial_z \mid m \in \mathcal{M}_f, \alpha, \beta \in \mathbb{R}\},$$

*Proof.* Let $m \in \mathcal{M}_f$ be such that $m \neq 0$. According to Lemma 5, $a_m \cos\langle m, \cdot\rangle + b_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f$. If $m_x \neq 0$,

$$\mathrm{ad}_{\partial_x}(a_m \cos\langle m, \cdot\rangle + b_m \sin\langle m, \cdot\rangle) = \underbrace{m_x}_{\neq 0}(-a_m \sin\langle m, \cdot\rangle + b_m \cos\langle m, \cdot\rangle) \in \mathfrak{L}_f.$$

Else we compute $\mathrm{ad}_{\partial_y}$ or $\mathrm{ad}_{\partial_z}$, and because $m \neq 0$,

$$-a_m \sin\langle m, \cdot\rangle + b_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f,$$

so for every $\alpha, \beta \in \mathbb{R}$,

$$(\alpha a_m + \beta b_m)\cos\langle m, \cdot\rangle + (\alpha b_m - \beta a_m)\sin\langle m, \cdot\rangle \in \mathfrak{L}_f,$$

and

$$\begin{aligned}&\mathrm{Lie}\,\{a_m \cos\langle m, \cdot\rangle + b_m \sin\langle m, \cdot\rangle, \partial_x, \partial_y, \partial_z\}\\ =&\mathrm{span}\,\{(\alpha a_m + \beta b_m)\cos\langle m, \cdot\rangle + (\alpha b_m - \beta a_m)\sin\langle m, \cdot\rangle, \partial_x, \partial_y, \partial_z \mid \alpha, \beta \in \mathbb{R}\}.\end{aligned}$$

Let $n \in \mathcal{M}_f$ be another mode of $f$, then $a_n \cos\langle n, \cdot\rangle + b_n \cos\langle n, \cdot\rangle \in \mathfrak{L}_f$ according to Lemma 5. If $\mathrm{span}\mathcal{M}_f$ is of dimension 1, then $m \wedge n = 0$ and $m^\perp = n^\perp$. If $\mathrm{span}\mathcal{M}_f$ is of dimension 2 and if $m \wedge n \neq 0$, then $\mathrm{span}\mathcal{M}_f = \mathrm{span}\,\{m, n\}$. But $\mathrm{span}\,\{a_m, b_m \mid m \in \mathcal{M}_f\}$ is of dimension 1 and $\langle k, a_k\rangle = 0$ for every $k \in \mathcal{M}_f$, so necessarily $\mathrm{span}\,\{a_m, b_m \mid m \in \mathcal{M}_f\} = \mathrm{span}\,\{m \wedge n\}$. In each case, $a_m, b_m, a_n, b_n \in \mathrm{span}\,\{m \wedge n\}$, and

$$\langle m, a_n\rangle = \langle n, a_m\rangle = \langle m, b_n\rangle = \langle n, b_m\rangle = 0.$$

Applying the third formula of Proposition 2,

$$[a_m \cos\langle m, \cdot\rangle + b_m \sin\langle m, \cdot\rangle, a_n \cos\langle n, \cdot\rangle + b_n \sin\langle n, \cdot\rangle] = 0,$$

and

$$\mathfrak{L}_f = \mathrm{span}\,\{(\alpha a_m + \beta b_m)\cos\langle m, \cdot\rangle + (\alpha b_m - \beta a_m)\sin\langle m, \cdot\rangle, \partial_x, \partial_y, \partial_z \mid m \in \mathcal{M}_f, \alpha, \beta \in \mathbb{R}\}.$$

□

**Remark 4.** *If $m \in \mathcal{M}_f$, then*

$$\forall p_m \in m^\perp, p_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f \iff \forall p_m \in m^\perp, p_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f,$$

*and*

$$(\forall p_m, q_m \in m^\perp, \quad p_m \cos\langle m, \cdot\rangle + q_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f) \iff (\forall p_m \in m^\perp, \quad p_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f).$$

**Theorem 8.** *Let $\#\mathcal{M}_f < \infty$. If* $\mathrm{span}\mathcal{M}_f = \mathbb{R}^3$ *and* $\mathrm{span}\,\{a_m, b_m \mid m \in \mathcal{M}_f\} = \mathbb{R}^3$*, then*

$$\mathrm{Lie}\left\{f + u \mid u \in \mathbb{R}^3\right\} = \mathrm{span}\left\{p_m \cos\langle m, \cdot\rangle, q_m \sin\langle m, \cdot\rangle, \partial_x, \partial_y, \partial_z \mid m \in \Gamma, p_m, q_m \in m^\perp\right\}.$$

Before going to the proof of Theorem 8, we need to introduce several propositions.

**Proposition 3.** *Let $m \in \mathcal{M}_f$ be such that $m \neq 0$. If there exists $a_m, b_m, a'_m, b'_m$ constant vector fields such that $a_m, a'_m \neq 0$,*

$$\begin{cases} a_m \cos\langle m, \cdot\rangle + b_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f, \\ a'_m \cos\langle m, \cdot\rangle + b'_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f, \end{cases}$$

*and such that $\begin{pmatrix} a'_m \\ b'_m \end{pmatrix} \neq \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} a_m \\ b_m \end{pmatrix}$ for every $\alpha, \beta \in \mathbb{R}$, then for every $p_m, q_m \in m^\perp$,*

$$p_m \cos\langle m, \cdot\rangle + q_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f.$$

*Proof.* According to Remark 4, this is sufficient to show that there exist two non-colinear vectors $p_m, q_m \in m^\perp$, such that

$$\begin{aligned} &(p_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f \quad \text{or} \quad p_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f), \\ \text{and} \quad &(q_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f \quad \text{or} \quad q_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f). \end{aligned}$$

According to Theorem 7, for every $\alpha, \beta \in \mathbb{R}$,

$$\begin{cases} (\alpha a_m + \beta b_m) \cos\langle m, \cdot\rangle + (\alpha b_m - \beta a_m) \sin\langle m, \cdot\rangle \in \mathfrak{L}_f, \\ (\alpha' a'_m + \beta' b'_m) \cos\langle m, \cdot\rangle + (\alpha' b'_m - \beta' a'_m) \sin\langle m, \cdot\rangle \in \mathfrak{L}_f. \end{cases} \tag{9}$$

<u>If $a'_m \wedge b'_m = 0$ and $a_m \wedge b_m = 0$:</u> Thanks to an adapted choice of $\alpha, \beta, \alpha', \beta'$, we obtain that

$$a_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f \quad \text{and} \quad a'_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f,$$

and necessarily $a_m \wedge a'_m \neq 0$, otherwise it would exist $\alpha, \beta \in \mathbb{R}$ such that $a'_m = \alpha a_m + \beta b_m$ and $b'_m = \alpha b_m - \beta a_m$.
<u>If $a'_m \wedge b'_m = 0$ and if $a_m \wedge b_m \neq 0$:</u> Thanks to an adapted choice of $\alpha'$ and $\beta'$, $a'_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f$. Moreover $a_m \wedge b_m \neq 0$, so there exist $\alpha, \beta$ such that

$$\begin{cases} \alpha a_m + \beta b_m = a'_m \\ (\alpha b_m - \beta a_m) \wedge a'_m \neq 0. \end{cases}$$

But $a'_m \wedge b'_m = 0$, so $(\alpha b_m - \beta a_m) \wedge b'_m \neq 0$. According to (9),

$$\begin{cases} a'_m \cos\langle m, \cdot\rangle + (\alpha b_m - \beta a_m) \sin\langle m, \cdot\rangle \in \mathfrak{L}_f, \\ a'_m \cos\langle m, \cdot\rangle + b'_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f, \end{cases} \tag{10}$$

So $(\alpha b_m - \beta a_m - b'_m) \sin\langle m, \cdot\rangle \in \mathfrak{L}_f$ and $(\alpha b_m - \beta a_m - b'_m) \wedge a'_m \neq 0$.
<u>If $a'_m \wedge b'_m \neq 0$ and $a_m \wedge b_m \neq 0$:</u> There exist $\alpha, \beta$ such that $\alpha a_m + \beta b_m = a'_m$ and then necessarily $\alpha b_m - \beta a_m \neq b'_m$. As explained in the previous case,

$$(\alpha b_m - \beta a_m - b'_m) \sin\langle m, \cdot\rangle \in \mathfrak{L}_f.$$

If $(\alpha b_m - \beta a_m) \wedge b'_m = 0$, then $b'_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f$, and by linear combination we also have $a'_m \cos\langle m, \cdot\rangle$, which is sufficient because $a'_m \wedge b'_m \neq 0$. Else, there exist $\gamma, \delta$ such that

$$\gamma b_m - \delta a_m = \alpha b_m - \beta a_m - b'_m,$$

and

$$\begin{cases} (\gamma a_m + \delta b_m) \cos\langle m, \cdot\rangle + (\gamma b_m - \delta a_m) \sin\langle m, \cdot\rangle \in \mathfrak{L}_f, \\ (\alpha b_m - \beta a_m - b'_m) \sin\langle m, \cdot\rangle \in \mathfrak{L}_f, \end{cases}$$

so $(\gamma a_m + \delta b_m) \cos\langle m, \cdot\rangle \in \mathfrak{L}_f$ and necessarily $(\gamma a_m + \delta b_m) \wedge (\gamma b_m - \delta a_m) \neq 0$, because $a_m \wedge b_m \neq 0$. □

**Proposition 4.** *Lets $m, n \in \mathcal{M}_f$ be such that $m \wedge n \neq 0$. If there exist $a_m, a'_m \in m^\perp$ and $c_n, d_n \in n^\perp$ such that*

$$a_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f, \quad a'_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f, \quad c_n \cos\langle n, \cdot\rangle + d_n \sin\langle n, \cdot\rangle \in \mathfrak{L}_f,$$

*and such that* $\operatorname{span}\{a_m, a'_m, c_n\} = \mathbb{R}^3$*, then $p_k \cos\langle k, \cdot\rangle + q_k \sin\langle k, \cdot\rangle \in \mathfrak{L}_f$ for every $p_k, q_k \in k^\perp$ and $k \in \langle m, n\rangle$, where $\langle m, n\rangle$ denotes the subgroup of $\mathbb{Z}^3$ generated by $m, n$.*

*Proof.* The vectors $\{a_m, a'_m, c_n\}$ generate $\mathbb{R}^3$ and $\operatorname{span}\{a_m, a'_m\} = m^\perp$, so $c_n \wedge (m \wedge n) \neq 0$. According to Theorem 7, for every $\alpha, \beta \in \mathbb{R}$,

$$(\alpha c_n + \beta d_n)\cos\langle n, \cdot\rangle + (\alpha d_n - \beta c_n)\sin\langle n, \cdot\rangle \in \mathfrak{L}_f.$$

Or $c_n \wedge d_n = 0$, and so we can chose $\alpha, \beta$ such that $\alpha d_n - \beta c_n = 0$, or $c_n \wedge d_n \neq 0$, and so we can chose $\alpha, \beta$ such that

$$(\alpha d_n - \beta c_n) \wedge (m \wedge n) = 0.$$

So without loss of generality we can assume that

$$c_n \cos\langle n, \cdot\rangle + d_n \sin\langle n, \cdot\rangle \in \mathfrak{L}_f, \tag{11}$$

with $c_n \wedge (m \wedge n) \neq 0$ and $d_n \in \operatorname{span}\{m \wedge n\}$. Then according to the third formula of Proposition 2 applied with $(a_m, b_m = 0, c_n, d_n)$, and because $\langle m, d_n\rangle = 0$,

$$p_{m+n} \cos\langle m+n, \cdot\rangle + q_{m+n} \sin\langle m+n, \cdot\rangle \in \mathfrak{L}_f,$$

with

$$\begin{cases} p_{m+n} = -\langle n, a_m\rangle d_n \in \operatorname{span}\{m \wedge n\} \\ q_{m+n} = -\langle m, c_n\rangle a_m + \langle n, a_m\rangle c_n. \end{cases}$$

Thanks to the same formula applied with $(a'_m, b'_m = 0, c_n, d_n)$,

$$p'_{m+n} \cos\langle m+n, \cdot\rangle + q'_{m+n} \sin\langle m+n, \cdot\rangle \in \mathfrak{L}_f,$$

with

$$\begin{cases} p'_{m+n} = -\langle n, a'_m\rangle d_n \in \operatorname{span}\{m \wedge n\} \\ q'_{m+n} = -\langle m, c_n\rangle a'_m + \langle n, a'_m\rangle c_n. \end{cases}$$

Because $\langle m, c_n\rangle \neq 0$ and $\operatorname{span}\{a_m, a'_m, c_n\} = \mathbb{R}^3$, then necessarily $q_{m+n} \wedge q'_{m+n} \neq 0$. Because $p_{m+n} \wedge p'_{m+n} = 0$, then necessarily for every $\alpha, \beta \in \mathbb{R}$,

$$(p'_{m+n}, q'_{m+n}) \neq (\alpha p_{m+n} + \beta q_{m+n}, \alpha q_{m+n} - \beta p_{m+n}).$$

According to Proposition 3,

$$a_{m+n} \cos\langle m+n, \cdot\rangle + b_{m+n} \sin\langle m+n, \cdot\rangle \in \mathfrak{L}_f, \tag{12}$$

for every $a_{m+n}, b_{m+n} \in (m+n)^\perp$. So by iteration of this computation we can obtain that $a_k \cos\langle k, \cdot\rangle \in \mathfrak{L}_f$ for every $a_k \in k^\perp$ and $k \in m\mathbb{N} + n\mathbb{N}$. Moreover, it is also verified that

$$a_m \cos\langle -m, \cdot\rangle \in \mathfrak{L}_f, \quad a'_m \cos\langle -m, \cdot\rangle \in \mathfrak{L}_f, \quad c_n \cos\langle -n, \cdot\rangle - d_n \sin\langle -n, \cdot\rangle \in \mathfrak{L}_f,$$

so in the same way we obtain that $a_k \cos\langle k, \cdot\rangle \in \mathfrak{L}_f$ for every $a_k \in k^\perp$ and $k \in m\mathbb{Z} + n\mathbb{Z}$. □

**Proposition 5.** *If* $\operatorname{span}\{a_m, b_m \mid m \in \mathcal{M}_f\} = \mathbb{R}^3$ *and* $\operatorname{span}\mathcal{M}_f = \mathbb{R}^3$*, and if there exists a non-zero mode $\ell \in \Gamma$, $p_\ell, p'_\ell \in \ell^\perp$ such that $p_\ell \wedge p'_\ell \neq 0$ and such that*

$$\begin{cases} p_\ell \cos\langle \ell, \cdot\rangle \in \mathfrak{L}_f, \\ p'_\ell \cos\langle \ell, \cdot\rangle \in \mathfrak{L}_f, \end{cases}$$

*then $p_k \cos\langle k, \cdot\rangle + q_k \sin\langle k, \cdot\rangle \in \mathfrak{L}_f$, for every $p_k, q_k \in k^\perp$ and $k \in \Gamma$.*

*Proof.* Because span $\{a_m, b_m \mid m \in \mathcal{M}_f\} = \mathbb{R}^3$ and span$\mathcal{M}_f = \mathbb{R}^3$, there exist a non-zero mode $m \in \mathcal{M}_f$ and $a_m, b_m \in m^\perp$ such that

$$a_m \cos\langle m, \cdot\rangle + b_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f,$$

and span $\{a_m, p_\ell, p'_\ell\} = \mathbb{R}^3$. So according to Proposition 4, $p_k \cos\langle k, \cdot\rangle \in \mathfrak{L}_f$ for every $p_k \in k^\perp$ and $k \in m\mathbb{Z} + \ell\mathbb{Z}$. Because $a_m \notin \ell^\perp$ and $\langle m, a_m\rangle = 0$, then $m \wedge \ell \neq 0$. Because span$\mathcal{M}_f = \mathbb{R}^3$, there exist a non-zero mode $\mathcal{M}_f$, and $a_n, b_n \in n^\perp, a_n \neq 0$ such that

$$a_n \cos\langle n, \cdot\rangle + b_n \sin\langle n, \cdot\rangle \in \mathfrak{L}_f,$$

and span $\{m, \ell, n\} = \mathbb{R}^3$. If $\langle m, a_n\rangle = \langle \ell, a_n\rangle = 0$, then $a_n = 0$. If we assume without loss of generality that $a_n \notin m^\perp$, according to Proposition 4, $p_k \cos\langle k, \cdot\rangle \in \mathfrak{L}_f$ for every $p_k \in k^\perp$ and $k \in m\mathbb{Z} + n\mathbb{Z}$. By iteration, we obtain that $p_k \cos\langle k, \cdot\rangle \in \mathfrak{L}_f$ for every $p_k \in k^\perp$ and $k \in m\mathbb{Z} + n\mathbb{Z} + \ell\mathbb{Z}$. Let $k' \in \Gamma$ be an other mode and $a_{k'}, b_{k'} \in {k'}^\perp, a_{k'} \neq 0$ such that

$$a_{k'} \cos\langle k', \cdot\rangle + b'_k \sin\langle k', \cdot\rangle \in \mathfrak{L}_f.$$

Necessarily there exists one mode in $\{m, n, \ell\}$, for example $m$, such that $a_{k'} \notin m^\perp$. Then $k' \wedge m \neq 0$ and $p_k \cos\langle k, \cdot\rangle \in \mathfrak{L}_f$ for every $p_k \in k^\perp$ and $k \in k'\mathbb{Z} + m\mathbb{Z}$. So in particular for every $k \in \mathcal{M}_f$, $p_k \cos\langle k, \cdot\rangle \in \mathfrak{L}_f$ for every $p_k \in k^\perp$. Let $k_1, k_2 \in \mathcal{M}_f$ be two non-zero modes. If $k_1 \wedge k_2 \neq 0$, according to the second formula of Proposition 2,

$$(\langle k_1, a_{k_2}\rangle a_{k_1} - \langle k_2, a_{k_1}\rangle a_{k_2}) \cos\langle k_1 + k_2, \cdot\rangle \in \mathfrak{L}_f, \tag{13}$$

for every $a_{k_1} \in k_1^\perp, a_{k_2} \in k_2^\perp$. So it is clear that $p_{k_1+k_2} \cos\langle k_1 + k_2, \cdot\rangle \in \mathfrak{L}_f$ for every $p_{k_1+k_2} \in (k_1+k_2)^\perp$. If $k_1 \wedge k_2 = 0$, then there exists $m \in \mathcal{M}_f$ such that $m \wedge k_1 \neq 0$ and such that for every $a \in k_1^\perp$ and $b \in m^\perp$,

$$\begin{cases} (\langle m, a\rangle b - \langle k_1, b\rangle a) \cos\langle k_1 + m, \cdot\rangle \in \mathfrak{L}_f, \\ (\langle -m, a\rangle b - \langle k_2, b\rangle a) \cos\langle k_2 - m, \cdot\rangle \in \mathfrak{L}_f, \end{cases}$$

and so we also prove that $p_k \cos\langle k, \cdot\rangle \in \mathfrak{L}_f$ for every $p_k \in k^\perp$ and $k \in \langle k_1, k_2\rangle$. This procedure can be generalized by recurrence and we obtain that $p_k \cos\langle k, \cdot\rangle \in \mathfrak{L}_f$ for every $p_k \in k^\perp$ and $k \in \Gamma$. □

*Proof of Theorem 8.* If span $\{a_m, b_m \mid m \in \mathcal{M}_f\} = \mathbb{R}^3$ and span$\mathcal{M}_f = \mathbb{R}^3$, according to Proposition 5, it is sufficient to prove that there exists a non-zero mode $\ell \in \Gamma$ such that $p_\ell \cos\langle \ell, \cdot\rangle \in \mathfrak{L}_f$ for every $p_\ell \in \ell^\perp$. There are two possible situations:

1. Either there exist $m, n \in \mathcal{M}_f$ such that $m \wedge n \neq 0$,

$$a_j \cos\langle j, \cdot\rangle + b_j \sin\langle j, \cdot\rangle \in \mathfrak{L}_f, \quad j \in \{m, n\},$$

   and such that span $\{a_m, b_m, a_n\} = \mathbb{R}^3$.

2. Or there exist $m, n, k \in \mathcal{M}_f$ such that span $\{m, n, k\} = \mathbb{R}^3$,

$$a_j \cos\langle j, \cdot\rangle + b_j \sin\langle j, \cdot\rangle \in \mathfrak{L}_f, \quad a_j \wedge b_j = 0, \quad j \in \{m, n, k\}, \tag{14}$$

   and such that span $\{a_m, a_n, a_k\} = \mathbb{R}^3$.

Let us consider both cases.

**1.** Because span $\{a_m, b_m, a_n\} = \mathbb{R}^3$, necessarily $a_m \wedge b_m \neq 0$. First we assume that $a_n \wedge b_n \neq 0$. According to Theorem 7, for every $\alpha, \beta \in \mathbb{R}$,

$$(\alpha a_j + \beta b_j) \cos\langle j, \cdot\rangle + (\alpha b_j - \beta a_j) \sin\langle j, \cdot\rangle \in \mathfrak{L}_f \quad j \in \{m, n\}.$$

Thanks to an adapted choice of $\alpha, \beta$, we can assume that $b_m, b_n \in \operatorname{span}\{m \wedge n\}$. According to the third formula of Proposition 2 applied with $(a_m, b_m, a_n, b_n)$,

$$p_{m+n} \cos\langle m+n, \cdot\rangle + q_{m+n} \sin\langle m+n, \cdot\rangle \in \mathfrak{L}_f,$$

with

$$\begin{cases} p_{m+n} = \langle m, a_n\rangle b_m - \langle n, a_m\rangle b_n \in \operatorname{span}\{m \wedge n\} \\ q_{m+n} = -\langle m, a_n\rangle a_m + \langle n, a_m\rangle a_n. \end{cases}$$

But it is also true that $a_n \cos\langle -n, \cdot\rangle - b_n \sin\langle -n, \cdot\rangle \in \mathfrak{L}_f$, so again we can apply the third formula of Proposition 2 with $(p_{m+n}, q_{m+n}, a_n, -b_n)$, and we obtain that $a'_m \cos\langle m, \cdot\rangle + b'_m \sin\langle m, \cdot\rangle \in \mathfrak{L}_f$, with

$$\begin{cases} a'_m = -\langle m, a_n\rangle^2 a_m \\ b'_m = -\langle m, a_n\rangle^2 b_m + \langle n, a_m\rangle\langle m, a_n\rangle b_n. \end{cases}$$

By assumption $a_n \wedge b_n \neq 0$, so $b_n \neq 0$. Moreover, $a_m, a_n \notin \operatorname{span}\{m \wedge n\}$, so $\langle n, a_m\rangle\langle m, a_n\rangle \neq 0$. Then $(a'_m, b'_m) \neq (\alpha a_m + \beta b_m, \alpha b_m - \beta a_m)$ for every $\alpha, \beta \in \mathbb{R}$ and $a_m, a'_m \neq 0$, so according to Proposition 3, $p_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f$ for every $p_m \in m^\perp$. If we assume that $a_n \wedge b_n = 0$, then $a_n \cos\langle n, \cdot\rangle \in \mathfrak{L}_f$. We can still assume that $b_m \wedge (m \wedge n) = 0$, and because $\operatorname{span}\{a_m, b_m, a_n\} = \mathbb{R}^3$, necessarily $a_n \wedge b_m \neq 0$. Finally we obtain that

$$p_{m+n} \cos\langle m+n, \cdot\rangle + q_{m+n} \sin\langle m+n, \cdot\rangle \in \mathfrak{L}_f,$$

with

$$\begin{cases} p_{m+n} = \langle m, a_n\rangle b_m \\ q_{m+n} = -\langle m, a_n\rangle a_m + \langle n, a_m\rangle a_n. \end{cases}$$

But $m \wedge (m+n) \neq 0$, $a_m \wedge b_m \neq 0$, $p_{m+n} \wedge q_{m+n} \neq 0$ and $\operatorname{span}\{p_{m+n}, q_{m+n}, a_m\} = \mathbb{R}^3$, so according to the previous computations, $a_{m+n} \cos\langle m+n, \cdot\rangle \in \mathfrak{L}_f$ for every $a_{m+n} \in (m+n)^\perp$.

**2.** If $a_j \cos\langle j, \cdot\rangle + b_j \sin\langle j, \cdot\rangle \in \mathfrak{L}_f$ and $a_j \wedge b_j = 0$, then according to Proposition 3 $a_j \cos\langle j, \cdot\rangle \in \mathfrak{L}_f$. So in this case there exist $m, n, k \in \mathcal{M}_f$ such that $\operatorname{span}\{m, n, k\} = \mathbb{R}^3$ and such that

$$a_m \cos\langle m, \cdot\rangle \in \mathfrak{L}_f, \quad a_n \cos\langle n, \cdot\rangle \in \mathfrak{L}_f, \quad a_k \cos\langle k, \cdot\rangle \in \mathfrak{L}_f,$$

with $\operatorname{span}\{a_m, a_n, a_k\} = \mathbb{R}^3$. According to the second formulas of Proposition 2, $p_{m+n} \cos\langle m+n, \cdot\rangle \in \mathfrak{L}_f$ and $p_{n+k} \cos\langle n+k, \cdot\rangle \in \mathfrak{L}_f$ with

$$p_{m+n} = \langle m, a_n\rangle a_m - \langle n, a_m\rangle a_n, \quad p_{n+k} = \langle n, a_k\rangle a_n - \langle k, a_n\rangle a_k.$$

If $\langle m, a_n\rangle = 0$ then $a_n \in \operatorname{span}\{m \wedge n\}$ and necessarily $\langle n, a_m\rangle \neq 0$. So $p_{m+n} \neq 0$ and by symmetry $p_{n+k} \neq 0$. Then according to the same formulas,

$$p_{m,n,k} \cos\langle m+n+k, \cdot\rangle \in \mathfrak{L}_f, \quad p_{n,k,m} \cos\langle n, k, m, \cdot\rangle \in \mathfrak{L}_f,$$

with

$$\begin{cases} p_{m,n,k} = \langle k, p_{m+n}\rangle a_k - \langle m+n, a_k\rangle p_{m+n}, \\ p_{n,k,m} = \langle m, p_{n+k}\rangle a_m - \langle n+k, a_m\rangle p_{n+k}. \end{cases}$$

If $\langle m+n, a_k\rangle = 0$ and $\langle k, p_{m+n}\rangle = 0$, then $a_k, p_{m+n} \in \operatorname{span}\{(m+n) \wedge k\}$, and so $a_k \wedge p_{m+n} = 0$. But $p_{m+n} \in \operatorname{span}\{a_m, a_n\}$ and $\operatorname{span}\{a_m, a_n, a_k\} = \mathbb{R}^3$, so necessarily $\langle m+n, a_k\rangle \neq 0$ or $\langle k, p_{m+n}\rangle \neq 0$ and $p_{m,n,k} \neq 0$. By symmetry $p_{n,k,m} \neq 0$. By computation we obtain that

$$\begin{cases} p_{m,n,k} = -\langle m+n, a_k\rangle\langle m, a_n\rangle a_m + \langle m+n, a_k\rangle\langle n, a_m\rangle a_n + (\langle m, a_n\rangle\langle k, a_m\rangle - \langle n, a_m\rangle\langle k, a_n\rangle) a_k, \\ \\ p_{n,k,m} = (\langle m, a_n\rangle\langle n, a_k\rangle - \langle m, a_k\rangle\langle k, a_n\rangle) a_m - \langle n+k, a_m\rangle\langle n, a_k\rangle a_n + \langle n+k, a_m\rangle\langle k, a_n\rangle a_k. \end{cases}$$

If $p_{m,n,k} \wedge p_{n,k,m} = 0$, then there exist a non-zero $\lambda \in \mathbb{R}$ such that

$$\begin{cases} -\langle m+n, a_k\rangle\langle m, a_n\rangle = \lambda(\langle m, a_n\rangle\langle n, a_k\rangle - \langle m, a_k\rangle\langle k, a_n\rangle) \\ \langle m+n, a_k\rangle\langle n, a_m\rangle = -\lambda\langle n+k, a_m\rangle\langle n, a_k\rangle \\ \langle m, a_n\rangle\langle k, a_m\rangle - \langle n, a_m\rangle\langle k, a_n\rangle = \lambda\langle n+k, a_m\rangle\langle k, a_n\rangle. \end{cases}$$

And so

$$\lambda\langle n+k, a_m\rangle\langle n, a_k\rangle\langle m, a_n\rangle = \lambda\langle n, a_m\rangle(\langle m, a_n\rangle\langle n, a_k\rangle - \langle m, a_k\rangle\langle k, a_n\rangle),$$

which leads to

$$\langle k, a_m\rangle\langle n, a_k\rangle\langle m, a_n\rangle = -\langle n, a_m\rangle\langle m, a_k\rangle\langle k, a_n\rangle. \tag{15}$$

Lets us prove that this cannot be verified in this case. By computation we obtain that

$$\det \begin{pmatrix} a_m^1 & a_m^2 & a_m^3 \\ a_n^1 & a_n^2 & a_n^3 \\ a_k^1 & a_k^2 & a_k^3 \end{pmatrix} \begin{pmatrix} m^1 & n^1 & k^1 \\ m^2 & n^2 & k^2 \\ m^3 & n^3 & k^3 \end{pmatrix} = \det \begin{pmatrix} 0 & \langle n, a_m\rangle & \langle k, a_m\rangle \\ \langle m, a_n\rangle & 0 & \langle k, a_n\rangle \\ \langle m, a_k\rangle & \langle n, a_k\rangle & 0 \end{pmatrix}$$
$$= \langle n, a_m\rangle\langle k, a_n\rangle\langle m, a_k\rangle + \langle k, a_m\rangle\langle m, a_n\rangle\langle n, a_k\rangle.$$

Equation (15) is verified if and only if this determinant is equal to zero, which is impossible because $\text{span}\,\{a_m, a_n, a_k\} = \text{span}\,\{m, n, k\} = \mathbb{R}^3$. Then necessarily $p_{m,n,k} \wedge p_{n,k,m} \neq 0$.

In both cases we conclude thanks to Proposition 4. □

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