The boundary Riemann solver coming from the real vanishing viscosity approximation

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Abstract

It is considered the hyperbolic-parabolic approximation

$$\begin{cases} v^\varepsilon_t + \tilde{A}\big(v^\varepsilon,\, \varepsilon v^\varepsilon_x\big) v^\varepsilon_x = \varepsilon \tilde{B}(v^\varepsilon) v^\varepsilon_{xx} \qquad v^\varepsilon \in \mathbb{R}^N \\ \tilde{\beta}(v^\varepsilon(t,\, 0)) \equiv \bar{g} \\ v^\varepsilon(0,\, x) \equiv \bar{v}_0 \end{cases}$$

of an hyperbolic boundary Riemann problem, where $\tilde{\beta}$ is a suitable smooth function that guarantees the well posedness of the hyperbolic-parabolic IBVP. The conservative case is, in particular, included in the previous formulation.

The hypotheses imposed on the matrices $\tilde{A}$ and $\tilde{B}$ are essentially those introduced by Kawashima and most of them are implied, in the conservative case, by the presence of a dissipative entropy. In particular, the boundary characteristic case is allowed, i.e. one eigenvalue of $\tilde{A}$ is allowed to be close to zero. Moreover, no hypothesis of invertibility is made on the viscosity matrix $\tilde{B}$.

It is also introduced on the structure of matrix $\tilde{A}$ a new condition of block linear degeneracy. Exhibiting counterexamples, it is shown that this condition prevents the appearance of pathological behaviors.

It is assumed that the approximating solutions v^ε converge in L^1_{loc} to a unique limit, which is supposed to depend continuously in L^1 with respect to the initial and the boundary data. Moreover, it is assumed that in the hyperbolic limit the disturbances travel with finite propagation speed.

Under these hypotheses, it is given a complete characterization of the boundary Riemann solver induced in the hyperbolic limit when the difference between the boundary and the initial datum is small.

The most interesting behavior is observed when the viscosity matrix $\tilde{B}$ is not invertible: indeed, in this case the function $\tilde{\beta}$ maps $\mathbb{R}^N$ into a lower dimensional space. This reflects the fact that when the viscosity is singular the hyperbolic-parabolic IBVP is in general ill posed if one prescribes a full boundary condition like $v^\varepsilon(t,\, 0) = \bar{v}_b \in \mathbb{R}^N$. Thanks to a perturbation technique, it is proved that the solution of the boundary Riemann problem is completely determined once the functions $\tilde{\beta}$ and the data $\bar{g}$ and $\bar{v}_0$ are assigned. In this way, it is obtained a precise characterization of the admissible boundary conditions imposed on the limiting hyperbolic system.

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1 Introduction

The aim of this work is the characterization of the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation

$$\begin{cases} v^\varepsilon_t + \tilde{A}\big(v^\varepsilon,\, \varepsilon v^\varepsilon_x\big) v^\varepsilon_x = \varepsilon \tilde{B}(v^\varepsilon) v^\varepsilon_{xx} \qquad v^\varepsilon \in \mathbb{R}^N \\ \tilde{\beta}(v^\varepsilon(t,\,0)) \equiv \bar{g} \\ v^\varepsilon(0,\,x) \equiv \bar{v}_0. \end{cases} \tag{1.1}$$

The precise expression of the function $\tilde{\beta}$ will be specified in the following, here it is enough to underline that it guarantees the well-posedness of the parabolic IBVP.

Even if the systems considered are not necessarily in conservation form, the class

$$v^\varepsilon_t + f(v^\varepsilon)_x = \varepsilon\big(\tilde{B}(v^\varepsilon) v^\varepsilon_x\big)_x \tag{1.2}$$

is included in the previous formulation as a particular case.

The study of the boundary Riemann solver has an interest in its own, but it also provides a reasonable admissibility criterium for the boundary datum $\bar{v}$ in the limit problem

$$\begin{cases} v_t + \tilde{A}(v,\,0) v_x = 0 \qquad v \in \mathbb{R}^N \\ v(0,\,x) = \bar{v}_0 \qquad v(t,\,0) = \bar{v}. \end{cases} \tag{1.3}$$

Indeed, simple example show that such a problem is, in general, ill posed if $\bar{v} \in \mathbb{R}^N$ and one does not impose additional conditions. More precisely, it is important to specify that here and in following the boundary datum in (1.3) is interpreted as the trace of the solution on the axis $x = 0$.

The hypotheses assumed are listed in Section 2. It is supposed that the approximating solutions v^ε converge in L^1_{loc} to a unique limit, which is moreover supposed to depend Lipschitz continuously in L^1 with respect to the initial and the boundary data.

A particular attention is paid to the hypotheses imposed on the structure of the matrices $\tilde{A}$ and $\tilde{B}$ and to their justification. More precisely, it will be made constant reference to the analysis performed in a series of works by Kawashima and Shizuta, [30, 31] and [32, 33]. Moreover, it is introduced a new condition of block linear degeneracy and it is shown that it is essential in order to avoid pathological behaviors both in the case of initial boundary value problems and of boundary free Cauchy problems.

The boundary characteristic case is, in particular, allowed, i.e. one eigenvalue of $\tilde{A}$ is allowed to be close to the speed of the boundary (0 in this case). Moreover, the presence of a non invertible viscosity matrix $\tilde{B}(v)$ is contemplated. The study of the Riemann solver is particularly interesting in this case, because one cannot prescribe in general a full boundary condition $\bar{v}_b \in \mathbb{R}^N$. More precisely, in this case it is in fact possible to assign only some components of the vector $v^\varepsilon(t, 0)$: this is the reason why one introduces the function $\tilde{\beta}$, mapping $\mathbb{R}^N$ into a lower dimensional space. When $\tilde{B}$ is invertible one can prescribe a full boundary condition: in this case the formulation (1.1) maintains its validity provided that $\tilde{\beta}(v^\varepsilon) = v^\varepsilon$.

The aim of this paper is to furnish a complete description of $v(t, x) = v(x/t)$, the self similar function satisfying (1.3) which is the limit as $\varepsilon \to 0^+$ of the approximating solutions v^ε of (1.1). In particular, it is obtained a characterization of the boundary datum $\bar{v}$ assumed by v; the analysis is then completed through the study of the boundary layer profiles converging to the state $\bar{v}$.

The characterization of the trace induced on the limit solution $v(t, x)$ also furnishes a reasonable admissibility criterium for the boundary datum imposed on the hyperbolic IBVP (1.3). It is interesting to stress that an admissibility criterium for the datum $\bar{v}$ was proposed in [23] in the case of systems in conservation form with only linearly degenerate or genuinely non linear fields: this criterium is, however, in general different from the one defined by the parabolic approximation (1.1).

On the other side, in other works it was considered a quite general boundary condition: it is made reference for example to [38] and to [1], [2] and [22]. More precisely, in those papers the boundary condition on (1.3) was assigned imposing the relation $\hat{\beta}(v(0, t)) = h(t)$, where h was the datum and the function $\hat{\beta}$ was not specified explicitly, but was required to satisfy suitable transversality conditions.

In the case of the boundary Riemann problem the datum is constant and hence the previous formulation reduces to $\hat{ß}(v(0,t)) = \bar{h}$: it turns out that the admissibility criterium found in the present paper is compatible with this way of assigning the boundary datum. More precisely, such an admissibility criterium could be actually seen as a way to obtain an explicit construction for the function $\hat{ß}$, once the functions $\tilde{A}$, $\tilde{B}$ and $\tilde{ß}$ are known: according to this interpretation, the boundary condition $\bar{h}$ is completely determined by the assignment of the data $\bar{v}_0$ and $\bar{g}$.

It is worth underlining, however, that the previous considerations apply completely only to the case of a non characteristic boundary, which occur when all the eigenvalues of the hyperbolic matrix $\tilde{A}(v,\, 0)$ are well separated from the speed of the boundary, which is zero in our problem. On the other side, when an eigenvalue of $\tilde{A}(v,\, 0)$ is allowed to be close to zero (boundary characteristic case), it becomes problematic to define the boundary condition in the hyperbolic IBVP (1.3) through a function $\hat{ß}$. The reasons of this fact will be explained in Sections 3.2.3 and 4.2.4, here however it is underlined that the dimension itself of the set of images of the function $\hat{ß}$ cannot be prescribed a priori in the boundary characteristic case.

It is remarked that, as it will be clearer in Section 2.2.1, similar difficulties could in principle arise also in the definition of the function $\tilde{ß}$ exploited to assign the boundary condition on the *hyperbolic-parabolic* IBVP (1.1). On the other side, these problems are prevented imposing the condition of block linear degeneracy. It is worth underlining again, however, that such a condition turns out to be have non only a technical, but also a deeper meaning, because it is essential in order to prevent the appearance of pathological behaviors.

Anyway, the previous considerations explain why the choice was to introduce the function $\tilde{ß}$ in the formulation of the *parabolic-hyperbolic* IBVP (1.1) and on the other side to state the *hyperbolic* IBVP in the form (1.3).

As regards problem (1.3), an existence result for problem was obtained in [27] and [38], relying on an algorithm of approximation of the solution based on an adaptation of the Glimm scheme introduced in [25].

These results were later improved by mean of the wave front tracking techniques introduced in [12] and later used in a series of papers ([13, 15, 16, 17, 18, 19, 20]) to establish the well posedness of the Cauchy problem. In particular, in [1] the existence results in [27] and [38] were substantially improved, while it is made reference to [22] for the well posedness of the initial boundary value problem (1.3) and to [2] for the study of the semigroup limit of wave front tracking approximation. In particular, in [2] it is extended to initial boundary value problems the definition of Standard Riemann Semigroup (SRS) which was introduced in [13] in the boundary free case and it is proved that, if there exists a SRS, then it is necessarily unique. Roughly speaking, this result guarantees that, under suitable hypotheses of stability, the solution of the boundary Riemann problem uniquely identifies the semigroup.

It is worth underling explicitly that all the results quoted so far deal with systems in conservation form; a comprehensive account of the stability and uniqueness results for the Cauchy problem for a system of conservation laws can be found in [14]. It is made reference, instead, to [21] and to [39] for a general introduction to systems of conservation laws.

As regards the study of the convergence of the approximated solutions defined by (1.1), no general result is up to now available.

Nevertheless, a complete result was achieved in [3] in the case of the vanishing viscosity approximation

$$v^{\varepsilon}_t + \tilde{A}(v^{\varepsilon}) v^{\varepsilon}_x = \varepsilon v^{\varepsilon}_{xx}. \tag{1.4}$$

More precisely, it was proved the convergence of the solutions v^{ε} and the uniqueness and the stability of the limit and it was characterized the Riemann solver induced by the vanishing viscosity approximation in the hyperbolic limit. The crucial tool is the introduction of a suitable decomposition of the gradient of the vanishing viscosity solutions and the consequent analysis exploits the techniques introduced in the boundary free case in [6, 7, 8, 9]. These techniques were also exploited in [5] to achieve a complete characterization of the solution of the Riemann problem in the boundary free case. Finally, it is made reference to [42] for the study of the vanishing viscosity approximation (1.4) of a 2×2 triangular system with Dirichlet data imposed on two boundaries.

Moreover, the general parabolic approximation

$$v^\varepsilon_t + f(v^\varepsilon)_x = \varepsilon\big(\tilde B(v^\varepsilon)v^\varepsilon_x\big)_x$$

was studied in numerous works. In particular, in [24] it was proved the existence of a $T > 0$ such that v^ε converges in $L^\infty\big((0, T); L^2(\mathbb{R}^+)\big)$ to a solution of

$$v_t + f(v)_x = 0$$

and it was given a precise characterization of the boundary condition induced in the hyperbolic limit. The viscosity matrix $\tilde B(v)$ was supposed to be invertible but in general different from the identity,

Moreover, in [40] it was introduced an Evans function machinery to study the stability of boundary layer profiles: the parabolic approximation dealt with was in conservation form (1.2) and it was considered the case of an invertible viscosity matrix $\tilde B$ and of a non characteristic boundary. However, the analysis was extended in a series of paper ([41] and [35, 36, 37]) to the boundary characteristic case and to very general parabolic approximations, with non invertible viscosity matrices.

The paper is organized as follows.

Section 2 introduces the hypotheses that will be exploited in the following. More precisely, it is assumed that the initial boundary value problem (1.1) is equivalent (in the sense specified therein) to

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ ß(u^\varepsilon(t,\, 0)) \equiv \bar g \qquad u^\varepsilon(0,\, x) \equiv \bar u_0, \end{cases} \tag{1.5}$$

where the matrices E, A and B and the function ß satisfy suitable assumptions whose precise statement can be found in Sections 2.1 and 2.2. Here, however, it is crucial to underline that those conditions are completely justified by physical examples. In particular, it will be made reference to the analysis performed in [30, 31] and [32, 33].

As concerns the function ß, when the viscosity B is invertible one can take $ß \equiv I$, but when B is allowed to be singular more restrictive conditions have to be prescribed. Making reference to Section 2.2.1 and in particular to Definition 2.1 for a more precise statement, here we will say heuristically that if the viscosity matrix B is not invertible, then (1.5) may be seen as the coupling of a hyperbolic and a parabolic part, and hence one can only prescribe the components of the entering characteristic fields of the hyperbolic subsystem. Definition 2.1 will be extended in Section 4.3 by Definition 4.1, which introduces a more general class of admissible functions ß.

As concerns the hypotheses imposed on the structure of the matrices A and B, a specific attention is paid to the case of a non invertible viscosity, which is the most complex among the two. It is introduced on the structure of the matrix A a new condition of block linear degeneracy: making reference to Hypothesis 2 for the precise statement, here it is important to underline that such a condition could actually be seen as a generalization of the assumptions considered in [37] and [5]. Examples 2.1 and 2.2 guarantee that the hypothesis of block linear degeneracy prevents the appearance of pathological behaviors, both in the case of initial boundary value problems and in the case of boundary free Cauchy problems. Finally, it is considered Example 2.3, which shows that the rank of the viscosity matrix has to be constant in order to avoid the presence of undesirable dynamics, and hence guarantees the importance of the corresponding condition introduced in [33].

Apart from the hypotheses on the structure of the matrices A and B, it is assumed the stability and the convergence of the approximating solutions. Moreover, the limit is supposed to be unique and to admit a finite propagation speed of the disturbances.

The precise statement of these hypotheses can be found in Section 2.3, but here it is worth stressing that even if the proof of the convergence and of the stability of the approximating solutions is still an open problem in the general case, the references listed before ensure that partial results are available (see in particular [3, 24] for the convergence and [35, 36, 37, 40, 41] for the stability). Moreover, it is underlined here that the hypothesis of stability and of finite propagation speed of the disturbances in the hyperbolic limit is essential to guarantee that the solution of a general boundary Riemann problem can be obtained putting together the solutions of simpler problems, namely those having data connected by travelling waves or by stationary solutions of the parabolic equation.

Finally, a key assumption is the smallness of the total variation, which in the case of the boundary Riemann problem (1.1) implies the smallness of the difference $|\bar{g} - \tilde{ß}(\bar{v}_0)|$.

The main results of the work, concerning the characterization of the boundary Riemann solver limit of the parabolic approximation (1.1), are exposed in Sections 3 and 4. Even if it is not the major point, in order to avoid misunderstanding in the use of the notations it is worth stressing explicitly that in those sections it is made reference to the formulation (1.5) instead of (1.1). Moreover, in order to hopefully simplify the exposition, in a first moment, i.e. in Section 3, it is considered only the limit induced by an approximation with non singular viscosity: in this case the initial-boundary value problem

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(t,\, 0) \equiv \bar{u}_b \qquad u^\varepsilon(0,\, x) \equiv \bar{u}_0, \end{cases} \tag{1.6}$$

is well posed for every datum $\bar{u}_b \in \mathbb{R}^N$.

Section 3.1 introduces some technical results, while the construction of the boundary Riemann solver is described in Section 3.2.

In particular, in Section 3.2.1 it is recalled the characterization of the Riemann solver in the boundary free case: it is made reference to [5] for the complete analysis, but some of the main results are briefly described. It is considered the equation satisfied by travelling wave solutions, i.e

$$B(U)U'' = \Big(A(U,\, U') - \sigma E(U)\Big)U', \tag{1.7}$$

and it is sketched the construction of the i-th curve of admissible states $T^i_{s_i}u^+$. Such a curve includes all the values that can be linked to u^+ through a sequence of rarefaction waves and shocks limit of a solution to (1.7) with σ_i close to the i-th eigenvalue of A.

After the most important preliminary results have been introduced, Sections 3.2.2 and 3.2.3 deal with the characterization of the boundary Riemann solver limit of the parabolic approximation (1.5) when the viscosity matrix B is invertible. Two distinct cases are considered: in Section 3.2.2 it is supposed that the boundary is non characteristic, which means that all the eigenvalues of the hyperbolic matrix A are well separated from zero. Namely, there exists a constant $c > 0$ such that

$$\lambda_1(u,\, u_x) < \cdots < \lambda_n(u,\, u_x) < -c < 0 < c < \lambda_{n+1}(u,\, u_x) < \cdots < \lambda_N(u,\, u_x).$$

Section 3.2.2, on the other side, deals with the boundary characteristic case: an eigenvalue of A is indeed allowed to have speed close to that of the boundary:

$$\lambda_k(u,\, u_x) \sim 0.$$

It is made reference to Proposition 3.1 for the precise characterization of the boundary Riemann solver in the non characteristic case, but in the following it will be given a sketch of the most important ideas involved in the construction. The initial datum $\bar{u}_0$ is connected to the hyperbolic trace on the axis $x = 0$ by the curves of admissible states $T^N_{s_N} \dots T^{n+1}_{s_{n+1}}$, whose construction is the same as in the boundary free case: however, because of the presence of the boundary, only the curves limit of travelling waves (1.7) with speed $\sigma_i > 0$ are needed. Moreover, it is worth underling that once the initial datum $\bar{u}_0$ is fixed this construction furnishes a reasonable admissibility criterium for the boundary datum in problem (1.3): the hyperbolic trace $\bar{u}$ has to be connected to the initial datum $\bar{u}_0$ by the curves $T^N_{s_N} \dots T^{n+1}_{s_{n+1}}$, in other words

$$\bar{u} = T^{n+1}_{s_{n+1}} \circ \cdots \circ T^N_{s_N}\bar{u}_0$$

for some $(N-n)$-dimensional vector $(s_{n+1}, \dots s_N)$.

In the conservative case

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon\big(B(u^\varepsilon)u^\varepsilon_x\big)_x \tag{1.8}$$

the set of admissible states is in this way uniquely determined, thanks to a result in [5] that guarantees the uniqueness of the curves $T^N_{s_N} \dots T^{n+1}_{s_{n+1}}$.

On the other side, the loss of boundary condition which one encounters when passing from the parabolic approximation to the hyperbolic limit is due to the boundary layer profiles. Being stationary solutions, they satisfy

$$B(U)U'' = A(U,\, U')U'. \tag{1.9}$$

Moreover, the boundary layer profiles connect the hyperbolic trace $\bar{u}$ to the boundary datum $\bar{u}_b$ assigned on the approximating solutions u^ε in (1.5):

$$U(0) = \bar{u}_b \qquad \lim_{x \to +\infty} U(x) = \bar{u}.$$

Hence the states $\bar{u}_b$ that can be connected to the hyperbolic trace $\bar{u}$ through boundary layer profiles lay on the stable manifold of system (1.9). The dimension of this manifold is proved to be equal to the number of negative eigenvalues of A: being all the eigenvalues of A well separated from zero, there is no interaction between travelling wave profiles and boundary layers.

Making reference to Proposition 3.1 for a more precise statement, it turns out that when the viscosity is invertible and the boundary is non characteristic the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation (1.6) is uniquely determined once the initial and boundary datum imposed on the parabolic IBVP are assigned: this result is later extended to all the cases allowed by the hypotheses introduced in Section 2.

Before considering the boundary characteristic case, it is interesting to make a remark concerning the problem of uniqueness: indeed, this issue marks a difference between the initial boundary value problem and the boundary free case. More precisely, the result in [5] that was quoted before guarantees that the Riemann solver induced in the hyperbolic limit is unique in the conservative case (1.8). On the other side, even in the case of a system in conservation form the limit of the parabolic approximation (1.5) *depends* on the choice of the viscosity matrix B because the stable manifold of system (1.9) does: an explicit example can be found in [24]. Roughly speaking, the problem in the case of the initial boundary value problem is that the there are no *intrinsic* hypothesis, defined directly on the limit equation

$$u_t + f(u)_x = 0,$$

which select the boundary data $\bar{u}_b$ that can be connected to a given hyperbolic trace $\bar{u}$ by a boundary layer profile. On the other side, in the boundary free case the admissible shocks satisfy the Rankine-Hugoniot conditions, which are defined in terms of the flux f. It is stressed explicitly that straightforward considerations guarantee that also when the boundary is characteristic or when the viscosity is allowed to be singular the boundary Riemann solver induced in the hyperbolic limit in general *depends* on the choice of the matrix B.

As concerns the boundary Riemann solver induced by a parabolic approximation with invertible viscosity in the boundary characteristic case, a characterization is given in Section 3.2.3. This problem is more complex then the one dealt with in Section 3.2.2 because of the possibility of an interaction between the travelling wave and the boundary profiles. However, as in the case of a non characteristic boundary, an important role is played by the curve of admissible states $T^N_{s_N} \dots T^{k+1}_{s_{k+1}}$, which connect the initial datum $\bar{u}_0$ to a state $\bar{u}_k$. A particular attention has to be paid to the construction of k-th curve $T^k_{s_k}\bar{u}_k$. Indeed, as far as the speed $\sigma_k(s)$ along $T^k_{s_k}$ is nonnegative the construction of the k-th curve is the same as in the boundary free case. In this way, the value $\bar{u}_k$ is connected to $\underline{u}_k = T^k_{\underline{s}_k}$, where

$$\underline{s}_k = \max\{s : \sigma_k(s) \geq 0\}.$$

To extend the curve $T^k_{s_k}$ for greater values of s_k one has to take into account a perturbation due the boundary layer profiles: more precisely, it will be studied the set of values $\bar{u}_b$ such that

$$U(0) = \bar{u}_b \qquad \lim_{x \to +\infty} U(x) = \underline{u} \tag{1.10}$$

for a solution U of (1.9). Since the center manifold of system (1.9) is non trivial, the set of boundary data $\bar{u}_b$ does not include only the stable manifold as in the case of a non characteristic boundary, but has a more complex structure. Making reference to Proposition 3.2 for the precise description, here it will be given only an heuristic idea: the solution connecting the state $\underline{u}$ to $\bar{u}_b$ may be split into the sum of its projection on the center manifold and its projection on the so called uniformly stable manifold, which contains all the orbits that when $x \to +\infty$ decay exponentially to some point belonging to a given manifold of equilibria. Correspondingly, the manifold connecting the state $\underline{u}$ to $\bar{u}_b$ is obtained from the composition of a curve, originated by the presence of the center component, and of a manifold, which corresponds to the projection of the solution on the uniformly stable manifold. Nevertheless, the decomposition of the solution is not complete, in the sense that the two components do not evolve independently, but there is an interaction in between.

Proposition 3.2 concludes the characterization of the boundary Riemann solver induced by a parabolic approximation with invertible viscosity: the case the matrix B is singular is instead dealt with in Section 4. As it was already underlined, this case is particulary interesting since one in general cannot prescribe a full boundary condition on the parabolic IBVP (1.5). More precisely, the assignment of the boundary datum $\bar{g}$ exploits the function ß satisfying Definition 2.1.

Before describing the construction of the Riemann solver, however, some preliminary results are introduced in Section 4.1. Moreover, in Section 4.2.2 is stated and proved Lemma 4.7, which plays a key role in the construction of the boundary Riemann solver. The result deals with the dimension of the stable manifold of

$$A(u,\, u_x)u_x = B(u)u_{xx}$$

in the case the matrix B is singular: the precise statement can be found in Section 4.2.2, here it is only underlined that the dimension of the manifold depends in the general on the structure of the matrices A and B. Nevertheless, it turns out to be equal to the difference between the number of boundary conditions imposed on the *parabolic* IBVP (1.5) (i.e. to the dimension of the set of images of the function ß) and the number of conditions that can be imposed on the datum $\bar{u}$ in the *hyperbolic* IBVP

$$\begin{cases} E(u)u_t + A(u,\, 0)u_x = 0 \\ u(t,\, 0) \equiv \bar{u} \qquad u(0,\, x) \equiv \bar{u}_0. \end{cases}$$

This is coherent with the previous construction, since the study of the manifold of boundary layer profiles was introduced in order to face the possibility of a loss of boundary condition when passing from the parabolic approximation into the hyperbolic limit. More precisely, Lemma 4.7 gives actually an answer to a question left open in [37] and it guarantees that the assumption called $(H5)$ therein is actually a consequence of Kawashima condition. The proof exploits a perturbation technique based on methods analogous to those introduced in [41]; moreover, the consequent study of perturbed eigenvalues owes much to the analysis carried on in [10].

Section 4.2.1 describes the construction of the i-th curve of admissible states, which was already studied in [5] to characterize the solution of the boundary free Riemann problem. As in the case of an invertible viscosity, one has to study the equation of travelling wave profiles

$$B(U)U'' = \Big(A(U,\, U') - \sigma E(U)\Big)U'$$

with a speed σ_i close to the i-th eigenvalue of the hyperbolic matrix A. The difficulties coming from the singularity of the viscosity matrix are mostly technical and in Section 4.2.1 it will be given only a sketch of the crucial ideas involved in the construction, making reference to [5] for a complete analysis. Nevertheless, for completeness it is underlined that the hypotheses assumed here do not coincide completely with those imposed in [5]. More precisely, in [5] the matrix A was allowed to be non symmetric, but instead of the block linear degeneracy it was assumed a more restrictive condition.

In Section 4.2.3 it is characterized the Riemann solver limit of the parabolic approximation (1.5) in the case the boundary is not characteristic and the viscosity matrix B is singular. Let n be the number of negative eigenvalues of the hyperbolic matrix A: the initial datum $\bar{u}_0$ is connected to the trace $\bar{u}$ of the hyperbolic limit on the axis $x = 0$ through the curves of admissible states $T^N_{s_N} \dots T^{n+1}_{s_{n+1}}$. The admissibility criterium induced on the boundary datum $\bar{u}$ imposed in the hyperbolic initial boundary value problem (1.3) is, therefore, analogous to the one described in the case of a non singular viscosity.

On the other side, a deep difference between the two cases can be found in the characterization of the set of boundary data $\bar{u}_b$ that can be connected to the hyperbolic trace $\bar{u}$ through boundary layer profiles, i.e. of the states $\bar{u}_b$ such that there exists a solution of system

$$A(u,\, u_x)u_x = B(u)u_{xx} \tag{1.11}$$

which satisfies

$$u(0) = \bar{u}_b \qquad \lim_{x \to +\infty} u(x) = \bar{u}.$$

Indeed, when the viscosity is invertible the dimension of the stable manifold of system (1.11) is equal to the number of negative eigenvalues of A, but it may be strictly smaller if B is singular: in this last case, it is no more true that all the states $\bar{u}_b$ and $\bar{u}_0$ close enough can be connected by travelling waves and boundary layer profiles. Conversely, once the initial datum $\bar{u}_0$ is fixed, one can show that there exists only one point $\bar{u}_b$ which can be connected by travelling waves and a boundary layer to $\bar{u}_0$ and satisfies the boundary conditions $ß(\bar{u}_b) = \bar{g}$: it is made reference to Proposition 4.1 for the precise statement. Here, however, it is underlined that this ensures that the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation (1.5) is completely determined once the data $\bar{g}$ and $\bar{u}_0$ are assigned.

In Section 4.2.4 it is given a characterization of the boundary Riemann solver induced when the viscosity B is singular and the boundary is characteristic. Even if this case is the most complex, the difficulties can be actually solved with the techniques introduced in Sections 3.2.3 and 4.2.3, hence only the crucial steps of the construction will be given. As in the case of a non singular viscosity, the initial datum $\bar{u}_0$ is connected to $\underline{u}$ through the curves of admissible states $T^N_{s_n} \dots T^{k+1}_{s_{k+1}}$ and through the section of the k-th curve corresponding to non negative values of σ_k. The state $\underline{u}$ is then connected to the boundary datum $\bar{u}_b$ by a solution of equation (1.11) such that

$$u(0) = \bar{u}_b \qquad \lim_{x \to +\infty} u(x) = \underline{u}.$$

As in the case of an invertible viscosity, the converging solution splits in the sum of a uniformly stable and a center component, which interact one with each other. Nevertheless, being the matrix B singular, the dimension of the uniformly stable manifold may be strictly less then N plus the number of negative eigenvalues of A. Hence it turns out that also in the boundary characteristic case there are restrictions to impose in order to have boundary data $\bar{u}_b$ that can be connected to a given initial datum $\bar{u}_0$: making reference to Proposition 4.2 for a more precise statement, here it is only underlined that the characterization of the Riemann solver described so far furnishes the unique point $\bar{u}_b$ which can be connected to u_0 and satisfies the boundary condition $ß(\bar{u}_b) = \bar{g}$. In particular, it follows that the limit of the approximation (1.5) is uniquely determined once the data $\bar{g}$ and $\bar{u}_0$ are assigned.

Finally, in Section 4.3, the way of assigning the boundary datum in the parabolic IBVP given by Definition 2.1 is extended to the more general Definition 4.1. More precisely, it is first introduced Lemma 4.9, a transversality result which is exploited in Section 4.2.3 and 4.2.4 to prove the uniqueness of the boundary Riemann solver induced by the parabolic approximation (1.5). The fundamental idea is that there is a common subspace of $\mathbb{R}^n$ which is transversal to $\nabla ß$ and to the manifold of points which can be connected to u_0 by a boundary Riemann solver. It follows that we can state in general which conditions should be satisfied by a function $ß$ if we want (1.5) to be well posed and the boundary Riemann problem to have a solution: these assumptions are collected in Definition 4.1.

As a remark, it is underlined that the way of assigning the boundary condition on the *hyperbolic-parabolic* system introduced with Definition 4.1 is formally the same that was considered in several previous works in order to define the Dirichlet datum in the *hyperbolic* initial boundary value problem. It is made reference for example to [1], [2], [22] and [38].

2 Hypotheses

This section introduces the hypotheses exploited in the work.

The exposition is organized as follows: Section 2.1 deals with the hypotheses exploited in Section 3 to study a parabolic approximation with invertible viscosity, while Section 2.2 introduces the assumptions needed in the case of a singular viscosity, which will be dealt with in Section 4; in this case we need to specify the function ß, while for invertible viscosity one can take $ß(u) = u$. An explicit construction of the function ß is introduced in Section 2.2.1 and will be later extended to a more general setting in Section 4.3.

In both Sections 2.1 and 2.2 it will be assumed that the initial boundary value problem (1.1) is equivalent (in the sense specified therein) to

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ ß(u^\varepsilon(t,\, 0)) \equiv \bar{g} \qquad u^\varepsilon(0,\, x) \equiv \bar{u}_0 \end{cases} \tag{2.1}$$

and the other hypotheses are actually introduced making reference to (2.1).

Moreover, in Section 2.2.2 three counter-examples are described. In particular, Example 2.1 and 2.2 deal with the condition of block linear degeneracy that is introduced in Hypothesis 2. They guarantee that such a condition is essential in order to prevent the appearance of pathological behaviors both in the case of initial boundary value problems and in the case of the boundary free Cauchy problem. On the other side, Example 2.3 shows that the rank of the viscosity matrix B has to be constant in order to prevent the presence of undesirable dynamics.

Finally, Section 2.3 deals with the hypotheses required in both the cases, when the viscosity matrix B is invertible and when it isn't. An hypothesis of strict hyperbolicity is assumed. Moreover, it is supposed that the approximating solutions u^ε defined by (2.1) have total variation uniformly bounded in t and ε and converge to a unique limit, which is moreover supposed to depend Lipschitz continuously in L^1 with respect to the initial and the boundary data. Finally, it is assumed an hypothesis of finite propagation speed of the disturbances in the hyperbolic limit.

As it was underlined in the introduction, a particular attention is paid to the justification of the assumptions required. In particular, all the hypotheses imposed on the structure of the matrices A and B are well justified on the basis of several works by Kawashima and Shizuta [30, 31] and [32, 33], while the assumptions regarding the converge of the approximating solutions u^ε defined by (2.1) and the stability of the limit are reasonable thanks to the results in [9, 24, 35, 36, 37, 40, 43].

2.1 The hypotheses assumed in the case of invertible viscosity

In this section it is considered the system

$$v_t + \tilde{A}(v,\, v_x)v_x = \tilde{B}(v)v_{xx} \tag{2.2}$$

in the case the viscosity matrix $\tilde{B}(v)$ is invertible. It is assumed that

Hypothesis 1. *There exists an (invertible) smooth change of coordinates $v = v(u)$ such that* (2.2) *is equivalent to system*

$$E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx}, \tag{2.3}$$

where

1. *for any u, the matrix $E(u)$ is real, symmetric and positive definite: there exists a constant $c_E(u)$ such that*
$$\forall\, \xi \in \mathbb{R}^N,\ \langle E(u)\xi,\, \xi\rangle \geq c_E(u)|\xi|^2.$$
2. *for any u the matrix $A(u,\ 0)$ is symmetric.*
3. *for any u, the viscosity matrix $B(u)$ is real and positive definite, in particular there exists a constant $c_B(u) > 0$ such that*
$$\forall\, \xi \in \mathbb{R}^N,\ \langle B(u)\xi,\, \xi\rangle \geq c_B(u)|\xi|^2.$$

In particular, the matrix $E(u)$ may be the differential $Dv(u)$ of the change of coordinates, but it may also be obtained multiplying $Dv(u)$ on the left by a suitable invertible matrix.

The initial boundary value problem (1.1) is equivalent to

$$\begin{cases} u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big) u^\varepsilon_x = \varepsilon B(u^\varepsilon) u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(t,\, 0) \equiv \bar{u}_b \\ u^\varepsilon(0,\, x) \equiv \bar{u}_0, \end{cases} \tag{2.4}$$

where $v(\bar{u}_0) = \bar{v}_0$ and $v(\bar{u}_b) = \bar{v}_b$. Indeed, one can verify that, thanks to the invertibility of the viscosity matrix, it is possible to assign a full boundary condition $\bar{u}_b \in \mathbb{R}^N$: in other words, the function ß is taken equal to the identity map from $\mathbb{R}^N$ into $\mathbb{R}^N$.

In the case system (2.2) is in conservation form

$$v_t + f(v)_v = \Big(\tilde{B}(v) v_x\Big)_x, \tag{2.5}$$

Hypothesis 1 is guaranteed by the presence of a dissipative and convex entropy.

For completeness it is recalled that the entropy η is dissipative for (2.5) if for any v there exists a constant $c_D(v)$ such that

$$\forall\, \xi \in \mathbb{R}^N,\ \langle D^2\eta(v)\xi,\, \tilde{B}(v)\xi\rangle \geq c_D(v)|\xi|^2 \tag{2.6}$$

It is known (see for example [26]) that if system (2.2) admits a convex entropy η, then there exists a symmetrizing and invertible change of variable $v = v(u)$. More precisely, if the inverse function $u(v)$ is defined by $u = \nabla\eta(v)$, then v satisfies system (2.3), with $A(u,\, 0)$ given by

$$A(u,\, 0) = Df\big(v(u)\big)\Big(D^2\eta\big(v(u)\big)\Big)^{-1} - \frac{\partial}{\partial x}\bigg(B\big(v(u)\big)\Big(D^2\eta\big(v(u)\big)\Big)^{-1}\bigg)\bigg|_{u_x=0} = Df\big(v(u)\big)\Big(D^2\eta\big(v(u)\big)\Big)^{-1}$$

and symmetric. Moreover,

$$E(u) = \Big(D^2\eta\big(v(u)\big)\Big)^{-1}$$

is symmetric (being the inverse of an hessian) and positive definite by convexity, while the dissipation condition (2.6) guarantees that

$$B(u) = \tilde{B}\big(v(u)\big)\Big(D^2\eta\big(v(u)\big)\Big)^{-1}$$

is positive definite. Hence Hypothesis 1 is satisfied.

2.2 The hypotheses assumed in the case of non invertible viscosity

The aim of this section is to introduce the hypotheses that will be needed in Section 4 to study system

$$v_t + \tilde{A}(v,\, v_x) v_x = \tilde{B}(v) v_{xx} \tag{2.7}$$

in the case the viscosity matrix is allowed to be singular.

Hypothesis 2. *There exists an (invertible) smooth change of coordinates $v = v(u)$ such that (2.7) is equivalent to system*

$$E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx}, \tag{2.8}$$

where

1. *the matrix $B(u)$ has constant rank $r < N$ and admits the block decomposition*

$$B(u) = \begin{pmatrix} 0 & 0 \\ 0 & b(u) \end{pmatrix} \tag{2.9}$$

with $b(u) \in \mathbb{M}^{r\times r}$. Moreover, for every u the matrix $b(u)$ is positive definite, namely there exists a constant $c_b(u) > 0$ such that

$$\forall\, \xi \in \mathbb{R}^r,\ \langle b(u)\xi,\, \xi\rangle \geq c_b(u)|\xi|^2.$$

2. *for any u the matrix $A(u,\,0)$ is symmetric. Moreover, the block decomposition of $A(u,\,u_x)$ corresponding to* (2.9) *takes the form*

$$\begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u,\,u_x) \end{pmatrix}, \tag{2.10}$$

with $A_{11} \in \mathbb{M}^{(N-r)\times(N-r)}$, $A_{12} \in \mathbb{M}^{(N-r)\times r}$, $A_{21} \in \mathbb{M}^{r\times(N-r)}$ and $A_{22} \in \mathbb{M}^{r\times r}$. Namely, only in the block A_{22} depends on u_x.

3. *a condition of block linear degeneracy holds. More precisely, for any $\sigma \in \mathbb{R}$ the dimension of $ker(A_{11}(u) - \sigma E_{11})$ does not depend on u, but only on σ.*

   This holds in particular for the limit cases in which the dimension of $ker(A_{11} - \sigma E_{11})$ is equal to $N-r$ or to 0.

4. *the so called Kawashima condition holds: for any u,*

$$ker\big(B(u)\big) \cap \{\text{eigenvectors of } E^{-1}(u)A(u,\,0)\} = \emptyset$$

5. *for any u, the matrix $E(u)$ is real, symmetric and positive definite: there exists a constant $c_E(u)$ such that*

$$\forall\, \xi \in \mathbb{R}^N,\ \langle E(u)\xi,\,\xi\rangle \geq c_E(u)|\xi|^2.$$

   In the following, we will denote by

$$E(u) = \begin{pmatrix} E_{11}(u) & E_{12}(u) \\ E_{21}(u) & E_{22}(u) \end{pmatrix}$$

   the block decomposition of $E(u)$ corresponding to (2.9).

The change of variable $v = v(u)$ guarantees that the initial-boundary value problem (1.1) is equivalent to

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A\big(u^\varepsilon,\,\varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ \text{ß}(u^\varepsilon(t,\,0)) \equiv \bar{g} \\ u^\varepsilon(0,\,x) \equiv \bar{u}_0, \end{cases} \tag{2.11}$$

where $v(\bar{u}_0) = \bar{v}_0$, $v(\bar{u}_b) = \bar{v}_b$, $\text{ß}(u^\varepsilon) = \tilde{\text{ß}}(v(u^\varepsilon))$.

Relying on the block decomposition of the viscosity matrix described in Hypothesis 2, in Section 2.2.1 the explicit definition of the function ß is introduced. Moreover, in Section 4.3 such a definition is extended to a more general formulation.

In order to justify the assumptions summarized in Hypothesis 2 it will be made reference to the works of Kawashima and Shizuta, in particular to [30] and to [33].

In particular, in [33] it is assumed that the system in conservation form

$$v_t + f(v)_x = \Big(\tilde{B}(v)v_x\Big)_x \tag{2.12}$$

admits a convex and dissipative entropy η which moreover satisfies

$$\Big(D^2\eta(v)\Big)^{-1}\tilde{B}(v)^T = \tilde{B}(v)\Big(D^2\eta(v)\Big)^{-1}.$$

If one performs the change of variables defined by $w = \nabla\eta(v)$, finds that system (2.7) is equivalent to

$$\hat{E}(w)w_t + \hat{A}(w)w_x = \Big(\hat{B}(w)w_x\Big)_x,$$

with $\hat{A}$ and $\hat{B}$ symmetric.

It is then introduced the assumption

Condition N: the kernel of $\hat{B}(w)$ does not depend on w.

and it is proved that *Condition N* holds if and only if there exists a change of variable $w = w(u)$ which ensures that system (2.12) is equivalent to

$$E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx}$$

for some $E(u)$ that satisfies condition 5 in Hypothesis 2 and some $A(u,\, u_x)$ and $B(u)$ as in condition 2 and 1 respectively.

Moreover, it is shown that *Condition N* is verified in the case of several systems with physical meaning.

On the other side, the fourth assumption in Hypothesis 2 is the so called Kawashima condition and was introduced in [30]. Roughly speaking, its meaning is to ensure that there exists an interaction between the parabolic and the hyperbolic component of system (2.7) and hence to guarantee the regularity of the solution of the associated initial boundary value problem (1.5).

The assumption of block linear degeneracy (the third in Hypothesis 2) was instead introduced on the basis of Example 2.1 and 2.2 in Section 2.2.2, which show that such an assumption is essential to avoid the appearance of pathological behaviors both in the case of initial boundary value problems and in the case of the boundary free Cauchy problem.

Finally, Example 2.3 in Section 2.2.2 deals with *Condition N*, and it guarantees that the hypothesis that the rank of B is constant is necessary in order to prevent the arising of undesired dynamics.

2.2.1 An explicit definition of boundary datum for the parabolic problem

Thanks to the block decomposition of the viscosity matrix introduced with Hypothesis 2, system (2.8) can be seen as the coupling between a hyperbolic and a parabolic component: the first corresponds to the first $N-r$ lines of the system, while the second component corresponds to the last r lines. The interaction between the two components is guaranteed by Kawashima condition (condition number 4 in Hypothesis 2), but the hyperbolic-parabolic structure plays a key role in the way one assigns the boundary condition on the parabolic IBVP

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ \text{ß}(u^\varepsilon(t,\, 0)) \equiv \bar{g} \\ u^\varepsilon(0,\, x) \equiv \bar{u}_0, \end{cases}$$

Before giving a more precise definition of the function ß, some notations have to be introduced. In the following, we will denote by n_{11} the number of strictly negative eigenvalues of A_{11}, by q the dimension of $\ker A_{11}$ and by n the number of strictly negative eigenvalues of A. Let $\vec{\zeta}_i(u,\, 0) \in \mathbb{R}^{N-r}$ be an eigenvector of $E_{11}^{-1}A_{11}(u)$ associated to a non positive eigenvalue $\eta_i \leq 0$. Let $Z_i(u,\, 0) \in \mathbb{R}^N$ be defined by

$$Z_i := \begin{pmatrix} \vec{\zeta}_i \\ 0 \end{pmatrix} \tag{2.13}$$

and finally let

$$\mathcal{Z}(u) := \text{span}\langle Z_1(u,\, 0), \dots, Z_{n_{11}+q}(u,\, 0)\rangle.$$

Define also the complementary subspace

$$\mathcal{W}(u) := \text{span}\Big\langle \begin{pmatrix} 0 \\ \vec{e}_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vec{e}_r \end{pmatrix}, \begin{pmatrix} \vec{w}_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \vec{w}_{N-r-n_{11}-q} \\ 0 \end{pmatrix} \Big\rangle,$$

where $\vec{e}_i \in \mathbb{R}^r$ are the vectors of a basis in $\mathbb{R}^r$ and $\vec{w}_j \in \mathbb{R}^{N-r}$ are the eigenvectors of $E_{11}^{-1}A_{11}$ associated to strictly positive eigenvalues.

Since $\mathcal{W}(u) \oplus \mathcal{V}(u) = \mathbb{R}^N$, every $u \in \mathbb{R}^N$ can be written as

$$u = u_1 + u_2, \quad u_1 \in \mathcal{W}(u), u_2 \in \mathcal{Z}(u). \tag{2.14}$$

A definition of boundary condition that guarantees the well-posedness of the parabolic IBVP (2.11) is

$$ß(u) = u_1. \tag{2.15}$$

Indeed, the datum imposed along the vectors $\vec{e}_1, \dots \vec{e}_r$ refers to the parabolic component of system (2.8), while the datum imposed along $\vec{w}_1, \dots \vec{w}_{N-r-n_{11}-q}$ refers to the hyperbolic one. It is possible to construct a solution to (2.11) at least for a small time interval if the initial and the boundary data $ß(u^\varepsilon(t, 0)) = g(t)$, $u^\varepsilon(0, x) = u_0(x)$ are sufficiently smooth.

In the following, however, it will be considered a slightly different way of assigning the boundary condition. Such a way is basically equivalent to (2.15) but allows to define ß as the projection on a *fixed* subspace, while the space $\mathcal{W}(u)$ in (2.14) in general *depends* on u.

More precisely, as it will be clear in the following the equivalence between the two definitions of boundary data is guaranteed if one knows a priori that u belongs to a suitable small neighborhood of a fixed state $\bar{u}_0$. Such a condition is, however, guaranteed by the assumption of small total variation which will be introduced in Hypothesis 4.

Let such a fix state $\bar{u}_0$ be given: one can for example take $\bar{u}_0 = \lim_{x \to 0^+} \bar{u}_0(x)$. If u is sufficiently close to $\bar{u}_0$, then by continuity $W(\bar{u}_0) \oplus Z(u) = \mathbb{R}^N$. Hence the decomposition

$$u = u_{01} + u_{02}, \quad u_{01} \in \mathcal{W}(\bar{u}_0), \; u_{02} \in \mathcal{Z}(u) \tag{2.16}$$

is uniquely defined.

It is now possible to introduce the following

Definition 2.1. The function ß which allows to assign the boundary datum in the parabolic IBVP (2.11) is

$$\begin{aligned} ß : \mathbb{R}^N &\longrightarrow \mathbb{R}^{N-n_{11}-q} \\ u \quad &\mapsto u_{10}, \end{aligned} \tag{2.17}$$

where u_{10} is the component of u along $W(\bar{u}_0)$, according to the decomposition (2.16).

One can check by straightforward considerations that once the component u_{01} defined by (2.16) is assigned, then it is uniquely determined the component u_1 in (2.14): in this sense, the definition of boundary datum determined by (2.14) and by (2.17) are completely equivalent.

Moreover, it is made reference to Section 4.3 for an extension to more general conditions.

2.2.2 Three fundamental examples

Example 2.1. This example deals with the equation

$$A(u, u_x)u_x = B(u)u_{xx}, \tag{2.18}$$

in the case the condition of block linear degeneracy (the third in Hypothesis 2) does not hold.

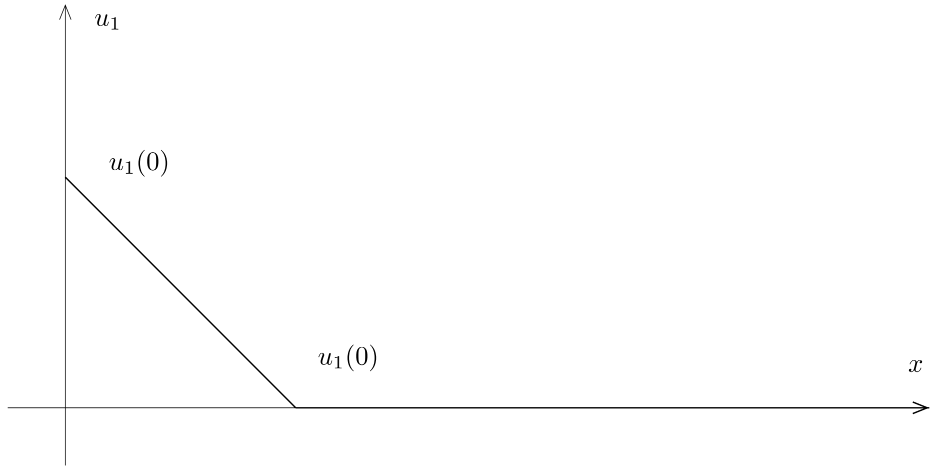
System (2.18) is satisfied by the stationary solutions of (2.8) One therefore expects the solution of (2.18) to have good regularity properties: the example which is going to be discussed show that if the condition of block linear degeneracy does not hold a function u satisfying (2.18) may have the graph illustrated by Figure 1 and hence be not $\mathcal{C}^1$. Moreover, the figure suggests that a pathological behavior typical of the solution of the porous-media equation may occur, namely it may happen that a solution is different from zero on a interval and then vanishes identically. Let

$$B(u) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

and

$$A(u, u_x) = \begin{pmatrix} u_1 & 1 \\ 1 & 0, \end{pmatrix}.$$

Figure 1: the graph of the function $u_1(x)$ in Example 2.1



being $u = (u_1,\, u_2)^T$. Then the assumption of block linear degeneracy is not satisfied in a neighborhood of $u_1 = 0$: to find out solutions of (2.18) showing pathological behaviors it seems therefore promising to impose the limit conditions

$$\lim_{x \to +\infty} u_1(x) = 0 \qquad \lim_{x \to +\infty} u_2(x) = 0.$$

In this case equation (2.18) writes

$$\begin{cases} u_1\big(u_{1x} + 1\big) = 0 \\ u_1 = u_{2x} \end{cases} \tag{2.19}$$

Since the equation satisfied by u_1 admits more solutions in a neighborhood of $u_1 = 0$, to introduce a selection principle we will use a perturbation technique.

Namely, we will approximate (2.18) with a system with non singular viscosity B^ν and we will denote by u^ν the corresponding solution. Then it holds

$$u^\nu_{xx} = \Big(B^\nu(u^\nu)\Big)^{-1}\Big(A(u^\nu,\, u^\nu_x) - \sigma E(u^\nu)\Big)u^\nu_x, \tag{2.20}$$

and hence the perturbed system admits a unique solution. In this example we will take

$$B^\nu = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}.$$

and we will study the behavior of the solutions of (2.20) such that

$$\begin{cases} u^\nu_1(0) = u_1(0) \\ u^\nu_2(0) = u_2(0) \end{cases} \qquad \begin{cases} \lim_{x \to +\infty} u^\nu_1(x) = 0 \\ \lim_{x \to +\infty} u^\nu_2(x) = 0. \end{cases}$$

In this case system (2.20) writes

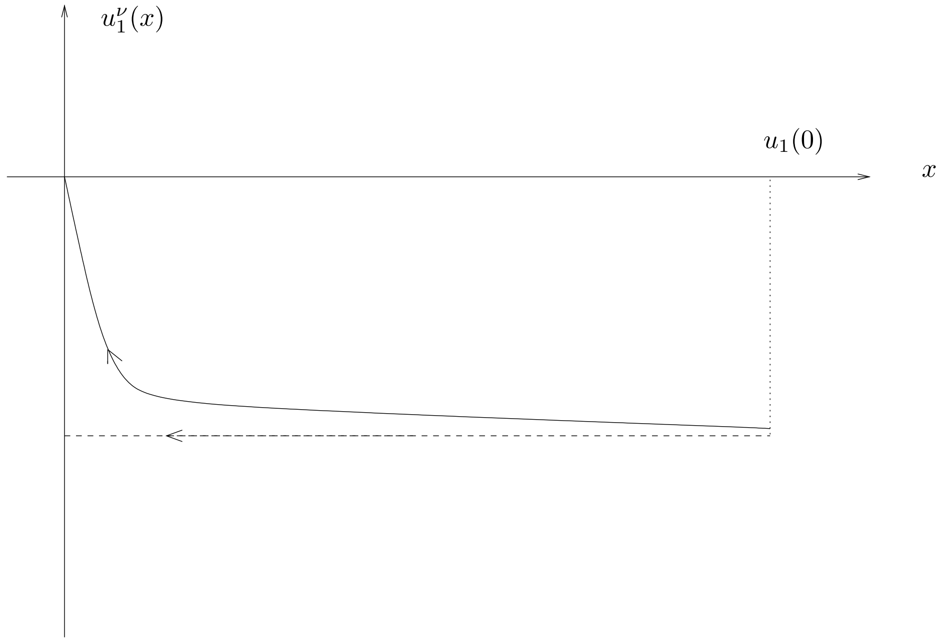
$$\begin{cases} u_{1xx} = u_1\big(u_{1x} + 1\big)/\nu \\ u_{2x} = u_1 \end{cases} \tag{2.21}$$

Letting $\nu \to 0^+$, it turns out that if $u_1(0) \geq 0$, then the solution u^ν_1 converges to

$$u_1(x) = \begin{cases} u_1(0) - x & x \leq u_1(0) \\ 0 & x > u_1(0). \end{cases} \tag{2.22}$$

The corresponding u_2 can then be computed by a simple integration, since $u_{2x} = u_1$.

Figure 2: the orbits $(u_1^\nu,\, u_{1x}^\nu)$ of the perturbed solutions in Example 2.1



To prove the convergence of u_1^ν one can prove that the orbits of $(u_1^\nu,\, u_{1x}^\nu)$ converge (in the sense that we will specify soon) to the set

$$\Big\{(u_1,\, -1) : u_1 \in]0,\, u_1(0)]\Big\} \cup \Big\{(0,\, 0)\Big\} \subseteq \mathbb{R}^2. \tag{2.23}$$

Indeed, this is sufficient to select (2.22) among all the solutions of (2.19), because it is the only one whose orbit does not contain other points.

From (2.21) one deduces

$$\nu\Big(\log|u_{1x}^\nu + 1|\Big)_x = u_1^\nu = \nu u_{1xx}^\nu - \Big(\frac{1}{2}(u_1^\nu)^2\Big)_x$$

and hence, integrating and imposing the limit conditions,

$$\frac{1}{2}(u_1^\nu)^2 = \nu\Big(u_{1x}^\nu - \log|u_{1x}^\nu + 1|\Big). \tag{2.24}$$

The orbits $(u_1^\nu,\, u_{1x}^\nu)$ belong to the set described by the previous relation, union the line $u_{1x}^\nu \equiv -1$. Since the aim is to study the solution satisfying

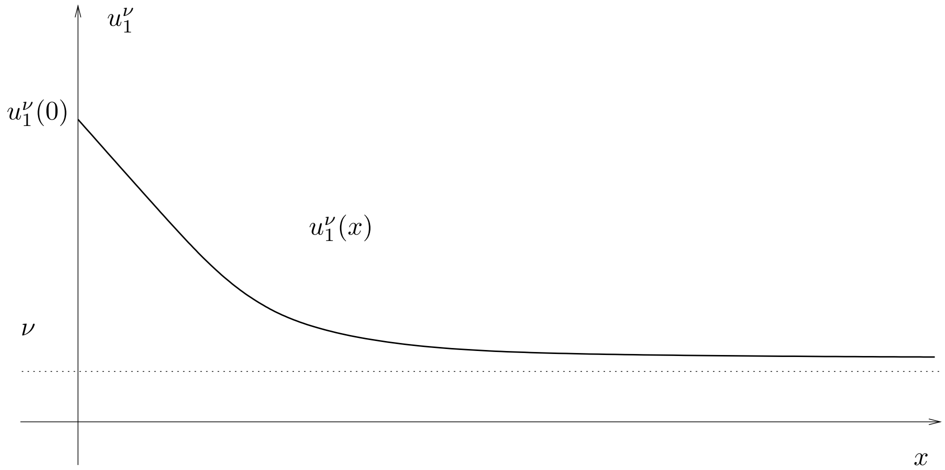
$$u_1^\nu(0) > 0 \qquad \lim_{x \to +\infty} u_1^\nu(x) = 0,$$

the attention is focused on the orbit illustrated in the Figure 2.1, which is the graph of a function $u_{1x}^\nu = u_{1x}^\nu(u_1^\nu)$ whose derivative in $u_1 = 0$ is equal to $-1/\nu$. Being uniformly bounded, this function admits a subsequence that converges pointwise when $\nu \to 0^+$: moreover, from the relation (2.24) it follows that the limit is given by

$$u_{1x}(u_1) = \begin{cases} -1 & 0 < u_1 \le u_1(0) \\ 0 & u_1 = 0 \end{cases}$$

whose graph is the set defined by (2.23). Hence on the basis of the previous considerations one deduces that the solution of (2.19) selected by means of the perturbation technique is (2.22).

Figure 3: the solution obtained perturbing the limit condition in Example 2.1



It is interesting to underline that the same limit solution (2.22) is obtained considering another kind of approximation: more precisely, it is introduced the family of systems

$$\begin{cases} u_1^\nu u_{1x}^\nu + u_{2x}^\nu = 0 \\ u_{1x}^\nu = u_{2xx}^\nu \\ \lim_{x\to+\infty} u_1^\nu(x) = \nu \qquad \lim_{x\to+\infty} u_2^\nu(x) = 0 \end{cases} \tag{2.25}$$

where ν is a positive parameter which is allowed to go to zero. In other words, while system (2.21) is obtained from (2.19) via a perturbation of the viscosity matrix, (2.25) is obtained letting the limit condition vary.

It turns out through straightforward computations that (2.25) rewrites as

$$\begin{cases} -u_1^\nu(x) + u_1^\nu(0) + \nu \log\left(\dfrac{u_1^\nu(0)-\nu}{u_1^\nu(x)-\nu}\right) = x \\ u_{2x}^\nu = u_1^\nu - \nu \end{cases}$$

and hence that $u_1^\nu(x)$ has the graph illustrated in Figure 3. Moreover, the equation satisfied by $\nu_1(x)$ is

$$u_{1x}^\nu = \frac{\nu - u_1^\nu}{u_1^\nu}$$

and hence the orbits $(u_1^\nu,\, u_{1x}^\nu)$ corresponding to an initial datum $u_1^\nu(0) > \nu$ converge when $\nu \to 0^+$ to the set defined by (2.23).

It follows that the limit solution selected by the approximation (2.25) is again (2.22).

Remark 2.1. An alternative interpretation of the previous considerations is the following.

Consider the family of systems

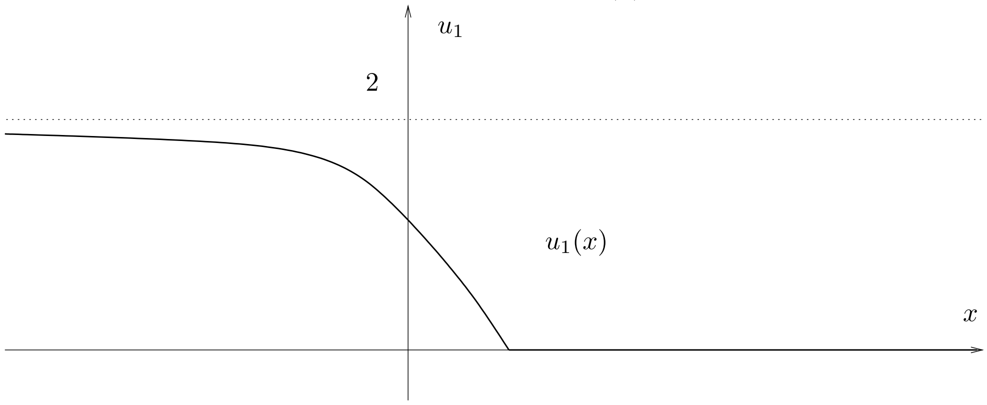
$$\begin{cases} u_1 u_{1x} + u_{2x} = 0 \\ u_{1x} = u_{2xx} \\ u_1(0) = \upsilon \qquad \lim_{x\to+\infty} u_1(x) = y \qquad \lim_{x\to+\infty} u_2(x) = 0 \end{cases} \tag{2.26}$$

parametrized by the limit value $y \in \mathbb{R}$. Let $\mathcal{U}_1(y)$ be the set of the values υ such that (2.26) admits a solution. The ODE satisfied by u_1 is

$$u_{1x} = \frac{y - u_1}{u_1}$$

when $u_1 \neq 0$. From a standard analysis it turns out that when $y > 0$, then $\mathcal{U}_1(y) =]0, +\infty[$. When $y = 0$, $\mathcal{U}_1(0) = [0, +\infty[$, while when $y < 0$, $\mathcal{U}_1(y) = \{y\}$.

Figure 4: the graph of the function $u_1(x)$ in Example 2.2



In other words, $\mathcal{U}_1$ is a manifold of dimension 1 when $y > 0$, a manifold with boundary and with dimension 1 when $y = 0$, while when $y < 0$ the dimension of the manifold drops to zero, i.e. the manifold reduces to a point.

Example 2.2. Example 2.1 guarantees that the condition of block linear degeneracy (the third in Hypothesis 2) is essential in order to avoid the presence of stationary solutions of (2.8) showing pathological behaviors. Consequently, it justifies the introduction of such a condition when system (2.8) is associated to an initial boundary value problem.

Example 2.2 shows that the assumption of block linear degeneracy is also necessary in order to prevent the appearance of travelling wave solutions of (2.8) with pathological dynamics. Hence, it provides a complete justification of this assumption also in the case system (2.8) is associated to a boundary free Cauchy problem.

For reasons that will be clearer in Sections 3.2.1 and 4.2.1, we are actually interested in the travelling wave solutions

$$B(u)u_{xx} = \Big(A(u,\, u_x) - \sigma E(u) \Big) u_x \tag{2.27}$$

such that the speed σ is close to an eigenvalue of $E^{-1}A$, i.e. such that one eigenvalue of $(A - \sigma E)$ is close to zero. Let

$$A(u,\, u_x) := \begin{pmatrix} u_1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B(u) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.28}$$

which are obtained from a system in conservation form

$$u_t + f(u)_x = \Big(B(u) u_x \Big)_x$$

where

$$f(u) = \Big(u_1^2/2 + u_2,\, u_1 + u_2,\, 0 \Big)^T$$

and $B(u)$ is the constant matrix defined by (2.28).

Since the matrix A defined by (2.28) has an eigenvalue identically equal to zero, it is interesting to consider the travelling wave (2.27) with speed $\sigma = 0$.

System 2.27 can then be rewritten as

$$\begin{cases} u_1 u_{1x} + u_{2x} = 0 \\ u_{1x} + u_{2x} = u_{2xx} \\ 0 = u_{3xx}. \end{cases} \tag{2.29}$$

As in the case considered in Example 2.1, the condition of block linear degeneracy is not satisfied in a neighborhood of $u_1 = 0$ and therefore to find out pathological behaviors it seems promising to study the travelling wave solutions such that

$$\lim_{x \to +\infty} u_1(x) = 0 \qquad \lim_{x \to +\infty} u_2(x) = 0 \qquad \lim_{x \to +\infty} u_3(x) = 0.$$

Moreover, we will suppose that $u_1(0) > 0$. Adding this conditions to system (2.29), one obtains

$$\begin{cases} u_1^2/2 + u_2 = 0 \\ u_1 + u_2 = u_{2x} \\ u_3 \equiv 0. \end{cases}$$

Hence in the following we will ignore the third component of the solution and we will study only the first two lines of the system, which can be rewritten as

$$\begin{cases} u_2 = -u_1^2/2 \\ u_1 u_{1x} = u_1^2/2 - u_1. \end{cases} \tag{2.30}$$

If $u_1 \neq 0$, the second line is equivalent to

$$u_{1x} = u_1/2 - 1$$

and hence by straightforward computations it follows that

$$u_1(x) = 2 + (u_1(0) - 2)e^{x/2},$$

where $0 < u_1(0) < 2$. This solution can be extended as far as $u_1 > 0$: when u_1 reaches the value zero, (2.30) admits more that one solution.

In order introduce a selection principle, we will consider the same perturbation argument introduced in the second part of Example 2.1. More precisely, let the following conditions be assigned

$$\lim_{x \to +\infty} u_1^\nu(x) = \nu \qquad \lim_{x \to +\infty} u_2^\nu(x) = 0 \qquad \lim_{x \to +\infty} u_3^\nu(x) = 0, \tag{2.31}$$

where ν is a small and positive parameter which is allowed to go to zero. The solution selected via this perturbation technique is

$$u_1(x) = \begin{cases} u_1(x) = 2 + (u_1(0) - 2)e^{x/2} & x \leq 2 \log \Big(2/\big(2 - u_1(0)\big)\Big) \\ 0 & x > 2 \log \Big(2/\big(2 - u_1(0)\big)\Big) \end{cases} \tag{2.32}$$

which has the graph illustrated in Figure 4.

Indeed, it turns out that, provided that the limit conditions (2.31) hold, system (2.29) is equivalent to

$$\begin{cases} u_2^\nu = \nu^2/2 - (u_1^\nu)^2/2 \\ u_1^\nu + u_2^\nu - \nu = u_{2x}^\nu \\ u_3^\nu \equiv 0 \end{cases}$$

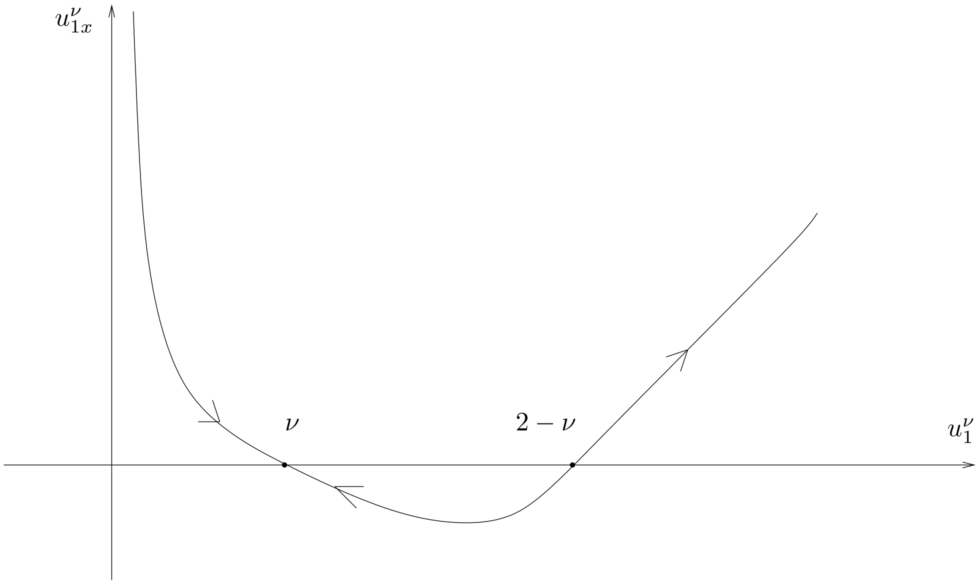
and hence the ODE satisfied by u_1^ν when $u_1^\nu \neq 0$ is

$$u_{1x}^\nu = \frac{(u_1^\nu)^2/2 - u_1^\nu + \nu - \nu^2/2}{u_1^\nu}.$$

It follows that the orbits $(u_1^\nu, \, u_{1x}^\nu)$ are those represented in Figure 5 and if $u_1(0) \in]0, \, 2[$ converge to the orbits $(u_1, \, u_{1x})$ of (2.32).

One can therefore conclude that, if the condition of block linear degeneracy is not satisfied, then a travelling wave solution of (2.8) may be not $\mathcal{C}^1$ and have may the graph illustrated by Figure 4, which show the pathological behavior typical of the solution of the porous media equation.

Figure 5: the orbits $(u_1^\nu,\, u_{1x}^\nu)$ of the solution u_1^ν of the perturbed problem in Example 2.2



Example 2.3. The aim of this example is to show that if in a system of the form (2.8) the rank of the matrix B is not constant, thus contradicting the first assumption of Hypothesis 2, then pathological behaviors may appear.

It is considered the system in conservation form

$$u_t + f(u)_x = \Big(B(u)u_x\Big)_x$$

with

$$f(u_1,\, u_2) = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} \qquad B(u_1,\, u_2) = \begin{pmatrix} \gamma u_1^{\gamma-1} & 0 \\ 0 & 1 \end{pmatrix}$$

for some $\gamma > 3$. In this case, the rank of $B(u)$ drops from 2 to 1 when u_1 reaches 0: to find out pathological behaviors it seems therefore natural to study the solution in a neighborhood of $u_1 = 0$. More precisely, the attention will be focused on boundary layer profiles

$$f(u)_x = \Big(B(u)u_x\Big)_x \tag{2.33}$$

such that

$$\lim_{x\to+\infty} u_1(x) = 0 \qquad u_1(x) \geq 0 \qquad \lim_{x\to+\infty} u_2(x) = 0 \tag{2.34}$$

and it will be shown that the graph of the first component u_1 has the shape illustrated in Figure 6. Hence u_1 is not $\mathcal{C}^1$ and presents a pathological behavior like the one typical of the solutions of the porous-media equation: it is different from zero on a interval and then vanishes identically.

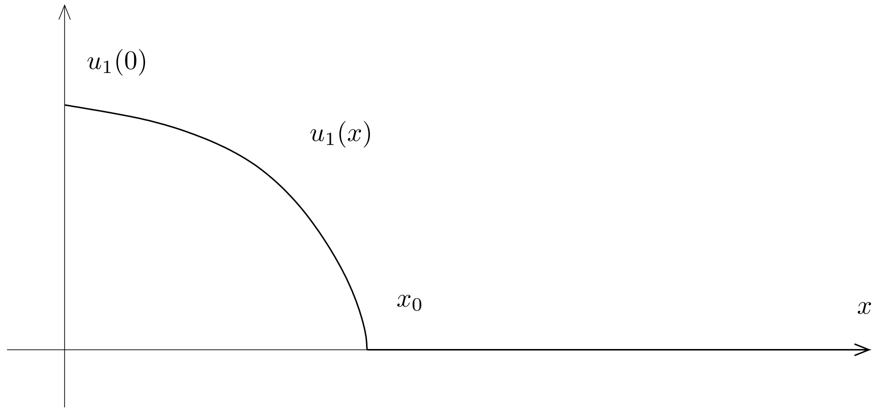
In this case equation (2.33) writes

$$\begin{cases} u_{2x} = \Big(u_1^\gamma\Big)_{xx} \\ u_1 = u_{2x} \end{cases} \tag{2.35}$$

and hence after some computations one obtains

$$\begin{cases} u_{1x} = -\sqrt{\dfrac{2}{\gamma(\gamma+1)} u_1^{(3-\gamma)}} \\ \\ u_2 = \displaystyle\int_{+\infty}^{x} u_1(y)dy. \end{cases} \tag{2.36}$$

Figure 6: the graph of the function $u_1(x)$ in Example 2.3



The equation satisfied by u_1 admits more than one solution in a neighborhood of $u_1 = 0$, but thanks to a perturbation technique analogous to that introduced to study Example 2.1 one selects the function

$$u_1(x) = \begin{cases} \sqrt[\gamma-1]{\dfrac{(\gamma-1)^2}{2\gamma(\gamma+1)}(x-x_0)^2} & x \le x_0 \\ \\ 0 & x > x_0, \end{cases}$$

where

$$x_0 = \sqrt{\frac{2\gamma(\gamma+1)}{(\gamma-1)^2} u_1(0)^{\gamma-1}}.$$

Indeed, let

$$B^\nu(u^\nu) = \begin{pmatrix} \nu + \gamma(u_1^\nu)^{\gamma-1} & 0 \\ 0 & 1 \end{pmatrix},$$

then the equation

$$u_{xx}^\nu = \Big(B^\nu(u^\nu)\Big)^{-1} f(u^\nu)_x$$

admits a unique solution satisfying (2.34). Moreover, this solution satisfies

$$(u_2^\nu)^2 = \nu(u_1^\nu)^2 + \frac{2\gamma}{\gamma+1}(u_1^\nu)^{\gamma+1}. \tag{2.37}$$

Indeed, from

$$\begin{cases} u_{2x}^\nu = \Big(\nu u_{1x}^\nu + \gamma(u_1^\nu)^{\gamma-1} u_{1x}^\nu\Big)_x \\ u_{1x}^\nu = u_{2xx}^\nu \end{cases}$$

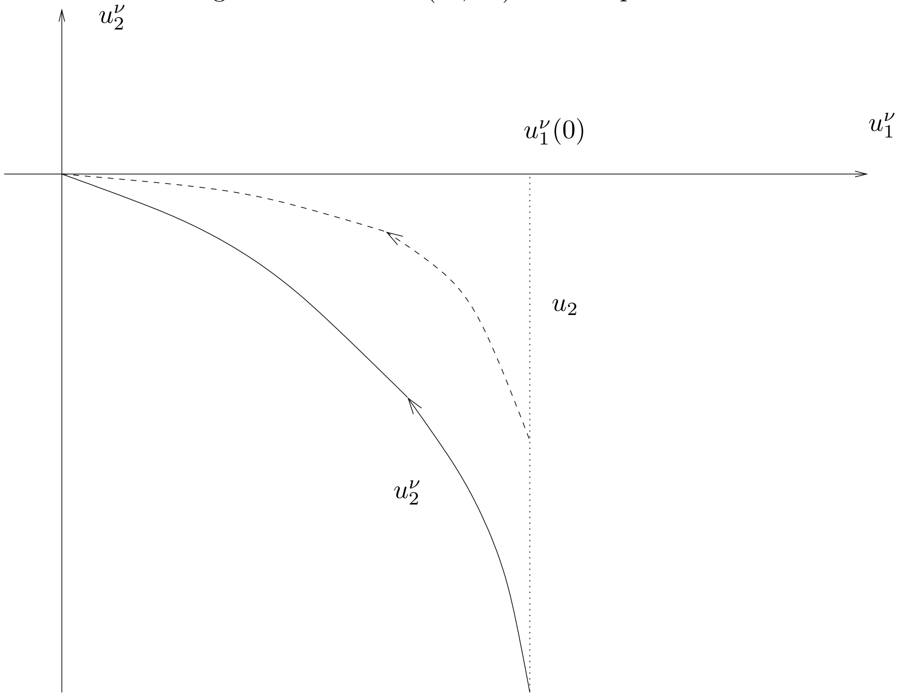
one obtains integrating

$$\begin{cases} u_2^\nu = \nu u_{1x}^\nu + \gamma(u_1^\nu)^{\gamma-1} u_{1x}^\nu \\ u_1^\nu = u_{2x}^\nu. \end{cases}$$

Multiplying the first line by $u_1 = u_{2x}$ and then integrating again one obtains (2.37). It follows that the orbit (u_1, u_2) is the graph of the function

$$u_2^\nu = -\sqrt{\nu u_1^\nu + \frac{2\gamma}{\gamma+1}(u_1^\nu)^{\gamma+1}},$$

Figure 7: the orbits (ν_1, ν_2) in Example 2.3



which is illustrated in Figure 7 and when $\nu \to 0^+$ converges to

$$u_2 = -\sqrt{\frac{2\gamma}{\gamma+1}(u_1)^{\gamma+1}}.$$

This shows that the perturbation technique selects (2.36) because it is the only solution of equation (2.35) whose orbits coincide with the graph of the above function.

2.3 General hypotheses

This section introduces the general hypotheses required in both cases, i.e. when the viscosity is invertible and when it is singular. In the statements of the hypotheses it will be made reference to the formulation (2.1).

First of all, an assumption of strict hyperbolicity is introduced:

Hypothesis 3. *If u belongs to a sufficiently small neighborhood of $\bar{u}_0$ then all the eigenvalues of the matrix $E^{-1}(u)A(u, 0)$ are real. Moreover, there exists a constant $c > 0$ such that*

$$\inf_u \left\{ \left| \lambda_i(u, 0) - \lambda_j(u, 0) \right| \right\} \geq c > 0 \quad \forall\, i \neq j.$$

In order to study the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation (1.5), an hypothesis of convergence is certainly needed. More precisely, it will be assumed the following:

Hypothesis 4. *Let*

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \\ ß(u^\varepsilon(t, 0)) = g(t) \quad u^\varepsilon(0, x) = \bar{u}_0(x) \end{cases}$$

be a parabolic initial boundary value problem satisfying g, $\bar{u}_0 \in L^1_{loc}$ and

$$\left| g(0) - ß\big(\bar{u}_0(0)\big) \right|, \ \mathrm{TotVar}\big\{g\big\}, \ \mathrm{TotVar}\big\{\bar{u}_0\big\} \leq \delta << 1. \tag{2.38}$$

Then it is assumed that there exits a time T_0, which depends only on the bound δ and on the matrices E, A and B, such that

$$\mathrm{TotVar}\big\{u^{\varepsilon}(t,\,\cdot)\big\} \le C\delta \quad \forall\,\varepsilon,\ t\in[0,\,T_0].$$

If the solutions are absolutely continuous, one can apply Helly's theorem and deduce from the uniform bound on the total variation a compactness result, which however does not ensure that the limit is unique. Here we will assume directly a result of convergence and of uniqueness of the limit:

$$\forall\, t\in[0,\,T_0] \quad u^{\varepsilon}(t,\,\cdot)\to u(t)\ L^1_{loc} \text{ when } \varepsilon\to 0^+ \qquad \mathrm{TotVar}\big\{u(t)\big\}\le C\delta.$$

We remark that the uniqueness of the limit is actually implied by the next hypotheses and the uniqueness results for the Standard Riemann Semigroup with boundary: it is made reference to [2] for the extension of the definition of SRS to initial boundary value problems, while an application of this notion to prove the uniqueness of the limit of the vanishing viscosity solutions was introduced in [9] in the case of the Cauchy problem.

It is also assumed that the data $\bar u_b$ and $\bar u_0$ imposed on the boundary Riemann problem fits indeed in the assumption (2.38):

Hypothesis 5. *It holds*

$$|\beta(\bar u_0)-\bar g| < \delta$$

for the same constant $\delta << 1$ that appears in Hypothesis 4.

An hypothesis of stability with respect to L^1 perturbations in the initial and boundary data is also introduced:

Hypothesis 6. *There exists a constant $L>0$ such that the following holds.*

Two families of parabolic initial boundary value problems are fixed:

$$\begin{cases} u^{1\,\varepsilon}_t + A(u^{1\,\varepsilon},\,\varepsilon u^{1\,\varepsilon}_x)u^{1\,\varepsilon}_x = \varepsilon B(u^{1\,\varepsilon})u^{1\,\varepsilon}_{xx} \\ \beta(u^{1\,\varepsilon}(t,\,0)) = \bar g^1(t). \\ u^{1\,\varepsilon}(0,\,x) = \bar u^1_0(x) \end{cases} \qquad \begin{cases} u^{2\,\varepsilon}_t + A(u^{2\,\varepsilon},\,\varepsilon u^{2\,\varepsilon}_x)u^{2\,\varepsilon}_x = \varepsilon B(u^{2\,\varepsilon})u^{2\,\varepsilon}_{xx} \\ \beta(u^{2\,\varepsilon}(t,\,0)) = \bar g^2(t). \\ u^{2\,\varepsilon}(0,\,x) = \bar u^2_0(x) \end{cases} \tag{2.39}$$

with $\bar u^1_0$, $\bar g^1$ and $\bar u^2_0$, $\bar g^2$ in L^1_{loc} and satisfying the assumption (2.38).

Then for all $t\in[0,\,T_0]$ and for all $\varepsilon>0$ it holds

$$\|u^{1\,\varepsilon}(t)-u^{2\,\varepsilon}(t)\|_{L^1} \le L\bigg(\|\bar u^1_0-\bar u^2_0\|_{L^1}+\|\bar g^1-\bar g^2\|_{L^1}\bigg).$$

From the stability of the approximating solutions and from the L^1_{loc} convergence one can deduce the stability of the limit. More precisely, let u^1 and u^2 the limits of the two approximations defined above, then

$$\|u^1(t)-u^2(t)\|_{L^1} \le L\bigg(\|\bar u^1_0-\bar u^2_0\|_{L^1}+\|\bar g^1-\bar g^2\|_{L^1}\bigg).$$

Finally, it is assumed that in the hyperbolic limit there is a *finite propagation speed* of the disturbances:

Hypothesis 7. *There exist constants $\beta, c>0$ such that the following holds.*

Let $\bar u^1_0$, $\bar g^1$ and $\bar u^2_0$, $\bar g^2$ in L^1_{loc} and bounded, let $u^{\varepsilon\,1}$ and $u^{\varepsilon\,2}$ be the solutions of (2.39) *and u^1 and u^2 the corresponding limits. If*

$$\bar g^1(t)=\bar g^2(t) \quad \forall\, t\le t_0 \qquad \bar u^1_0(x)=\bar u^2_0(x) \quad \forall\, x\le b$$

then

$$|u^{\varepsilon 1}(x,\,t)-u^{\varepsilon 2}(x,\,t)| \le \mathcal{O}\Big(e^{-c\min\big\{|x-\max\{0,\,\beta(t-t_0)\}|,|x-b+\beta t|\big\}/\epsilon}\Big) \quad \forall x\in[\max\{0,\,\beta(t-t_0)\},\,b-\beta t].$$

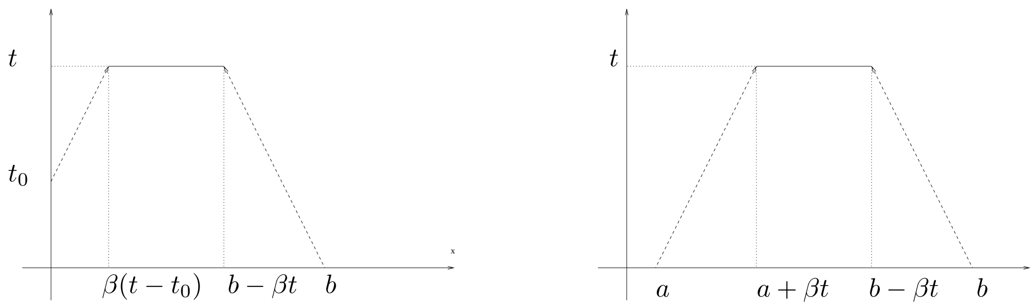
Analogously, if

$$\bar u^1_0(x)=\bar u^2_0(x) \quad \forall\, x\in[a,\,b]$$

then

$$|u^{\varepsilon 1}(x,\,t)-u^{\varepsilon 2}(x,\,t)| \le \mathcal{O}\Big(e^{-c\min\big\{|x-a-\beta t|,|x-b+\beta t|\big\}/\epsilon}\Big) \quad \forall\, x\in[a+\beta t,\,b-\beta t].$$

Figure 8: the finite propagation speed in the hyperbolic limit



Remark 2.2. An intuitive description of Hypothesis 7 is provided in Figure 8. This condition may appear a bit technical, but its meaning is roughly speaking to guarantee that the solutions of a general boundary Riemann solver can be obtained piecing together the solutions of simpler problems, i.e. those having data connected by a travelling wave or by a stationary solution of the parabolic equation. These conditions imply indeed the finite speed of propagation for the hyperbolic limit.

3 The case of invertible viscosity

This section deals with the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation (1.1) in the case the viscosity matrix $\tilde{B}$ is invertible. The precise hypotheses that are assumed are listed in Section 2.3 and 2.1: in particular, those hypotheses guarantee that it is sufficient to study system

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(t,\, 0) \equiv \bar{u}_b \\ u^\varepsilon(0,\, x) \equiv \bar{u}_0, \end{cases} \tag{3.1}$$

where the matrices E, A and B satisfy Hypotheses 1 and 3.

An immediate but crucial observation is that (3.1) is well posed for a general boundary condition $\bar{u}_b \in \mathbb{R}^N$. On the other side, immediate examples show that the hyperbolic initial boundary value problem

$$\begin{cases} E(u)u_t + A\big(u,\, 0\big)u_x = 0 \qquad u \in \mathbb{R}^N \\ u(t,\, 0) \equiv \bar{u} \\ u(0,\, x) \equiv \bar{u}_0, \end{cases} \tag{3.2}$$

is in general ill posed if $\bar{u} \in \mathbb{R}^N$. The study of the Riemann solver limit of the parabolic approximation (3.1) will provide a precise admissibility criterium for the boundary datum $\bar{u}$ in (3.2).

The crucial tool in the analysis is the study of systems

$$-\sigma E(U)U' + A(U,\, U')U' = B(U)U''$$

and

$$A(U,\, U')U' = B(U)U'',$$

satisfied respectively by travelling wave and stationary solutions. The most important results are summarized in Propositions 3.1 and 4.2: the first one deals with the case of a non characteristic boundary, while the seconds describes the Riemann solver in the boundary characteristic case.

The exposition is organized as follows. In Section 3.1 some preliminary results are recalled: they were already introduced in [24] and [4]. In Section 3.2 it is sketched the construction of the Riemann solver induced by the parabolic approximation

$$E(u^\varepsilon)u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx}$$

in the boundary free case and it is made reference to [5] for the complete analysis. In Section 4.2.3 it is achieved a complete characterization of the Riemann solver limit of (3.1) in the case of a non characteristic boundary. Finally, the study is completed in Section 3.2.3 with the analysis of the boundary characteristic case.

3.1 Preliminary results

The first result guarantees that the dimension of the center, stable and unstable spaces is the same for $E^{-1}A$ and for $E^{-1}B$. Even if it is not unknown (see [24]), for completeness the proof, based on a homotopy argument, will be given.

Lemma 3.1. *Assume that Hypotheses 1 and 3 hold. Then for any given u and u_x*

1. $B^{-1}(u)A(u,\, u_x)\vec{\xi} = 0 \iff A(u,\, u_x)\vec{\xi} = 0 \iff E^{-1}(u)A(u,\, u_x)\vec{\xi} = 0 \qquad \vec{\xi} \in \mathbb{R}^N$
2. *the number of eigenvalues of the matrix $B^{-1}(u)A(u,\, 0)$ with negative (respectively positive) real part is equal to the number of eigenvalues of $E^{-1}(u)A(u,\, 0)$ with negative (respectively positive) real part.*

Proof. The first point is actually an immediate observation.

To simplify the notation, in the proof of the second point, we will write A, B and E instead of $A(u,\, 0)$, $B(u)$ and $E(u)$ respectively. One can define the continuous path

$$\begin{aligned} F : [0,\, 1] &\to \mathbb{M}^{N\times N} \\ s &\mapsto (1-s)B + sE, \end{aligned}$$

which satisfies the following condition: for every $s \in [0,\, 1]$, $F(s)$ is positively definite and hence invertible. Indeed,

$$\forall \vec{\xi} \in \mathbb{R}^N \quad \langle F(s)\vec{\xi},\, \vec{\xi}\rangle \geq \min\{c_B(u),\, c_E(u)\}|\vec{\xi}|^2. \tag{3.3}$$

Moreover, by classical results (see for example [28]) from the continuity of the path $F(s)^{-1}A$ it follows the continuity of the corresponding paths of eigenvalues $\lambda_1(s)\dots\lambda_N(s)$.

Let $k-1$ be the number of eigenvalues of $F(0)^{-1}A = B^{-1}A$ with negative real part:

$$Re(\lambda_1(0)) < \dots Re(\lambda_{k-1}(0)) < 0 = Re(\lambda_k(0)) < Re(\lambda_{k+1}(0)) < \dots Re(\lambda_N(0)).$$

Because of the continuity of $\lambda_1(s)\dots\lambda_i(s)$, to prove that the number of negative eigenvalues of $F(1)^{-1}A = E^{-1}A$ is $(k-1)$ it is sufficient to prove that for any $i = 1\dots(k-1)$, $\lambda_i(s)$ cannot cross the imaginary axis. Moreover, the case $\lambda_i(s) = 0$ is excluded because otherwise the corresponding eigenvector $\vec{\xi}$ should satisfy $F(s)^{-1}A\vec{\xi}(s) \equiv 0$ for any s and hence $\lambda_i(s) \equiv 0$. It remains therefore to consider the possibility that $\lambda_i(s)$ is purely imaginary.

By contradiction, assume that there exist $\bar{s} \in [0,\, 1]$, $\lambda \in \mathbb{R}\setminus\big\{0\big\}$, $r \in \mathbb{C}^N$ such that

$$F(\bar{s})^{-1}Ar = i\lambda r : \tag{3.4}$$

just to fix the ideas, it will be supposed that $\lambda > 0$. Moreover, let

$$r = r_1 + ir_2, \quad r_1,\; r_2 \in \mathbb{R}^N.$$

Then from (3.4) one gets

$$\begin{aligned} Ar_1 &= -\lambda F(\bar{s})r_2 \\ Ar_2 &= \lambda F(\bar{s})r_1 \end{aligned}$$

and hence from (3.3) and from the symmetry of A it follows

$$\langle Ar_1,\ r_2\rangle \le -\lambda \min\{c_B(v),\ c_E(v)\}|r_2|^2$$
$$\langle Ar_1,\ r_2\rangle = \langle Ar_2,\ r_1\rangle \ge \lambda \min\{c_B(v),\ c_E(v)\}|r_1|^2,$$

which is a contradiction since $\lambda > 0$. This ends the proof of Lemma 3.1. □

In the following, given a matrix C we will denote by $V^s(C)$, $V^c(C)$ and $V^u(C)$ the stable, center and unstable subspace respectively. Lemma 3.1 ensures that, if Hypothesis 1 holds, then for all u, u_x

$$V^c\big(E^{-1}(u)A(u,\,0)\big) = V^c\big(A(u,\,0)\big) = V^c\big(B^{-1}(u)A(u,\,0)\big)$$
$$dimV^u\big(E^{-1}(u)A(u,\,0)\big) = dimV^u\big(B^{-1}(u)A(u,\,0)\big)$$
$$dimV^s\big(E^{-1}(u)A(u,\,u_x)\big) = dimV^s\big(B^{-1}(u)A(u,\,u_x)\big)$$

Thanks to the following lemma, which is a direct application of Lemma 7.1 in [4], one can conclude that actually

$$V^u\Big(E^{-1}(u)A(u,\,0)\Big) \oplus V^c\Big(E^{-1}(u)A(u,\,0)\Big) \oplus V^s\Big(B^{-1}(u)A(u,\,0)\Big) = \mathbb{R}^N. \tag{3.5}$$

Lemma 3.2. *Assume that Hypotheses 1 and 3 hold, then*

$$V^u\Big(E^{-1}(u)A(v,\,0)\Big) \cap V^s\Big(B^{-1}(u)A(v,\,0)\Big) = \{0\}.$$

3.2 Construction of the boundary Riemann solver in the case of invertible viscosity

3.2.1 The Riemann solver in the boundary free case

The Riemann solver induced in the boundary free case by the parabolic approximation (2.3) is described in [5], but, since we will need it in the following, in this section the crucial steps of the construction will be recalled.

The crucial point is the characterization of the i-th curve of admissible states starting from a datum u^+. Such a characterization is based on the analysis of the dynamics of the equation satisfied by the travelling waves:

$$B(U)U'' = \Big(A(U,\,U') - \sigma E(U)\Big)U' \tag{3.6}$$

or, equivalently,

$$\begin{cases} u' = v \\ B(u)v' = \Big(A(u,\,v) - \sigma E(u)\Big)v \\ \sigma' = 0. \end{cases} \tag{3.7}$$

Let $\lambda_i(u^+,\,0)$ be an eigenvalue of the matrix $E^{-1}(u^+)A(u^+,\,0)$ and $r_i(u^+,\,0)$ the corresponding eigenvector; if one linearizes the previous system around the equilibrium point $(u^+,\,0,\,\lambda_i(u^+,\,0))$ finds that the center space is given by

$$V^c = \Big\{(u,\,sr_i(u^+,\,0),\,\sigma) : u \in \mathbb{R}^N,\ s,\,\sigma \in \mathbb{R}\Big\}.$$

The center manifold theorem guarantees that in a small enough neighborhood of $(u^+,\,0,\,\lambda_i(u^+,\,0))$ V^c parametrizes an invariant manifold that contains all the orbits of the system (3.7) that do not blow exponentially neither at $x \to +\infty$ nor at $x \to -\infty$. More specifically, in the Appendix A.1 it is proved that system (3.7) reduced on the center stable manifold takes the form

$$\begin{cases} u' = v_i\tilde{r}_i(u,\,v_i,\,\sigma_i) \\ v_i' = \phi_i(u,\,v_i,\,\sigma_i)v_i \\ \sigma_i' = 0, \end{cases} \tag{3.8}$$

where $\phi_i(u,\, v_i,\, \sigma_i)$ and $\tilde{r}_i(u,\, v_i,\, \sigma_i)$ are functions whose explicit expression can be found in Appendix A.1. Nevertheless, it is important to underline that

$$\frac{\partial \phi_i}{\partial \sigma_i}\bigg|_{(u^+,\, 0,\, \sigma_i)} < 0, \tag{3.9}$$

as it is shown in section A.1. Hence one can apply the machinery developed in [5], Section 2, and obtain the i-th curve of admissible states starting from u^+:

$$u(s_i) = T^i_{s_i} u^+.$$

More precisely, let

$$f_i(\tau,\, u,\, v_i,\, \sigma_i) := \int_0^\tau \Big(\phi_i(u,\, v_i,\, \sigma_i) + \sigma_i \Big) d\zeta.$$

Then $u(s_i)$ is the value at time $\tau = s_i$ of the map $(u(\tau),\, v_i(\tau),\, \sigma_i(\tau))$, which is the solution of the fixed point problem

$$\begin{cases} u(\tau) = u^+ + \displaystyle\int_0^\tau \tilde{r}_i(u(\xi),\, v_i(\xi),\, \sigma_i(\xi)) d\xi \\ v_i(\tau) = f_i(\tau,\, u,\, v_i,\, \sigma_i) - \mathrm{conc} f_i(\tau,\, u,\, v_i,\, \sigma_i) \\ \sigma_i(\tau) = \dfrac{d}{d\tau} \mathrm{conc} f_i(\tau,\, u,\, v_i,\, \sigma_i). \end{cases} \tag{3.10}$$

In the previous expression $\mathrm{conc} f_i$ denotes the concave envelope of the function f_i. The analysis carried on in [5] guarantees that (3.10) admits indeed a unique Lipschitz continuous solution $(u(\tau),\, v_i(\tau),\, \sigma_i(\tau))$ and that

$$\frac{\partial T^i}{\partial s}\bigg|_{s=0} = r_i(u,\, 0).$$

3.2.2 The case of non characteristic boundary

This section deals with the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation (3.1) when all the eigenvalues of the matrix $E^{-1}(u)A(u,\, 0)$ are well separated from 0. More precisely, it will be assumed that

Hypothesis 8. *The boundary is non characteristic for problem* (3.1), *i.e. there exists a constant* $c > 0$ *such that, for all* u *belonging to a small enough neighborhood of* $\bar{u}_0$, *it holds*

$$\lambda_1(u,\, 0) < \cdots < \lambda_n(u,\, 0) \le -c < 0 < c \le \lambda_{n+1}(u,\, 0) < \cdots < \lambda_N(u,\, 0). \tag{3.11}$$

Hence here and in the following n *denotes the number of negative eigenvalues of the matrix* $E^{-1}A$, *while there are* $N - n$ *positive eigenvalues.*

The initial datum $\bar{u}_0$ is connected to some state $\bar{u}$ by waves of positive speed and the corresponding curves of admissible states $T^N \dots T^{n+1}$ are the same curves which were introduced in the boundary free case and whose construction has been briefly recalled in Section 3.2.1:

$$\bar{u} = T^{n+1}_{s_{n+1}} \circ T^{n+2}_{s_{n+2}} \cdots T^N_{s_N} \bar{u}_0. \tag{3.12}$$

The state $\bar{u}$ is the trace on the axis $x = 0$ of the solution in the hyperbolic limit: this means, in particular, that the admissibility criterium induced by the parabolic approximation (3.1) on the boundary datum in problem

$$\begin{cases} u_t + E(u)^{-1} A(u,\, 0) u_x = 0 \\ u(t,\, 0) = \bar{u} \qquad u(0,\, x) = \bar{u}_0 \end{cases}$$

is that (3.12) has to hold for some vector $(s_{n+1}, \dots s_N)$. Through the change of variable $v = v(u)$ described in the statement of Hypothesis 1, this induces an admissibility criterium on the the boundary datum $\bar{v}$ of problem (1.3):

$$\bar{v} = v\Big(T^{n+1}_{s_{n+1}} \circ T^{n+2}_{s_{n+2}} \cdots T^{N}_{s_N} v^{-1}(\bar{v}_0)\Big).$$

for some vector $(s_{n+1}, \dots s_N)$.

The state $\bar{u}_b$ can be connected to $\bar{u}$ only if there exists a boundary layer profile, i.e. a stationary solution of (3.1) such that

$$\begin{cases} A(U, U_x)U_x = B(U)U_{xx} \\ U(0) = \bar{u}_b \\ \lim_{x \to +\infty} U(x) = \bar{u}. \end{cases}$$

or, equivalently,

$$\begin{cases} U_x = p \\ p_x = B(U)^{-1}A(U,\, p)p \\ U(0) = \bar{u}_b \qquad \lim_{x \to +\infty} U(x) = \bar{u}. \end{cases} \tag{3.13}$$

The stable manifold $\mathcal{M}^s$ of the previous system computed around the equilibrium point $(\bar{u},\, 0)$ has dimension equal to the number of negative eigenvalues of $B(\bar{u})^{-1}A(\bar{u},\, 0)$, and hence, by Lemma 3.1, to the number of negative eigenvalues of $A(\bar{u},\, 0)$, i.e. to n. If one denotes by π_1 the projection

$$\begin{aligned} \pi_1 : \mathbb{R}^N \times \mathbb{R}^N &\to \mathbb{R}^N \\ (x_1, \cdots x_N,\, y_1, \cdots y_N) &\mapsto (x_1, \cdots x_N), \end{aligned}$$

then the problem (3.13) admits solution if and only if $\bar{u}_b \in \pi_1\big(\mathcal{M}^s(\bar{u})\big)$. Moreover, let

$$\begin{aligned} \phi(\bar{u}) : \mathbb{R}^n &\to \mathbb{R}^N \times \mathbb{R}^N \\ (s_1, \dots s_n) &\mapsto (U(s_1, \dots s_n),\, p\,(s_1, \dots s_n)) \end{aligned}$$

be the map that parametrizes $\mathcal{M}^s(\bar{u})$, then the columns of the differential $D\phi(\bar{u})\big|_{s_1 = \dots s_n = 0}$ are

$$\begin{pmatrix} \chi_i \\ \\ \lambda_i \chi_i \end{pmatrix}, \quad \chi_i \text{ eigenvalue of } B^{-1}(\bar{u})A(\bar{u},\, 0) \text{ associated to } \lambda_i(\bar{u},\, 0) < 0, \quad i = 1 \dots n$$

because the stable manifold is tangent to the stable space of the system (3.13) in the origin.

The Riemann solver is therefore implicitly defined by the relation

$$\bar{u}_b = \pi_1\Big(\phi(s_1, \dots s_n)\Big) \circ T^{n+1}_{s_{n+1}} \circ \dots T^{N}_{s_n} \bar{u}_0. \tag{3.14}$$

Let $\vec{s}$ be the vector $(s_1,\, s_2, \dots s_N)$: one can check using standard techniques that

$$D_{\vec{s}=\vec{0}}\Bigg(\pi_1 \circ \phi(s_1, \dots s_n) \circ T^{n+1}_{s_{n+1}} \circ \dots T^{N}_{s_n}\Bigg) = \big(\ \chi_1 \big| \dots \big| \chi_n \,\big|\, \zeta_{n+1} \big| \dots \big| \zeta_N\ \big),$$

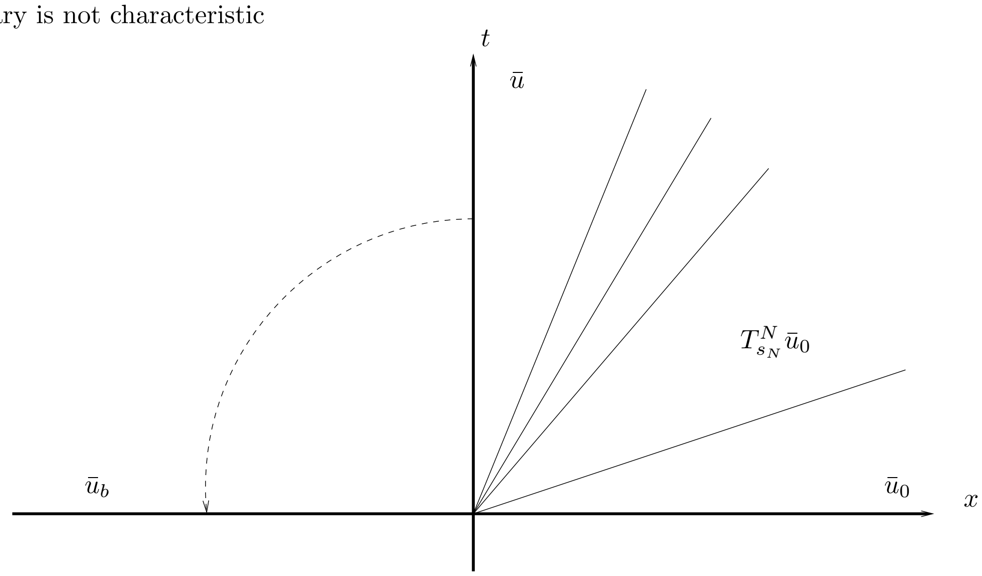
where $\chi_1 \dots \chi_n$ are the eigenvectors of $B^{-1}(\bar{u}_0)A(\bar{u}_0,\, 0)$ associated to negative eigenvalues and $\zeta_{n+1} \dots \zeta_N$ are the eigenvectors of $E^{-1}(\bar{u}_0)A(\bar{u}_0,\, 0)$ associated to positive eigenvalues. Thanks to Lemma 3.2, the columns are independent and hence the map $\pi_1 \circ \phi(s_1, \dots s_n) \circ T^{n+1}_{s_{n+1}} \circ \dots T^{N}_{s_n}$ is invertible in a neighborhood of $\vec{s} = 0$.

The observations performed so far are illustrated by Figure 3.2.2 and are summarized in the following

Proposition 3.1. *If Hypotheses 1, 3, 5, 6, 7 and 8 hold then the Riemann solver induced by the parabolic approximation*

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(t,\, 0) \equiv \bar{u}_b \qquad u^\varepsilon(0,\, x) \equiv \bar{u}_0 \end{cases}$$

Figure 9: the solution of the boundary Riemann problem when the viscosity is invertible and the boundary is not characteristic



in the hyperbolic limit is implicitly defined by the relation

$$\bar{u}_b = \pi_1\Big(\phi(s_1, \dots s_n)\Big) \circ T^{n+1}_{s_{n+1}} \circ \dots T^N_{s_n}\bar{u}_0,$$

where $T^N_{s_N} \dots T^{n+1}_{s_{n+1}}$ *are the curves of admissible states described in Section 3.2.1 and* $\pi_1(\phi)$ *is the first component of the map that parametrizes the stable manifold of equation* (3.13)*.*

Moreover, the state

$$\bar{u} = T^{n+1}_{s_{n+1}} \circ \dots T^N_{s_n}\bar{u}_0 \tag{3.15}$$

is the trace induced on the axis $x = 0$ *in the hyperbolic limit. Hence the admissibility criterium induced by the parabolic approximation* (3.1) *on the boundary datum* $\bar{u}$ *of problem*

$$\begin{cases} E(u)u_t + A(u, 0)u_x = 0 & u \in \mathbb{R}^N \\ u(t, 0) \equiv \bar{u} \qquad u(0, x) \equiv \bar{u}_0 \end{cases}$$

is that (3.15) *has to hold for some vector* $(s_{n+1}, \dots s_N)$*.*

3.2.3 The boundary characteristic case

This section deals with the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation (3.1) in the case the viscosity is invertible, but an eigenvalue of the hyperbolic matrix is allowed to be close to zero. More precisely, it will be assumed that

Hypothesis 9. *The boundary is characteristic for problem* (3.1)*, namely one eigenvalue of* $E^{-1}(u)A(u, u_x)$ *is close to zero*

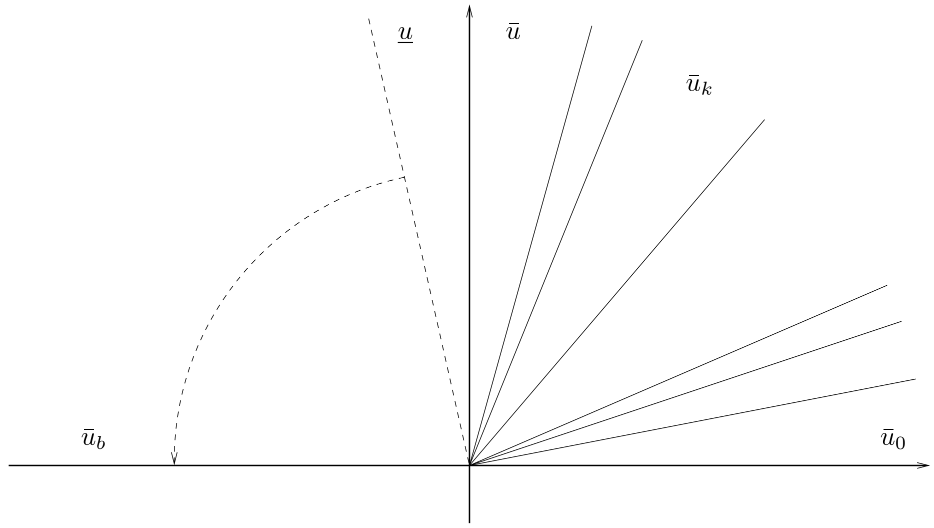
$$\lambda_k(u, u_x) \sim 0.$$

On the other side, because of strict hyperbolicity (Hypothesis 3) the other eigenvalues of $E^{-1}(u)A(u, u_x)$ are well separated from zero, in the sense of (3.11): there exist a constant $c > 0$ such that for all u in a close enough neighborhood of $\bar{u}_0$, it holds

$$\lambda_1(u, 0) < \dots < \lambda_{k-1}(u, 0) \le -c < 0 < c \le \lambda_{k+1}(u, 0) < \dots < \lambda_N(u, 0). \tag{3.16}$$

Hence here and in the following $(k - 1)$ denotes the number of strictly negative eigenvalues of $E^{-1}A(u, 0)$, while there are $N - k$ strictly positive eigenvalues.

Figure 10: the solution of the boundary Riemann problem in the boundary characteristic case when the viscosity is invertible



A key point in the construction described in Section 3.2.2 in the case of a non characteristic boundary is the clear separation between the waves entering in to the domain and the waves flowing out. Roughly speaking, the waves entering in the domain correspond to travelling waves, while the waves flowing out are associated to the boundary layer profiles. On the other side, the difficulties in the study of the boundary characteristic Riemann solver arise exactly from the possibility of an interaction between the two parts of construction described in Section 3.2.2: this interaction is now allowed by the lack of a rigid separation between positive and negative speeds. In particular, in this case it seems difficult to write the boundary data for the limiting hyperbolic system in the form $\hat{\beta}(u) = h(t)$. Indeed, the dimension of the vector $h(t)$ should be equal to the number of waves entering into the domain: when the boundary is characteristic, however, this number cannot be prescribed a priori, but should be allowed to depend on the solution u itself.

The travelling waves whose speed σ_i is strictly positive, however, are certainly not affected by the interaction with the boundary. Hence the initial datum $\bar{u}_0$ is connected to a status $\bar{u}_k$ through the waves of the families $N, \dots k+1$, whose construction is the same as in the boundary free and in the boundary non characteristic case and was described in Section 3.2.1:

$$\bar{u}_k = T^{k+1}_{s_{k+1}} \cdots \circ T^N_{s_N} \bar{u}_0.$$

On the other side, the determination of the k-th curve of admissible states $\tilde{T}^k_{s_k}$ in the boundary characteristic case is the most delicate step in the construction: before describing it, some notations should be introduced. In the following, $T^k_{s_k} \bar{u}_k$ will denote the k-th curve of admissible states introduced in the boundary free case. The corresponding curve of speeds, which will be denoted by $\sigma_k(s_k)$, is by construction decreasing being the derivative of a concave function. There are therefore three possibilities:

1. $\sigma_k(0) < 0$; in this case we set $\bar{u} = \underline{u} = \bar{u}_k$
2. $\sigma_k(0) = 0$, then let $\underline{s}_k := \max\{s : \sigma_k(s) = 0\}$. In this case we set $\bar{u} = \bar{u}_k$ and $\underline{u} = T^k_{\underline{s}_k} \bar{u}_k$
3. $\sigma_k(0) > 0$, then let $\bar{s}_k := \min\{s : \sigma_k(s) = 0\}$ and as before $\underline{s}_k = \max\{s : \sigma_k(s) = 0\}$. In this case we set $\bar{u} = T^k_{\bar{s}_k} \bar{u}_k$ and $\underline{u} = T^k_{\underline{s}_k} \bar{u}_k$.

The first two possibilities can actually be seen as particular cases of the third and hence only the last one will be dealt with explicitly.

The driving idea in the construction of the k-th curve of admissible states $\tilde{T}^k_{s_k}$ is the following. As far as $\sigma_k(s_k) > 0$, the waves of the k-th family are entering the domain, they do not interact with the stationary solutions and the behavior is the same as in the boundary free case. Hence the curve of admissible states $\tilde{T}^k_{s_k}\bar{u}_k$ coincides with $T^k_{s_k}\bar{u}_k$, as far as $\sigma_k(s_k) > 0$, i.e. as far as $s_k < \bar{s}_k$. The state $\bar{u} = \tilde{T}^k_{\bar{s}_k} = T^k_{\bar{s}_k}$ represents the trace of the hyperbolic limit on the axis $x = 0$. In this way, one obtains that the admissibility criterium induced by the parabolic approximation (3.1) on the boundary datum $\bar{u}$ in problem

$$\begin{cases} E(u)u_t + A(u,\, 0)u_x = 0 \\ u(t,\, 0) \equiv \bar{u} \qquad u(0,\, x) \equiv \bar{u}_0 \end{cases} \tag{3.17}$$

is that

$$\bar{u} = T^k_{\bar{s}_k} \circ T^{k+1}_{s_{k+1}} \circ \dots T^N_{s_N}\bar{u}_0$$

for some vector $(\bar{s}_k, s_{k+1}, \dots s_N)$ such that $\sigma_k(s_k) > 0$ for any $s < \bar{s}_k$.

Being $\underline{s}_k$ the maximal value such that $\sigma_k(s_k) = 0$ and $\underline{u} = T^k_{\underline{s}_k}\bar{u}_k$ the corresponding state, one can check that the solution of

$$\begin{cases} u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \\ u^\varepsilon(0,\, x) \equiv \bar{u} \qquad u^\varepsilon(t,\, 0) \equiv \underline{u} \end{cases}$$

converges to the constant $\bar{u}$. Here, however, the speed of convergence is $\exp(-\alpha\sqrt{\varepsilon}x)$ for some suitable constant α, while in Section 3.2.2 it was described the case of non characteristic boundary layer profiles, which behave like $\exp(-\alpha\varepsilon x)$. Nevertheless, this convergence is enough to guarantee that the curve of admissible states $\tilde{T}^k_{s_k}\bar{u}_k$ coincides with $T^k_{s_k}\bar{u}_k$ for every $s_k \leq \underline{s}_k$.

On the other side, once reached the state $\underline{u}$ in the boundary free case the curve $T^k_{s_k}\bar{u}_k$ would describe shock and rarefaction waves with strictly negative speed $\sigma_k < 0$, but such a behavior makes no sense when dealing with initial boundary value problems. Hence the k-th curve of admissible states $\tilde{T}^k_{s_k}\bar{u}_k$ in $\underline{s}_k$ "shifts" from system

$$B(U)U'' = \Big(A(U,\, U') - \sigma E(U)\Big)U',$$

which describes the travelling waves, to system

$$B(U)U'' = A(U,\, U')U', \tag{3.18}$$

satisfied by the stationary solutions.

Such a behavior can be formalized considering in the fixed point problem (3.10) the monotone concave envelope $\mathrm{mon}f_i$ instead of the the concave envelope:

$$\begin{cases} u(\tau) = \bar{u}_k + \displaystyle\int_0^\tau \tilde{r}_i(u(\xi),\, v_i(\xi),\, \sigma_i(\xi))d\xi \\ v_i(\tau) = f_i(\tau,\, u,\, v_i,\, \sigma_i) - \mathrm{mon}f_i(\tau,\, u,\, v_i,\, \sigma_i) \\ \sigma_i(\tau) = \dfrac{d}{d\tau}\mathrm{mon}f_i(\tau,\, u,\, v_i,\, \sigma_i). \end{cases} \tag{3.19}$$

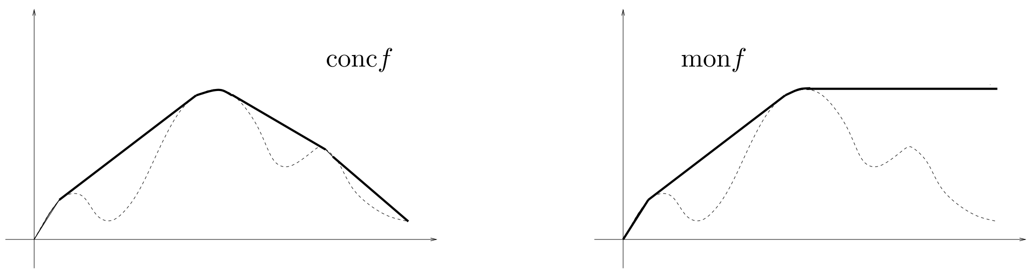
The monotone concave envelope of a function f on a given interval $[0,\, s]$ is defined as

$$\mathrm{mon}f(\tau) = \inf\{g(\tau) \text{ s.t. } g : [0,\, s] \to \mathbb{R},\ g \geq f,\ g \text{ nondecreasing and concave}\}$$

and Figure 11 illustrates the concave and the monotone concave envelope of a given function f. The following lemma, whose proof is an application of the contraction map theorem and can be found in Appendix A.2, guarantees that if, s_k is small enough, then system (3.19) admits a solution defined on the interval $[0,\, s_k]$

Lemma 3.3. *If s_k is small enough, then there exist and are unique three Lipschitz functions $(u(\tau),\, v(\tau),\, \sigma(\tau))$ defined on the interval $[0,\, s_k]$ which satisfy* (3.19).

Figure 11: The concave and the monotone concave envelope of a given function f



This result completes the description of the boundary Riemann solver when $k = 1$, i.e. when the matrix $E^{-1}A$ (and hence also the matrix $B^{-1}A$) does not have strictly negative eigenvalues. Indeed, one can define $\tilde{T}^k_{s_k}\bar{u}_k$ as the value of $u(\tau)$ computed in the instant $\tau = s_k$, where $(u(\tau),\, v(\tau),\, \sigma(\tau))$ satisfies (3.19). Considerations entirely analogous to those performed in [5] in the boundary free case ensure that the curve $\tilde{T}^k_{s_k}$ is Lipschitz continuous and that

$$\frac{\partial \tilde{T}^k_{s_k}\bar{u}_k}{\partial s_k}\bigg|_{s_k=0} = r_k(\bar{u}_k,\, 0),$$

where $r_k(\bar{u}_k,\, 0)$ is the eigenvector of $E^{-1}A(\bar{u}_k,\, 0)$ associated to λ_k. The same considerations as in Section 3.2.2 guarantee that the map

$$\bar{u}_b = \tilde{T}^k_{s_k} \circ T^2_{s_2} \circ \dots T^N_{s_N}\bar{u}_0$$

implicitly defines the boundary Riemann solver and is invertible in a small enough neighborhood of $(s_k,\, s_2, \dots s_N) = \vec{0}$.

In the general case $k > 1$, instead, one has to make further considerations. First of all, it is worth recalling from Section 3.2.2 that, even in the case of a non characteristic boundary, one cannot isolate a *single i-th curve* of admissible states when $i \le n$, i.e. if λ_i is a negative eigenvalue of $E^{-1}A$. Conversely, the boundary data $\bar{u}_b$ which can be connected in the non characteristic case to the hyperbolic trace $\bar{u}$ lay on a smooth manifold that is tangent in zero to the stable subspace. The dimension of the manifold is equal to the number n of negative eigenvalues of $E^{-1}A$ and the map which parametrizes it depends *simultaneously* from n components $s_1 \dots s_n$.

In the boundary characteristic case one observes a more complex situation, but still finds a smooth manifold which contains the orbits of the boundary layer profiles exponentially converging and whose dimension is equal to the number of strictly negative eigenvalues of $E^{-1}A$. Nevertheless, one has to take into account a perturbation due to the orbits that converge with slower speed: as it will be clearer in the following, this perturbation leads to the definition of a curve of admissible states $\tilde{T}^k_{\tilde{s}_k}$ which in $\tilde{s}_k = \underline{s}_k$ has a $\mathcal{C}^1$ connection with $T^k_{s_k}\bar{u}_k$.

In order to make the previous considerations more precise one has to study the set of admissible initial conditions $\bar{u}_b$ for the problem

$$\begin{cases} u_x = p \\ p_x = B^{-1}(u)A(u,\, p)p \\ u(0) = \bar{u}_b \quad \lim_{x \to +\infty} u(x) = \underline{u}. \end{cases} \tag{3.20}$$

It is clearly not restrictive to study system (3.20) reduced on the center stable manifold around $(\underline{u},\, 0)$: dimension of this manifold is equal to N plus the number of non positive eigenvalues of $B^{-1}(\underline{u})A(\underline{u},\, 0)$ and hence to $N + k$ by Lemma 3.1. It is worth underling explicitly that we are studying the set of orbits converging to $\underline{u}$ but we are not prescribing an exponential speed of convergence: hence such a set includes but does not coincide in general with the stable manifold.

As it is shown in the appendix, Section A.3, system (3.20) reduced on the center-stable manifold takes the form

$$\begin{cases} u_x = R^{cs}(u,\, V^{cs})V^{cs} \\ V^{cs}_x = \Lambda^{cs}(u,\, V^{cs})V^{cs}, \end{cases} \tag{3.21}$$

where $R^{cs} \in \mathbb{M}^{N\times k}$, $V^{cs} \in \mathbb{R}^k$ and $\Lambda^{cs} \in \mathbb{M}^{k\times k}(\underline{u},\, 0)$ is negatively semidefinite when $u = \underline{u}$ and $V^{cs} = 0$.

The center stable manifold contains all the orbits that when $x \to +\infty$ do not blow up with exponential speed. Roughly speaking, one would like to isolate the center component, which refers to the dynamic which do not blow up exponentially neither at $-\infty$ nor at $+\infty$, from the exponential decreasing component, which takes into account all the orbits that converge with exponential speed to $(\underline{u},\, 0)$ or to some close equilibrium. Actually, it will not be achieved a complete separation, since there will be an interaction between the two components.

Nevertheless, it is possible to formalize the previous heuristic considerations considering a decomposition of the orbits laying on the center stable manifold on the center and on the uniformly stable manifold. Making reference to Appendix A.4 for all the technical details, it turns out that system (3.21) is equivalent to

$$\begin{cases} u_{kx} = \check{r}_k(u_k + u_s,\, V^s,\, v_k)v_k \\ v_{kx} = \check{\lambda}_k(u_k + u_s,\, V^s,\, v_k)v_k \end{cases} \quad \text{and} \quad \begin{cases} u_{sx} = \check{R}^s(u_k + u_s,\, V^s,\, v_k)V^s \\ V_{sx} = \check{\Lambda}^s(u_s + u_k,\, V^s,\, v_k)V^s, \end{cases} \tag{3.22}$$

with $V^s \in \mathbb{R}^{k-1}$ and $v_k \in \mathbb{R}$. The exact definition of the generalized eigenvectors $\check{R}^s(u_k+u_s,\, V^s,\, v_k)$ belonging to $\mathbb{R}^{N\times(k-1)}$ and $\check{r}_k(u_k + u_s,\, V^s,\, v_k) \in \mathbb{R}^N$ and of the generalized eigenvalues $\check{\Lambda}^s(u_k + u_s,\, V^s,\, v_k) \in \mathbb{M}^{(k-1)\times(k-1)}$ and $\check{\lambda}_k(u_k + u_s,\, V^s,\, v_k) \in \mathbb{R}$, as well as the computations that lead to (3.22), can be found in the Appendix A.4, but here it is important to underline that $\check{\lambda}_k$ is close to 0, while $\check{\Lambda}^s$ is negative definite.

Moreover, a key feature is the presence of a coupling between the two systems, in the sense that the coefficients of the first depends also on the solution of the second, and viceversa: this can be expressed saying that in the boundary characteristic case when characterizing the k-th curve of admissible states $\tilde{T}^k_{s_k}\bar{u}_k$ one has to take into account a perturbation due to the stable component. On the other side, the equivalence between (3.21) and systems (3.22) can be seen roughly speaking as a diagonalization, although incomplete because of the coupling. This diagonalization is possible because (3.16) guarantees that there is a strict separation between the negative eigenvalues of Λ^{cs} and the eigenvalue close to zero.

Relying on the formulation (3.22), Lemma 3.4 furnishes a precise characterization of the set of admissible boundary conditions in problem (3.20). However, one has to specify preliminarily the limit conditions induced on the component u_k, u_s:

$$\begin{cases} \lim_{x\to\infty} u_k(x) = \underline{u} \\ \lim_{x\to\infty} v_k(x) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \lim_{x\to\infty} u_s(x) = 0 \\ \lim_{x\to\infty} V^s(x) = 0 \end{cases} \tag{3.23}$$

Lemma 3.4. *There exits $\nu > 0$ and a Lipschitz continuous function*

$$\begin{aligned} \phi : B_\nu(0) \subseteq \mathbb{R}^{k-1} \times [\underline{s}_k,\, \underline{s}_k + \nu[\ &\to \mathbb{R}^N \times \mathbb{R}^N \\ (s_1, \dots s_{k-1},\, s_k) \ &\mapsto \Big(\phi_1(s_1, \dots s_{k-1},\, s_k),\, \phi_2(s_1, \dots s_{k-1},\, s_k)\Big) \end{aligned}$$

such that the following properties holds:

1. *systems* (3.22) *admit a solution satisfying the limit conditions* (3.23) *if and only if*
$$u_k(0) \in \phi_1(B_\delta(0) \times [\underline{s}_k,\, \underline{s}_k + \nu[) \qquad u_s(0) \in \phi_2(B_\delta(0) \times [\underline{s}_k,\, \underline{s}_k + \nu[);$$
2. *denoting with $\vec{\chi}_1 \dots \vec{\chi}_{k-1}, \vec{\chi}_k$ the eigenvectors of $B^{-1}(\underline{u})A(\underline{u},\, 0)$ associate to non positive eigenvalues, it holds*

$$\begin{aligned} &D_{s_1\dots s_{k-1}}\phi_1\Big|_{s_1=\dots s_{k-1}=s_k=0} = 0 \qquad \frac{\partial}{\partial s_k}\phi_1\Big|_{s_1=\dots s_{k-1}=s_k=0} = \vec{\chi}_k \\ &D_{s_1\dots s_{k-1}}\phi_2\Big|_{s_1=\dots s_{k-1}=s_k=0} = (\vec{\chi}_1|\dots|\vec{\chi}_{k-1}) \qquad \frac{\partial}{\partial s_k}\phi_2\Big|_{s_1=\dots s_{k-1}=s_k=0} = 0 \end{aligned} \tag{3.24}$$

Proof. The proof is organized into three steps:

1. the case $u_s \equiv 0$ $V_s \equiv 0$ was already considered in Lemma 3.3, which guarantees that the curve of admissible states $\tilde{T}^k_{s_k}$ is well defined also for $s_k > \underline{s}_k$: we will denote by $u^0_k(\tau)$, $v^0_k(\tau)$ the corresponding solution of the fixed point problem (3.19).

   More precisely, the situation is the following: since when $|s_k - \underline{s}_k| < \nu$

$$\int_{\underline{s}_k}^{s} \hat{\lambda}_k \Big(u^0_k(\varsigma) + v^0_k(\varsigma) \Big) d\varsigma < 0$$

   and hence $v^0_k(s) < 0$, the solution of the ODE

$$\begin{cases} \dfrac{d\alpha}{d\tau} = \dfrac{1}{v^0_k(\tau)} \\ \tau(0) = s_k \end{cases} \tag{3.25}$$

   defines an invertible change of variable from the interval $]\underline{s}_k,\, s_k]$ to the semi-line $[0,\, +\infty[$. Let $\beta(x)$ be its inverse: by direct computations one can verify that the function $u^0_k(x) = u\big(\beta(x)\big)$ is a solution of (3.22) (the second system is trivially satisfied) and (3.23). In other words, as for the curve of admissible states in the boundary free case the change of variable β transforms systems (3.22) with condition (3.23) into the fixed point problem

$$\begin{cases} u^0_k(\tau) = \underline{u} + \displaystyle\int_{\underline{s}}^{\tau} \tilde{r}_i(u^0_k(\xi),\, v^0_k(\xi),\, 0) d\xi \\ v^0_k(\tau) = \displaystyle\int_{\underline{s}_k}^{\tau} \hat{\lambda}_k \Big(u^0_k(\varsigma),\, v^0_k(\varsigma) \Big) d\varsigma. \end{cases}$$

2. In the general case u_s and V^s are not identically zero, but the construction of the solution to (3.22) and (3.23) is based on the same step as before. More precisely, we will consider the same change of variable as before and we will show that the presence of a small perturbation due to the non zero component $(u_s,\, V^s)$ on the uniformly stable manifold does not compromise the construction.

   Let $(u_k,\, v_k)$ and $(u_s,\, V^s)$ solve (3.22) and (3.23). If

$$v_k(\tau) < 0 \qquad \forall\, x \tag{3.26}$$

   then the ODE (3.25) defines an invertible change of variables from the interval $]\underline{s}_k,\, s_k]$ into the semi-line $[0,\, +\infty[$. As before, we will denote by β the inverse of α, which transforms (3.22) and (3.23) into the fixed point problem

$$\begin{cases} u_k(\tau) = T^1(u_k,\, v_k,\, V^s,\, u_s)(\tau) = \underline{u} + \displaystyle\int_{\underline{s}_k}^{\tau} \check{r}_k \Big(u_k(\varsigma) + u_s\big(\alpha(\varsigma)\big),\, V^s\big(\alpha(\varsigma)\big),\, v_k(\varsigma) \Big) d\varsigma \\ v_k(\tau) = T^2(u_k,\, v_k,\, V^s,\, u_s)(\tau) = \displaystyle\int_{\underline{s}_k}^{\tau} \check{\lambda}_k \Big(u_k(\varsigma) + u_s\big(\alpha(\varsigma)\big),\, V^s\big(\alpha(\varsigma)\big),\, v_k(\varsigma) \Big) d\varsigma \\ u_s(x) = T^3(u_k,\, v_k,\, V^s,\, u_s)(\tau) = \displaystyle\int_{x}^{\infty} \check{R}_s \Big(u_k\big(\beta(\xi)\big) + u_s(\xi),\, V^s(\xi),\, v_k\big(\beta(\xi)\big) \Big) V^s(\xi) d\xi \\ V^s(x) = T^4(u_k,\, v_k,\, V^s,\, u_s)(\tau) = V_s(0) \exp \Big(\displaystyle\int_{0}^{x} \check{\Lambda}^s \Big(u_k\big(\beta(\xi)\big) + u_s(\xi),\, V^s(\xi),\, v_k\big(\beta(\xi)\big) \Big) \Big). \end{cases} \tag{3.27}$$

   Hence, to conclude the proof of the existence of a solution of (3.22) that satisfy (3.23) it is sufficient to check that for every fixed $V_s(0) \in B_\delta(0) \subseteq \mathbb{R}^{k-1}$ and for every s_k sufficiently close to $\underline{s}_k$ T is a contraction in the subspace of

$$\mathcal{C}^0(]\underline{s}_k,\, s_k]) \times \mathcal{C}^0(]\underline{s}_k,\, s_k]) \times \mathcal{C}^0([0,\, +\infty[) \times \mathcal{C}^0([0,\, +\infty[)$$

defined by the relations

$$|u_k - u_k^0|_\infty \leq \frac{|v_k^0|_\infty}{4} \quad |v_k - v_k^0|_\infty \leq \frac{|v_k^0|_\infty}{4} \quad |u^s(x)| \leq \delta e^{-cx} \quad |V^s(x)| \leq \delta e^{-cx}.$$

In the previous expression $c > 0$ is the *separation speed* satisfying (3.16). Moreover, it is worth underling explicitly that with this definition condition (3.26) is automatically satisfied.

The proof that the map T defined by (3.27) maps the space into itself and that it is a contraction follows then by straightforward computations: the key point is that the matrix $\breve{\Lambda}^s$ is negatively definite and hence one obtains for $|V^s(x)|$ an exponential speed of decay.

The map ϕ is then defined as

$$\phi_1(V^s(0),\, s_k) = u_k(\tau = s_k) \qquad \phi_2(V^s(0),\, s_k) = u_s(x = 0).$$

3. (Differential properties)

   The tangency conditions (3.24) are clear if takes into account that when $u_k \equiv \underline{u}$, $v_k \equiv 0$ then (3.27) characterizes the stable manifold of (3.20) around the equilibrium point $(\underline{u},\, 0)$, while the case $u_s \equiv 0$, $V^s \equiv 0$ was already dealt with. On the other side, the Lipschitz continuity of the map ϕ with respect to $V^s(0)$ follows from the Lipschitz continuity of T with respect to the parameter $V^s(0)$, while the regularity with respect to s_k can be obtained via considerations analogous (but easier) to those performed in the boundary free case in [5].

□

The notation we are going to introduce makes reference to Lemma 3.4 and is essential to formulate the precise characterization of the boundary Riemann solver. First of all, it will be used the notation $\phi(s_1,\, \dots, s_{k-1},\, s_k,\, \underline{u})$ in order to underline the dependence of ϕ from the value $\underline{u}$, which is the limit assigned in (3.23). Moreover, let

$$\tilde{T}^k_{s_k}(s_1 \dots s_{k-1},\, \bar{u}_k) := \begin{cases} T^k_{s_k}(\bar{u}_k) & s_k \leq \underline{s}_k \\ \phi_1(s_1, \dots s_{k-1},\, s_k,\, \underline{u}) & s_k > \underline{s}_k \end{cases} \tag{3.28}$$

and

$$\tilde{\phi}_2(s_1 \dots s_{k-1},\, s_k,\, \bar{u}_k) := \begin{cases} \phi_2\big(s_1 \dots s_{k-1},\, \underline{s}_k,\, T^k_{\underline{s}_k}(\bar{u}_k)\big) + T^k_{\underline{s}_k}(\bar{u}_k) & s_k \leq \underline{s}_k \\ \phi_2(s_1, \dots s_{k-1},\, s_k,\, \underline{u}) + \tilde{T}^k_{s_k}(\bar{u}_k) & s_k > \underline{s}_k \end{cases} \tag{3.29}$$

The underling ideas are the followings: as far as $s_k \leq \underline{s}_k$ the center and the uniformly stable component are independent and the situation is analogous to the non characteristic case. Indeed, the k-th curve of admissible state has the same construction described in Section 3.2.1 and does not interact with the manifold of boundary layer profiles. On the other side, to extend the k-th curve of admissible states $\tilde{T}^k_{s_k}$ to $s_k > \underline{s}_k$ one has make reference to (3.22) and hence to face an interaction between the uniformly stable and the center component.

Putting all the previous considerations together one obtains that the boundary datum $\bar{u}_b$ satisfies

$$\bar{u}_b = \psi(s_1, \dots s_N) := \tilde{\phi}_2(s_1 \dots s_{k-1},\, s_k,\, \cdot) \circ \tilde{T}^k_{s_k}(s_1 \dots s_{k-1},\, \cdot) \circ T^{k+1}_{s_{k+1}} \circ \dots T^N_{s_N} \bar{u}_0.$$

Since

$$D\psi \Big|_{s_1 = \dots = s_N = 0} = \Big(\chi_1 \,|\, \dots \,|\, \chi_k \,|\, \zeta_{k+1} \,|\, \dots \,|\, \zeta_N \Big),$$

where $\chi_1 \dots \chi_k$ are the eigenvectors of $B^{-1}(\bar{u}_0)A(\bar{u}_0,\, 0)$ associated to non positive eigenvalues and $\zeta_{n+1} \dots \zeta_N$ are the eigenvectors of $E^{-1}(\bar{u}_0)A(\bar{u}_0,\, 0)$ associated to the eigenvalues $\lambda_i > 0$. Thanks to Lemmata 3.1 and 3.2, the map ψ is invertible in a small enough neighborhood of $(s_1, \dots s_N) = \vec{0}$.

The considerations performed in this section are illustrated by Figure 3.2.3 and are summarized in the following

Proposition 3.2. *If Hypotheses 1, 3, 5, 6, 7 and 9 hold then the Riemann solver induced by the parabolic approximation*

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} & u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(t,\, 0) \equiv \bar{u}_b \qquad u^\varepsilon(0,\, x) \equiv \bar{u}_0 \end{cases}$$

in the hyperbolic limit is implicitly defined by the relation

$$\bar{u}_b = \tilde{\phi}_2(s_1 \dots s_{k-1},\, s_k,\, \cdot) \circ \tilde{T}^k_{s_k}(s_1 \dots s_{k-1},\, \cdot) \circ T^{k-1}_{s_{k-1}} \circ \dots T^N_{s_N}\bar{u}_0$$

where $T^N_{s_N} \dots T^{k+1}_{s_{k+1}}$ are the curves of admissible states described in Section 3.2.1 and $\tilde{T}^k_{s_k}$ and $\tilde{\phi}_2$ are defined by (3.28) *and* (3.29) *respectively.*

Moreover, the admissibility criterium induced by the parabolic approximation (3.1) *on the boundary datum $\bar{u}$ of problem*

$$\begin{cases} E(u)u_t + A(u,\, 0)u_x = 0 & u \in \mathbb{R}^N \\ u(t,\, 0) \equiv \bar{u} \qquad u(0,\, x) \equiv \bar{u}_0 \end{cases}$$

is that

$$\bar{u} = T^k_{\bar{s}_k} \circ T^{k+1}_{s_{k+1}} \circ \dots T^N_{s_N}\bar{u}_0$$

for some vector $(\bar{s}_k, s_{k+1}, \dots s_N)$ such that $\sigma_k(s_k) > 0$ for any $s < \bar{s}_k$. In the previous expression, we denoted by $\sigma_k(s_k)$ the curve of speeds associated to the curve of admissible states $T^k_{s_k}$ of the boundary free Riemann problem.

4 The case of incomplete viscosity

This section deals with the characterization of the Riemann solver induced in the hyperbolic limit by the parabolic approximation (1.1) in the case the viscosity matrix $\tilde{B}$ is not invertible. The precise hypotheses assumed are discussed in Sections 2.3 and 2.2: in particular, they guarantee that it is sufficient to study the family of systems

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} & u^\varepsilon \in \mathbb{R}^N \\ ß(u^\varepsilon(t,\, 0)) \equiv \bar{g} \qquad u^\varepsilon(0,\, x) \equiv \bar{u}_0, \end{cases} \tag{4.1}$$

with E, A and B satisfying Hypothesis 2, ß given by Definition 2.1.

As it was underlined in the introduction, the case of a singular viscosity is particularly interesting since one in general cannot in prescribe a full boundary condition even in the *parabolic* IBVP. Moreover, several physical examples belong to this case. As in the case of an invertible viscosity it will be achieved the exact characterization of the trace on the axis $x = 0$ of the solution u limit of the parabolic approximation (4.1) and hence it will be obtained an admissibility criterium for the boundary datum $\bar{u}$ in the hyperbolic problem

$$\begin{cases} E(u)u_t + A(u,\, 0)u_x = 0 & u \in \mathbb{R}^N \\ u(t,\, 0) \equiv \bar{u} \qquad u(0,\, x) \equiv \bar{u}_0 \end{cases}$$

The crucial point will be the study of the systems

$$B(U)U'' = \Big(A(U,\, U') - \sigma E(U)\Big)U' \tag{4.2}$$

and

$$B(U)U'' = A(U,\, U')U', \tag{4.3}$$

satisfied respectively by the travelling wave and by the stationary solutions. However, the study is in this case complicated by the singularity of the matrix B: in particular, some additional work is required to reduce the previous systems to the explicit form

$$U'' = F(U,\, U') \tag{4.4}$$

for a suitable function F.

Moreover, a key point in the analysis is the determination of the dimension of the stable manifold of (4.3): to this purpose, it is followed the same approach as in [41] and it is introduced a suitable perturbation technique. The most important results obtained are summarized in Proposition 4.1 (case of a non characteristic boundary) and Proposition 4.2 (boundary characteristic case). Several results stated therein were actually already known, at least in particular cases: it is made reference mainly to [36, 37] and [41]. Here, however, it is given a complete analysis for all the systems satisfying the conditions listed in Hypothesis 2: as it is underlined in Section 2.2, these conditions are well justified, in the sense specified therein. In particular, here it is given an answer to a question which was left open in [37] and which concerns the dimension of the stable manifold of (4.3).

The exposition is organized as follows: Section 4.1 introduces some preliminary results, while Section 4.2 concerns the characterization of the boundary Riemann solver. More precisely, in Section 4.2.1 it is recalled the construction in the boundary free case and it is made reference to [5] for the complete analysis. Section 4.2.2 introduces a key lemma on the dimension of the stable manifold of system (4.3). This result is then applied in Sections 4.2.3 and 4.2.4 to achieve the characterization of the boundary Riemann solver in the case of a non characteristic and of a characteristic boundary respectively.

4.1 Preliminary results

This section introduces the preliminary results needed in Section 4.2. The exposition is organized as follows: Section 4.1.1 gives some results about the structure of the matrix $A(u,\, u_x)$, all implied by Hypothesis 2 and in particular by Kawashima condition. In Section 4.1.2 it is shown how systems (4.2) and (4.3) can be reduced to an explicit form like (4.4).

Some notations have to be introduced: given a real parameter σ, let

$$A(u,\, u_x,\, \sigma) := A(u,\, u_x) - \sigma E(u). \tag{4.5}$$

Consequently, both systems (4.2) and (4.3) may be written in the form

$$A(u,\, u_x,\, \sigma) u_x = B(u) u_{xx}. \tag{4.6}$$

Finally, let

$$\begin{pmatrix} A_{11}(u,\, \sigma) & A_{12}(u,\, \sigma) \\ A_{21}(u,\, \sigma) & A_{22}(u,\, u_x,\, \sigma) \end{pmatrix} = \begin{pmatrix} A_{11}(u) - \sigma E_{11}(u) & A_{12}(u) - \sigma E_{12}(u) \\ A_{21}(u) - \sigma E_{21}(u) & A_{22}(u,\, u_x) - \sigma E_{22}(u) \end{pmatrix} \tag{4.7}$$

and

$$\Xi = (w,\, z)^T, \quad w \in \mathbb{R}^{N-r},\; z \in \mathbb{R}^r \tag{4.8}$$

be the block decompositions of $A(u,\, u_x,\, \sigma)$ and of $\Xi \in \mathbb{R}^N$ corresponding to (2.9).

4.1.1 Some results about the structure of $A(u,\, u_x,\, \sigma)$

Even if it is not explicitly specified in the statement, all the results of this section presuppose that Hypothesis 2 holds.

The first observation is immediate, but useful:

Lemma 4.1. *Let $A_{ij}(u,\, u_x,\, \sigma)$ be the blocks of $A(u,\, u_x,\, \sigma)$ defined by (4.7). Then*

$$A_{12}(u,\, \sigma) = A_{21}^T(u,\, \sigma) \qquad A_{11}^T(u,\, \sigma) = A_{11}(u,\, \sigma)$$

Some further notations are required: let $P_0(u,\, \sigma)$ denote the projection of $\mathbb{R}^{N-r}$ on $\ker A_{11}(u,\, \sigma)$. Thanks to the third condition in Hypothesis 2, the dimension of this subspace is constant in u: we will denote it by q. The operator P_0 can be identified with a matrix $P_0(u,\, \sigma) \in \mathbb{M}^{q\times(N-r)}$. Similarly, the projection on the subspace orthogonal to $\ker A_{11}(u,\, \sigma)$ is identified with a matrix $P_\perp(u,\, \sigma) \in \mathbb{M}^{(N-r-q)\times(N-r)}$. The decomposition (4.8) is hence refined setting

$$\bar{w} = P_0 w \qquad \tilde{w} = P_\perp w \qquad \bar{w} \in \mathbb{R}^q,\;\; \tilde{w} \in \mathbb{R}^{N-r-q}. \tag{4.9}$$

Using the previous notations and recalling Lemma 4.1 the product $A(u,\,u_x,\,\sigma)\vec{\Xi}$ writes

$$\begin{pmatrix} A_{11}(u,\,\sigma) & A_{21}(u,\,\sigma)^T \\ A_{21}(u,\,\sigma) & A_{22}(u,\,u_x,\,\sigma) \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \left(A^I_{21}\right)^T(u,\,\sigma) \\ 0 & \tilde{A}_{11}(u,\,\sigma) & \left(A^{II}_{21}\right)^T(u,\,\sigma) \\ A^I_{21}(u) & A^{II}_{21}(u,\,\sigma) & A_{22}(u,\,\sigma,\,u_x) \end{pmatrix}\begin{pmatrix} \bar{w} \\ \tilde{w} \\ z \end{pmatrix}, \tag{4.10}$$

where

$$A^I_{21}(u,\,\sigma) = A_{21}(u,\,\sigma)P^T_0(u,\,\sigma) \in \mathbb{M}^{r\times q} \qquad A^{II}_{21}(u,\,\sigma) = A_{21}(u,\,\sigma)P^T_\perp(u,\,\sigma) \in \mathbb{M}^{r\times(N-r-q)}$$

and

$$\tilde{A}_{11}(u,\,\sigma) = P_\perp(u,\,\sigma)A_{11}(u,\,\sigma)P^T_\perp(u,\,\sigma) \in \mathbb{M}^{(N-r-q)\times(N-r-q)}$$

It is now possible to state the following result, a consequence of Kawashima condition:

Lemma 4.2. *The matrix $A^I_{21}(u,\,\sigma) \in \mathbb{M}^{r\times q}$ has rank q. Thus, in particular, $q \le r$.*

Proof. First of all, one observes that

$$E^{-1}A(u,\,0)\Xi = \sigma\Xi \iff A(u,\,0,\,\sigma)\Xi = 0$$

and hence Kawashima condition writes

$$\ker\big(A(u,\,0,\,\sigma)\big) \cap \ker\big(B(u)\big) = \big\{0\big\} \qquad \forall\, u,\,\sigma. \tag{4.11}$$

We claim that this implies

$$A^I_{21}\bar{w} = 0 \Rightarrow \bar{w} = 0. \tag{4.12}$$

Indeed, suppose by contradiction that there exists $\bar{w} \neq 0$ such that $A^I_{21}\bar{w} = 0$. Then

$$\begin{pmatrix} 0 & 0 & (A^I_{21})^T \\ 0 & \tilde{A}_{11} & (A^I I_{21})^T \\ A^I_{21} & A^I I_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \bar{w} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}\begin{pmatrix} \bar{w} \\ 0 \\ 0 \end{pmatrix},$$

which contradicts (4.11). Hence (4.12) holds.

From (4.12) one then deduces that A^I_{21} has q independent columns and hence it posses a submatrix of rank q. $\square$

In the following we will isolate an invertible submatrix of $A^I_{21}(u,\,\sigma)$ and we will denoted it by $a_{11}(u,\,\sigma)$. More precisely, a_{11} is obtained from $A^I_{21}(u,\,\sigma)$ selecting q independent rows. By continuity, one can suppose that the rows selected are the same for every u.

With considerations analogous to the ones that led to (4.10), one can write

$$z = \bar{P}^T(\sigma)\bar{z} + \tilde{P}^T(\sigma)\tilde{z},$$

where $\bar{P} \in \mathbb{M}^{q\times r}$ is the projection such that $a_{11} = A^I_{21}\bar{P}^T$ and $\tilde{P} \in \mathbb{M}^{(r-q)\times r}$ is the projection on the subspace orthogonal to the image of $\mathbb{R}^r$ through $\bar{P}$.

Thanks to the previous considerations, one obtains that $A(u,\,u_x,\,\sigma)\Xi$ writes

$$\begin{pmatrix} 0 & 0 & a^T_{11}(u,\,\sigma) & a^T_{21}(u,\,\sigma) \\ 0 & \tilde{A}_{11}(u,\,\sigma) & a^T_{12}(u,\,\sigma) & a^T_{22}(u,\,\sigma) \\ a_{11}(u,\,\sigma) & a_{12}(u,\,\sigma) & \alpha_{11}(u,\,u_x,\,\sigma) & \alpha^T_{21}(u,\,u_x,\,\sigma) \\ a_{21}(u) & a_{22}(u,\,\sigma) & \alpha_{21}(u,\,u_x,\,\sigma) & \alpha_{22}(u,\,u_x,\,\sigma) \end{pmatrix}\begin{pmatrix} \bar{w} \\ \tilde{w} \\ \bar{z} \\ \tilde{z}, \end{pmatrix} \tag{4.13}$$

with

$$a_{12} \in \mathbb{M}^{q\times(N-r-q)}, \quad a_{22} \in \mathbb{M}^{(r-q)\times(N-r-q)}$$

$$\alpha_{11} \in \mathbb{M}^{q\times q}, \quad \alpha_{21} \in \mathbb{M}^{(r-q)\times q}, \quad \alpha_{22} \in \mathbb{M}^{(r-q)\times(r-q)}$$

$$\bar{z} \in \mathbb{R}^q \quad \tilde{z} \in \mathbb{R}^{r-q}$$

The corresponding decomposition of $B(u)$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_{11}(u) & b_{12}(u) \\ 0 & 0 & b_{21}(u) & b_{22}(u) \end{pmatrix} \tag{4.14}$$

with

$$b_{11} \in \mathbb{M}^{q\times q}, \;\; b_{12} \in \mathbb{M}^{q\times (r-q)}, \;\; b_{21} \in \mathbb{M}^{(r-q)\times q}, \;\; b_{22} \in \mathbb{M}^{(r-q)\times(r-q)}.$$

It is worth underling explicitly that formulations (4.13) and (4.14) include in particular the limit cases $q = 0$ and $q = N - r$: indeed, it is enough to assume $w = \tilde{w}$, $z = \tilde{z}$ and $w = \bar{w}$, $z = \bar{z}$ respectively.

Moreover, the observation that it is not restrictive to assume that $\bar{P}$ and $\tilde{P}$ do not depend on u is essential in order to avoid the appearance of terms like $D\bar{P}u_x$ or $D\tilde{P}u_x$.

Lemma 4.3. *The matrix*

$$a_{21}a_{11}^{-1}\Big(b_{11}(a_{11}^T)^{-1}a_{21}^T - b_{12}\Big) - b_{21}(a_{11}^T)^{-1}a_{21}^T + b_{22}$$

is invertible.

Proof. By contradiction, assume that there exists an $(r-q)$-dimensional vector $\vec{\xi} \neq 0$ such that

$$\Big(a_{21}a_{11}^{-1}\Big(b_{11}(a_{11}^T)^{-1}a_{21}^T - b_{12}\Big) - b_{21}(a_{11}^T)^{-1}a_{21}^T + b_{22}\Big)\vec{\xi} = 0.$$

It follows that

$$\begin{pmatrix} a_{21}a_{11}^{-1} & 0 \\ 0 & I_{r-q} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \big(a_{11}^T\big)^{-1}a_{21}^T & 0 \\ 0 & I_{r-q} \end{pmatrix} \begin{pmatrix} \vec{\xi} \\ -\vec{\xi} \end{pmatrix} = \begin{pmatrix} \vec{v} \\ \vec{v} \end{pmatrix} \tag{4.15}$$

for some suitable $\vec{v} \in \mathbb{R}^{r-q}$. Let

$$\Xi = \begin{pmatrix} \vec{\xi} \\ -\vec{\xi} \end{pmatrix} \qquad \Upsilon = \begin{pmatrix} \vec{v} \\ \vec{v} \end{pmatrix} \qquad D = \begin{pmatrix} (a_{11}^{-1})^T a_{21}^T & 0 \\ 0 & I_{r-q} \end{pmatrix},$$

then (4.15) reads

$$0 = \langle \Upsilon, \, \Xi \rangle = \langle D^T b D \Xi, \, \Xi \rangle = \langle bD\Xi, \, D\Xi \rangle.$$

Since $D\,\Xi \neq 0$, this is a contradiction because b is positively definite. □

4.1.2 The explicit form of system (4.6)

The following result guarantees that system (4.6), and hence systems (4.2) and (4.3), can be reduced to an explicit form like (4.16)

Lemma 4.4. *For every $u_0 \in B_{\delta_0}(\bar{u}_0)$ and $u_x \in B_{\delta_0}(0)$ system is equivalent to*

$$\begin{cases} u_x = \Big(\bar{w}(u, \, \tilde{z}, \, \sigma), \; \tilde{w}(u, \, \tilde{z}, \, \sigma), \; \bar{z}(u, \, \tilde{z}, \, \sigma), \; \tilde{z}\Big)^T \\ \tilde{z}_x = f(u, \, \tilde{z}, \, \sigma) \end{cases} \tag{4.16}$$

for suitable functions

$$\bar{w} : \mathbb{R}^N \times \mathbb{R}^{r-q} \times \mathbb{R} \to \mathbb{R}^q \qquad \tilde{w} : \mathbb{R}^N \times \mathbb{R}^{r-q} \times \mathbb{R} \to \mathbb{R}^{N-r-q} \qquad \bar{z} : \mathbb{R}^N \times \mathbb{R}^{r-q} \times \mathbb{R} \to \mathbb{R}^q$$
$$f : \mathbb{R}^N \times \mathbb{R}^{r-q} \times \mathbb{R} \to \mathbb{R}^{r-q}.$$

These functions satisfy

$$D_u\bar{z}\big|_{u=\bar{u}_0, \, \tilde{z}=0, \, \sigma} = 0 \qquad D_\sigma\bar{z}\big|_{u=\bar{u}_0, \, \tilde{z}=0, \, \sigma} = 0 \qquad D_{\tilde{z}}\bar{z}\big|_{u=\bar{u}_0, \, \tilde{z}=0} = -(a_{11}^T)^{-1}a_{21}^T \tag{4.17}$$

and

$$
\begin{aligned}
&D_u\tilde{w}\big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\sigma}=0 \qquad D_\sigma\tilde{w}\big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\sigma}=0\\
&D_{\tilde{z}}\tilde{w}\big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\sigma}=-\tilde{A}_{11}^{-1}\big(a_{12}^TD_{\tilde{z}}\bar{z}+a_{22}^T\big)=-\tilde{A}_{11}^{-1}\big(-a_{12}^T(a_{11}^T)^{-1}a_{21}^T+a_{22}^T\big).
\end{aligned}
\tag{4.18}
$$

Moreover,

$$
\begin{aligned}
&D_u\bar{w}\big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\sigma}=0 \qquad D_\sigma\bar{w}\big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\sigma}=0\\
D_{\tilde{z}}\bar{w}\big|_{u=\bar{u}_0,\,\tilde{z}=0}&=-a_{11}^{-1}\Big(a_{12}D_{\tilde{z}}\tilde{w}+\alpha_{11}D_{\tilde{z}}\bar{z}+\alpha_{12}^T\Big)+a_{11}^{-1}b_{11}D_{\tilde{z}}\bar{z}D_{\tilde{z}}f+a_{11}^{-1}b_{12}D_{\tilde{z}}f\\
&=a_{11}^{-1}\Big(a_{12}\tilde{A}_{11}^{-1}\big(a_{22}^T-a_{12}^T(a_{11}^T)^{-1}a_{21}^T\big)+\alpha_{11}(a_{11}^T)^{-1}a_{21}^T-\alpha_{12}\Big)+a_{11}^{-1}\Big(b_{12}\\
&\quad-b_{11}(a_{11}^T)^{-1}a_{21}^T\Big)\Big(a_{21}a_{11}^{-1}\Big(b_{11}(a_{11}^T)^{-1}a_{21}^T-b_{12}\Big)-b_{21}(a_{11}^T)^{-1}a_{21}^T\\
&\quad+b_{22}\Big)^{-1}\Big(a_{21}a_{11}^{-1}\Big(a_{12}\tilde{A}_{11}^{-1}\big(a_{22}^T-a_{12}^T(a_{11}^T)^{-1}a_{21}^T\big)+\alpha_{11}(a_{11}^T)^{-1}a_{21}^T-\alpha_{12}\Big)\\
&\quad+a_{22}\tilde{A}_{11}^{-1}\big(a_{12}^T(a_{11}^T)^{-1}a_{21}^T-a_{22}^T-\alpha_{21}(a_{11}^T)^{-1}a_{21}^T+\alpha_{22}\big)\Big)
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
&D_uf\big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\sigma}=0 \qquad D_\sigma f\big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\sigma}=0\\
D_{\tilde{z}}f\big|_{u=\bar{u}_0,\,\tilde{z}=0}&=\Big(b_{21}D_{\tilde{z}}\bar{z}+b_{22}-a_{21}\Big(a_{11}^{-1}b_{11}D_{\tilde{z}}\bar{z}+a_{11}^{-1}b_{12}\Big)\Big)^{-1}\Big(-a_{21}a_{11}^{-1}\Big(a_{12}D_{\tilde{z}}\tilde{w}+\alpha_{11}D_{\tilde{z}}\bar{z}\\
&\quad+\alpha_{12}^T\Big)+a_{22}D_{\tilde{z}}\tilde{w}+\alpha_{21}D_{\tilde{z}}\bar{w}+\alpha_{22}\Big)\\
&=\Big(a_{21}a_{11}^{-1}\Big(b_{11}(a_{11}^T)^{-1}a_{21}^T-b_{12}\Big)-b_{21}(a_{11}^T)^{-1}a_{21}^T+b_{22}\Big)^{-1}\Big(a_{21}a_{11}^{-1}\Big(a_{12}A_{11}^{-1}\big(a_{22}^T\\
&\quad-a_{12}^T(a_{11}^T)^{-1}a_{21}^T\big)+\alpha_{11}(a_{11}^T)^{-1}a_{21}^T-\alpha_{21}^T\Big)+a_{22}\tilde{A}_{11}^{-1}\Big(a_{12}^T(a_{11}^T)^{-1}a_{21}^T\\
&\quad-a_{22}^T\Big)-\alpha_{21}(a_{11}^T)^{-1}a_{21}^T+\alpha_{22}\Big).
\end{aligned}
\tag{4.20}
$$

Proof. Using notation (4.13) and (4.14), equation rewrites as

$$
\begin{pmatrix}
0 & 0 & a_{11}^T(u,\,\sigma) & a_{21}^T(u,\,\sigma)\\
0 & \tilde{A}_{11}(u,\,\sigma) & a_{12}^T(u,\,\sigma) & a_{22}^T(u\,\sigma)\\
a_{11}(u,\,\sigma) & a_{12}(u,\,\sigma) & \alpha_{11}(u,\,\bar{w},\,\tilde{w},\,\bar{z},\,\tilde{z},\,\sigma) & \alpha_{21}^T(u,\,\bar{w},\,\tilde{w},\,\bar{z},\,\tilde{z},\,\sigma)\\
a_{21}(u,\,\sigma) & a_{22}(u,\,\sigma) & \alpha_{21}(u,\,\bar{w},\,\tilde{w},\,\bar{z},\,\tilde{z},\,\sigma) & \alpha_{22}(u,\,\bar{w},\,\tilde{w},\,\bar{z},\,\tilde{z}\,\sigma)
\end{pmatrix}
\begin{pmatrix}\bar{w}\\ \tilde{w}\\ \bar{z}\\ \tilde{z}\end{pmatrix}
$$
$$
=\begin{pmatrix}
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & b_{11}(u) & b_{12}(u)\\
0 & 0 & b_{21}(u) & b_{22}(u)
\end{pmatrix}
\begin{pmatrix}\bar{w}_x\\ \tilde{w}_x\\ \bar{z}_x\\ \tilde{z}_x\end{pmatrix}
$$

Hence from the first line it follows

$$a_{11}^T(u,\,\sigma)\bar{z}+a_{21}^T(u,\,\sigma)\tilde{z}=0$$

and therefore

$$\bar{z}(u,\,\tilde{z})=-\big(a_{11}^T(u,\,\sigma)\big)^{-1}a_{21}^T(u,\,\sigma)\tilde{z}.$$

The second line then reads

$$\tilde{A}_{11}(u,\,\sigma)\tilde{w}-a_{12}^T(u)\big(a_{11}^T(u,\,\sigma)\big)^{-1}a_{21}^T(u,\,\sigma)\tilde{z}+a_{22}^T(u,\,\sigma)\tilde{z}=0$$

and hence thanks to the invertibility of $\tilde{A}_{11}(u)$

$$\tilde{w}(u,\,\tilde{z}) = \tilde{A}_{11}^{-1}(u,\,\sigma)\Big(a_{12}^T(u)\big(a_{11}^T(u,\,\sigma)\big)^{-1}a_{21}^T(u,\,\sigma) + a_{22}^T(u,\,\sigma)\Big)\tilde{z}.$$

Moreover, one has

$$\big(\bar{z}(u,\,\tilde{z})\big)_x = -\big(a_{11}^T(u,\,\sigma)\big)^{-1}a_{21}^T(u,\,\sigma)\tilde{z}_x - D_u\Big(\big(a_{11}^T(u,\,\sigma)\big)^{-1}a_{21}^T(u,\,\sigma)\Big)u_x = \bar{z}_x(u,\,\bar{w},\,\tilde{z},\,\tilde{z}_x)$$

and hence the third line reads

$$a_{11}(u)\bar{w} + a_{12}(u)\tilde{w}(u,\,\tilde{z}) + \alpha_{11}(u,\,\bar{w},\,\tilde{z})\bar{z}(u,\,\tilde{z}) + \alpha_{21}^T(u,\,\bar{w},\,\tilde{z})\tilde{z} = b_{11}(u)\bar{z}_x(u,\,\bar{w},\,\tilde{z},\,\tilde{z}_x) + b_{12}(u)\tilde{z}_x$$

and therefore, thanks to the invertibility of a_{11}, in a small enough neighborhood of $(u = \bar{u}_0,\,\bar{w} = 0,\,\tilde{z} = 0,\,\tilde{z}_x = 0)$ it is implicitly defined a map $\bar{w} = \bar{w}(u,\,\tilde{z},\,\tilde{z}_x)$ such that

$$\begin{aligned}
&D_u\bar{w}\big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\tilde{z}_x=0} = 0 \qquad\qquad D_{\tilde{z}_x}\bar{w}\big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\tilde{z}_x=0} = a_{11}^{-1}\Big(b_{12} - b_{11}(a_{11}^T)^{-1}a_{21}^T\Big)\\
&D_{\tilde{z}}\bar{w}\big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\tilde{z}_x=0} = a_{11}^{-1}\Big(a_{12}A_{11}^{-1}\big(a_{22}^T - a_{12}^T(a_{11}^T)^{-1}a_{21}^T\big) + \alpha_{11}(a_{11}^T)^{-1}a_{21}^T - \alpha_{21}^T\Big).
\end{aligned}$$

Then the fourth line reads

$$a_{21}(u)\bar{w}(u,\,\tilde{z},\,\tilde{z}_x) + a_{22}(u)\tilde{w}(u,\,\tilde{z}) + \alpha_{21}(u,\,\tilde{z},\,\tilde{z}_x)\bar{z}_x(u,\,\tilde{z},\,\tilde{z}_x) + \alpha_{22}(u,\,\tilde{z},\,\tilde{z}_x)\tilde{z} = b_{21}(u)\bar{z}_x(u,\,\tilde{z},\,\tilde{z}_x) + b_{22}(u)\tilde{z}_x$$

Hence thanks to Lemma 4.3, it is implicitly defined a map $\tilde{z}_x = f(u,\,\tilde{z})$, which satisfies the hypotheses described in the statement of the lemma. One can then chose δ in Hypothesis 5 small enough to have that all the functions found so far are well defined for $u \in B_\delta(\bar{u})$ and $u_x \in B_\delta(0)$ and hence complete the proof of the lemma. □

In order to simplify the notations, we set

$$\begin{aligned}
\underline{a}(u,\,\tilde{z},\,\sigma) :=\;& a_{21}a_{11}^{-1}\Big(a_{12}\tilde{A}_{11}^{-1}\big(a_{22}^T - a_{12}^T(a_{11}^T)^{-1}a_{21}^T\big) + \alpha_{11}(a_{11}^T)^{-1}a_{21}^T - \alpha_{21}^T\Big) + a_{22}\tilde{A}_{11}^{-1}\big(a_{12}^T(a_{11}^T)^{-1}a_{21}^T\\
&- a_{22}^T - \alpha_{21}(a_{11}^T)^{-1}a_{21}^T + \alpha_{22}\big) \in \mathbb{M}^{(r-q)\times(r-q)}\\
\underline{b}(u,\,\sigma) :=\;& a_{21}a_{11}^{-1}\Big(b_{11}(a_{11}^T)^{-1}a_{21}^T - b_{12}\Big) - b_{21}(a_{11}^T)^{-1}a_{21}^T + b_{22} \in \mathbb{M}^{(r-q)\times(r-q)}
\end{aligned} \tag{4.21}$$

Hence

$$D_{\tilde{z}}f\Big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\sigma} = \underline{b}^{-1}(\bar{u}_0,\,0,\,\sigma)\underline{a}(\bar{u}_0,\,0,\,\sigma)$$

The proof of Lemma 4.4 also implies the following result:

Lemma 4.5. *Given $\vec{\xi} \in \mathbb{R}^{r-q}$, the condition $\underline{a}(u,\,0\,\sigma)\vec{\xi} = \lambda\underline{b}(u,\sigma)\vec{\xi}$ is equivalent to $A(u,\,0,\,\sigma)\;\Xi = \lambda B(u)\;\Xi$, where $\Xi \in \mathbb{R}^N$ is given by*

$$\Xi = \begin{pmatrix}
a_{11}^{-1}a_{12}\tilde{A}_{11}^{-1}\Big(a_{22}^T - a_{12}^T(a_{11}^T)^{-1}a_{21}^T\Big)\vec{\xi} + a_{11}^{-1}\alpha_{11}(a_{11}^T)^{-1}a_{21}^T\vec{\xi} - a_{11}^{-1}\alpha_{21}^T\vec{\xi} + \lambda\Big(a_{11}^{-1}b_{12} - a_{11}^{-1}b_{11}(a_{11}^{-1})^Ta_{21}^T\Big)\vec{\xi}\\
\tilde{A}_{11}^{-1}\Big(a_{12}^T(a_{11}^T)^{-1}a_{21}^T - a_{22}^T\Big)\vec{\xi}\\
-(a_{11}^T)^{-1}a_{21}^T\vec{\xi}\\
\vec{\xi}.
\end{pmatrix} \tag{4.22}$$

The case $\lambda = 0$ and hence $\underline{a}\vec{\xi} = 0$ is, in particular, included in the previous formulation.

Remark 4.1. For simplicity, in the following it will be assumed that $\underline{b}^{-1}\underline{a}$ is diagonalizable. Hence, in particular, one can find $(r-q)$ independent vectors $\Xi_1, \ldots \Xi_{r-q}$ such that

$$(A - \lambda_i B)\Xi_i = 0.$$

4.2 Construction of the boundary Riemann solver in the case of non invertible viscosity

In this section it is described the boundary Riemann solver induced by the parabolic approximation (1.5) when the viscosity matrix B is singular.

The exposition is organized as follows: in Section 4.2.1 it is briefly recalled the construction in the boundary free case and it is made reference to [5] for the complete analysis. It is worth underling, however, that the assumptions considered in [5] are not exactly those listed in Hypothesis 2: in particular, in [5] the matrix A is not suppose to be symmetric. On the other side, here it is not imposed an hypothesis of invertibility on the block A_{11}, while in [5] it is.

Section 4.2.2 introduces a key lemma on the dimension of the stable manifold of system

$$B(U)U'' = A(U,\, U')U'.$$

Section 4.2.3 describes the construction of the Riemann solver when the boundary is not characteristic, while the boundary characteristic case is dealt with in Section 4.2.4.

4.2.1 The equation of travelling wave profiles

In this section it is briefly recalled the construction of the i-th curve of admissible states, which is the key point in the characterization of the Riemann solver induced by the parabolic approximation

$$E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx}$$

in the boundary free case. The fundamental steps in the construction are the same which were illustrated in the case of an invertible viscosity matrix in 3.2.1. Moreover, a complete analysis can be found in [5], hence here we will give just a very fast sketch.

The starting point is the study of the system satisfied by travelling wave profiles:

$$B(U)U'' = \Big(A(U,\, U') - \sigma E(U)\Big)U'. \tag{4.23}$$

Under the additional assumptions described in Remark 4.2, one can apply to Lemma 4.4, this system is equivalent to

$$\begin{cases} u_x = \Big(\bar{w}(u,\, \tilde{z},\, \sigma),\ \tilde{w}(u,\, \tilde{z},\, \sigma),\ \bar{z}(u,\, \tilde{z},\, \sigma),\ \tilde{z}\Big)^T \\ \tilde{z}_x = f(u,\, \tilde{z},\, \sigma) \\ \sigma_x = 0. \end{cases} \tag{4.24}$$

If one linearizes this system around the equilibrium point $(u^+,\, 0,\, \sigma)$ finds that the Jacobian is represented by the $(N + r - q + 1) \times (N + r - q + 1)$ matrix

$$\begin{pmatrix} 0 & \underline{c} & 0 \\ 0 & \underline{b}^{-1}\bar{a}(u^+,\, 0,\, \sigma_i) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the exact expression of the matrix $\underline{c} \in \mathbb{R}^{N \times (r-q)}$ is not important here. Let $\lambda_i(u^+,\, 0)$ be a value of σ such there exists $\vec{\xi}_i(u^+,\, 0,\, \lambda_i) \in \mathbb{R}^{r-q}$ a non trivial solution of

$$\underline{b}^{-1}\bar{a}(u^+,\, 0,\, \lambda_i)\vec{\xi}_i(u^+,\, 0,\, \lambda_i) = 0, \tag{4.25}$$

then the center space has dimension $N + 2$ and is given by

$$\tilde{V}^c = \Big\{(u,\, v\,\vec{\xi}_i(u^+),\, \sigma_i),\ u \in \mathbb{R}^N,\ v,\, \sigma \in \mathbb{R}\Big\} \subseteq \mathbb{R}^{N+r-q+1}.$$

Thanks to Lemma 4.5, the equation (4.25) has a non trivial solution if and only the corresponding $\Xi_i(u^+, 0, \lambda_i)$, given by (4.22), is an eigenvalue of $E^{-1}(u^+)A(u^+, 0)$ associated to $\lambda_i(u^+, 0)$:

$$\mathcal{A}(u^+, 0, \lambda_i)\,\Xi_i(u^+, 0, \lambda_i) = 0 \iff \Big(A(u^+, 0) - \sigma_i E(u^+)\Big)\,\Xi_i(u^+, 0, \lambda_i) = 0$$
$$\iff \Big(E^{-1}(u^+)A(u^+, 0) - \lambda_i I\Big)\,\Xi_i(u^+, 0, \lambda_i) = 0.$$

Hence, coming back to the original system (4.23), one finds that the $(N+2)$-dimensional center manifold is tangent in $(u^+, 0, \lambda_i)$ to the subspace

$$V^c = \Big\{(u,\, v\,\Xi_i(u^+, 0, \lambda_i),\, \sigma),\ u \in \mathbb{R}^N,\ v,\, \sigma \in \mathbb{R}\Big\} \subseteq \mathbb{R}^{2N+2}. \tag{4.26}$$

As in it shown in the Appendix A.5, system (4.23) reduced one the center manifold takes the form

$$\begin{cases} u_x = v_i \Xi_i^\sharp(u, v_i, \sigma_i) \\ v_{ix} = \phi_i(u, v_i, \sigma_i) \\ \sigma_{ix} = 0 \end{cases} \tag{4.27}$$

where ϕ_i is a suitable function satisfying $(\partial\phi_i/\partial\sigma_i)(u^+, 0, \lambda_i(u^+)) < 0$ and $\Xi_i^\sharp\big(u^+, 0, \lambda_i\big) = \Xi_i(u^+, 0, \lambda_i)$.

Hence one can apply the same construction sketched in Section 3.2.1 and obtains the i-th curve of admissible states $T^i_{s_i}u^+$, which satisfies

$$\left.\frac{\partial T^i_{s_i}u^+}{\partial s_i}\right|_{s_i=0} = \Xi_i(u^+, 0, \lambda_i). \tag{4.28}$$

Remark 4.2. By construction, the curve $T^i_{s_i}u^+$ contains all the states which can be linked to u^+ by a sequence of rarefaction waves and shocks such that the states separated by the shocks are also limits states for a solution of (4.24) that lays on the center manifold.

This solution is also a travelling wave for equation (4.23) provided that the equivalence between (4.23) and (4.24) holds. More precisely, in the statement of Lemma 4.4 σ was supposed to be a fixed parameter, while in system (4.24) it is allowed to vary. It turns out that one can repeat all the passages performed in the proof of Lemma 4.4 provided that the dimension of the kernel of $A_{11}(u) - \sigma E_{11}(u)$ does not vary in a small enough neighborhood of $\sigma = \lambda_i(u^+)$.

Is should be underlined that this condition is indeed quite restrictive: more precisely, it can be proven that it holds if and only if the kernel of $A_{11}(u^+) - \lambda_i(u^+)E_{11}(u^+)$ is trivial. However, the authors are going to describe how to avoid this condition in the forthcoming paper [11].

4.2.2 A lemma on the dimension of the stable manifold

The aim of this section is to introduce Lemma 4.7, which specifies the dimension of the stable manifold of the system

$$B(U)U'' = A(U, U')U', \tag{4.29}$$

satisfied by boundary layer profiles. In this way, it is given an answer to a point left open in [37]: more precisely, it is shown that the condition called there Hypothesis (H5) is actually a consequence of Kawashima condition.

Thanks to Lemma (4.4), system (4.29) is equivalent to

$$\begin{cases} u_x = \Big(\bar{w}(u, \tilde{z}, 0),\ \tilde{w}(u, \tilde{z}, 0),\ \bar{z}(u, \tilde{z}, 0),\ \tilde{z}\Big)^T \\ \tilde{z}_x = f(u, \tilde{z}, 0). \end{cases} \tag{4.30}$$

Hence, linearizing around the equilibrium point $(\bar{u}, 0)$, one finds that the Jacobian is represented by the matrix

$$\begin{pmatrix} 0 & \underline{m} \\ 0 & \underline{b}^{-1}\underline{a}(\bar{u}, 0, 0) \end{pmatrix}$$

where $\underline{a}$ and $\underline{b}$ are defined by (4.21) and $\underline{m}$ is a $N \times (r-q)$ dimensional matrix whose exact expression is not important here. Hence the stable manifold of system (4.30) is tangent in $(\bar{u},\, 0)$ to

$$\tilde{V}^s = \Big\{ \sum_{i=1}^{d} v_i \Big(\tilde{m}_i \vec{\theta}_i(\bar{u}),\, \vec{\theta}_i(\bar{u}) \Big)^T \,:\, v_i \in \mathbb{R} \Big\} \subseteq \mathbb{R}^{N+r-q},$$

where $\vec{\theta}_1, \ldots, \vec{\theta}_d \in \mathbb{R}^{r-q}$ are the generalized eigenvectors of $\underline{b}^{-1}\underline{a}$ associated to negative eigenvalues $\mu_i(\bar{u},\, 0) < 0$. All the matrices $\tilde{m}_i$ belong to $\mathbb{M}^{N \times (r-q)}$ and their exact expression has no importance here, while the dimension d of this manifold is the object of Lemma 4.7. Thanks to Lemma 4.5, the stable manifold of the original system (4.29) is then tangent in $(\underline{u},\, 0)$ to the space

$$\Big\{ \sum_{i=1}^{d} v_i \Big(M_i \Theta_i(\bar{u}),\, \Theta_i(\bar{u}) \Big)^T,\; v_i \in \mathbb{R} \Big\} \subseteq \mathbb{R}^{2N}, \tag{4.31}$$

where the precise expression of the matrices $M_i \in \mathbb{M}^{N \times N}$ is not important here, while $\Theta_i(\bar{u},\, 0) \in \mathbb{R}^N$ is given by formula (4.22) and satisfies

$$\Big(A(\bar{u},\, 0) - \mu_i(\bar{u},\, 0) B(\bar{u}) \Big) \Theta_i(\bar{u},\, 0) = 0 \qquad \mu_i(\bar{u},\, 0) < 0 \tag{4.32}$$

Before stating the lemma, some notations have to be introduced: let $k-1$ denotes the number of strictly negative eigenvalues of $A(u,\, u_x)$, i.e. of the eigenvalues that satisfy

$$\lambda_1(u,\, u_x) < \lambda_2(u,\, u_x) < \cdots < \lambda_{k-1}(u,\, u_x) < -c < 0$$

for some constant $c > 0$. The number $(k-1)$ does not depend on u and u_x because of the strict hyperbolicity (Hypothesis 3). Moreover, let

$$A(u,\, u_x) = \begin{pmatrix} A_{11}(u) & A_{21}^T(u) \\ A_{21}(u) & A_{22}(u,\, u_x) \end{pmatrix} \qquad\qquad E(u) = \begin{pmatrix} E_{11}(u) & E_{21}^T(u) \\ E_{21}(u) & E_{22}(u,\, u_x) \end{pmatrix} \tag{4.33}$$

be the block decompositions of A and E and let n_{11} be the number of negative eigenvalues of A_{11}. Since E is positive definite, also E_{11} is and hence by Lemma 3.1 n_{11} is equal to the number of strictly negative eigenvalues of $E_{11}^{-1}(u) A_{11}(u)$. Hence, by the third assumption in Hypothesis 2, n_{11} does not depend upon u. Finally, as usual q denotes the dimension of the kernel of $A_{11}(u)$, which as well does not depend on u.

Before introducing Lemma 4.7, it is necessary to state a preliminary result, which will be exploited in the proof of the lemma:

Lemma 4.6. *Let Hypothesis 2 hold, then for every $\bar{u} \in B_\delta(\bar{u}_0)$ the polynomial*

$$\det \Big(A(\bar{u},\, 0) - \mu B(\bar{u}) \Big) \tag{4.34}$$

has degree $(r-q)$ with respect to μ.

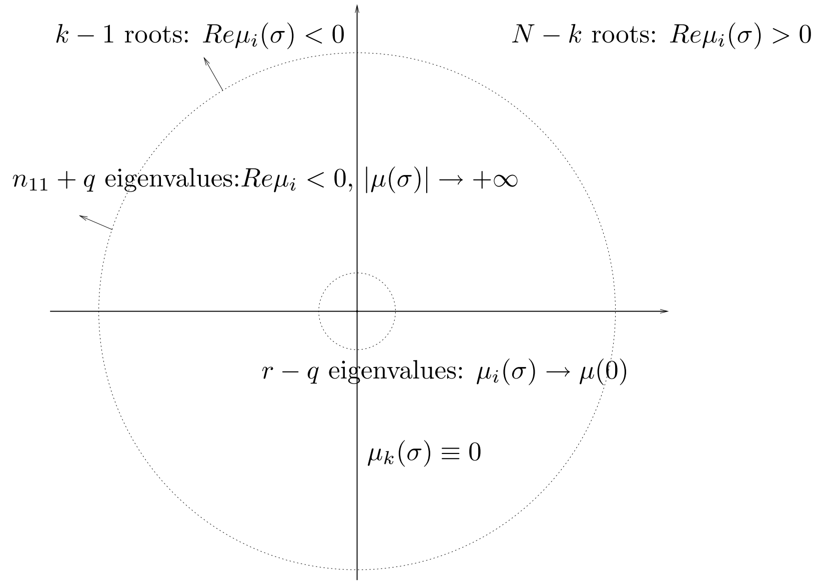
The proof follows immediately from Lemma 4.5, which guarantees that

$$\det \Big(A(\bar{u},\, 0) - \mu B(\bar{u}) \Big) = 0 \iff \det \Big(\underline{b}^{-1}(\bar{u},\, 0) \underline{a}(\bar{u},\, 0,\, 0) - \mu I_{r-q} \Big) = 0,$$

where the matrices $\underline{a},\, \underline{b} \in \mathbb{M}^{(r-q) \times (r-q)}$ are defined by (4.21) and $\underline{b}$ is invertible by Lemma 4.3. The symbol I_{r-q} denotes the identity matrix of dimension $(r-q)$.

Lemma 4.7. *Let Hypotheses 2 and 3 hold. Then the dimension d of the stable space* (4.31) *of system* (4.29) *is $d = k - 1 - n_{11} - q$.*

Figure 12: the behavior of the roots of (4.36) when the perturbation $\sigma \to 0^+$



Remark 4.3. As it will be clear in the following, one expects the dimension of the stable manifold to be equal to the number of boundary conditions that disappear when passing from the parabolic approximation to the hyperbolic limit. In other words, recalling Definition 2.1, d should be equal to

$$N - n_{11} - q - \Big(N - (k-1) \Big) :$$

in this way one obtains an heuristic justification of the validity of Lemma 4.7.

Proof. Since the state $\bar{u}$ is fixed, in the following for simplicity the matrices $A(\bar{u}, 0)$ and $B(\bar{u})$ will be denoted by A and B respectively. To prove the lemma one has to show that the equation

$$\det \Big(A - \mu_{\underline{i}} B \Big) = 0 \tag{4.35}$$

admits exactly $(k - 1 - n_{11} - q)$ roots with negative real part.

The proof is organized into six steps:

1. First, it is introduced a perturbation technique. More precisely, it is considered the problem

$$\det \Big(A - \mu_i(\sigma)(B + \sigma I) \Big) = 0. \qquad \sigma > 0, \tag{4.36}$$

where σ is a positive parameter which is allowed to go to 0. The symbol I denotes the N-dimensional identity matrix. Just to fix the ideas and to consider the most general case, it will be supposed that the matrix A is singular:

$$\lambda_1 < \dots < \lambda_{k-1} < 0 = \lambda_k < \lambda_{k+1} < \dots < \lambda_N$$

Once the value of the parameter σ is fixed, the solutions $\mu_i(\sigma)$ of (4.36) are the roots of a polynomial of degree N. Moreover, the matrix $(B + \sigma I)$ is positive definite and hence Lemma 3.1 guarantees that $(k-1)$ roots have strictly negative real part, $(N-k)$ roots have strictly positive real part and one root is zero.

If one let the parameter σ vary, (4.36) defines N algebraic functions $\mu_1(\sigma), \dots \mu_N(\sigma)$: since the coefficients of the polynomial (4.36) are analytic with respect to σ, it is known (see for example [34]) that the functions μ_i are analytic everywhere in the complex plane a part for some so called exceptional points that are certainly of finite number in every compact subset. Moreover, for every fixed function μ_i only two behaviors are possible in an exceptional point $\sigma = \bar{\sigma}$:

- $\mu_i(\sigma)$ is continuous in $\bar{\sigma}$.
- $\lim_{\sigma \to \bar{\sigma}} |\mu_i(\sigma)| = +\infty$.

In the case of the algebraic functions defined by (4.36), the point $\sigma = 0$ is an exceptional point since in $\sigma = 0$ the degree of the polynomial drops from N to $r - q$: we will denote by $\mu_{\underline{i}}(0)$, $\underline{i} = 1, \dots (r - q)$ the $(r - q)$ roots of (4.35).

Thanks to the known results recalled before, however, one deduces that there are

- $r - q$ functions $\mu_{\underline{i}}(\sigma)$ such that

$$\lim_{\sigma \to 0} \mu_{\underline{i}}(\sigma) = \mu_{\underline{i}}(0) \qquad i = 1, \dots r.$$

- $N - r + q$ functions $\mu_{\underline{i}}(\sigma)$ such that

$$\lim_{\sigma \to 0} |\mu_{\underline{i}}(\sigma)| = +\infty \qquad i = r + 1, \dots N.$$

The situation is summarized in Figure 12.

Moreover, restricting to a small enough neighborhood, it is not restrictive to suppose that $\sigma = 0$ is the only singular point.

2. As a second step we prove that to obtain the lemma it is sufficient to show that the number of functions $\mu_{\underline{i}}(\sigma)$ continuous in $\sigma = 0$ and such that $Re\mu_{\underline{i}}(\sigma) < 0$ when $\sigma \in \mathbb{R}^+$ is equal to $k - 1 - n_{11} - q$. Since the number of roots of (4.36) with negative is equal to $k - 1$, this is equivalent to show that there are exactly $n_{11} + q$ functions $\mu_{\underline{i}}(\sigma)$ such that $Re\mu_{\underline{i}}(\sigma) < 0$ and $|\mu_{\underline{i}}(\sigma)| \uparrow +\infty$ when $\sigma \to 0^+$.

   The only case that has to be excluded is the possibility that in the limit a function $\mu_{\underline{i}}(\sigma)$ with $Re\mu_{\underline{i}}(\sigma) < 0$ has zero real part.

   First, one observes that equation (4.35) does not admit purely imaginary roots: by contradiction, assume that there exists $\mu \in \mathbb{R}$, $\mu \neq 0$ and $v = v_r + iv_i$, v_r, $v_i \in \mathbb{R}^N$ such that

$$A(v_r + iv_i) = i\mu B(v_r + iv_i). \tag{4.37}$$

   Just to fix the ideas, one can assume $\mu > 0$: as in the proof of Lemma 3.1, it turns out that

$$0 \geq -\mu \langle Bv_i, \, v_i \rangle = \langle Av_r, \, v_i \rangle = \langle Av_i, \, v_r \rangle = \mu \langle Bv_r, \, v_r \rangle \geq 0 \tag{4.38}$$

   because B is positive semidefinite. Hence the only possibility is that v_i and v_r both belong to $\ker B$, but this contradicts (4.37) and hence there are no purely imaginary roots of (4.35).

   Hence one is left to exclude the possibility that a function $\mu_{\underline{i}}(\sigma)$ continuous in $\sigma = 0$ and such that $Re\mu_{\underline{i}}(\sigma) < 0$ converges to 0. From the strict hyperbolicity of the matrix $E^{-1}A$ (Hypothesis 3) and from Lemma 3.1 it follows that 0 is an eigenvalue of A with multiplicity one and hence $\mu_{\underline{k}} = 0$ is a root of (4.35) with multiplicity one. Moreover, again from Lemma 3.1, one deduces that the algebraic function $\mu_{\underline{k}}(\sigma)$ is identically equal to 0. Since the multiplicity of 0 as a root of (4.35) is one, it cannot happen that other functions $\mu_{\underline{i}}(\sigma)$, $\underline{i} \neq \underline{k}$ converge to zero when $\sigma \to 0^+$.

3. As a third step, we perform a change of variable. More precisely, the number of solutions $\mu_{\underline{i}}(\sigma)$ of (4.36) such that $Re\mu_{\underline{i}}(\sigma) < 0$ and $|\mu_{\underline{i}}(\sigma)| \uparrow +\infty$ when $\sigma \to 0^+$ is equal to the number of roots $x_{\underline{i}}(\sigma)$ of

$$\det \Big(B + \sigma I - x_{\underline{i}}(\sigma) A \Big) = 0 \qquad \underline{i} = 1, \dots N - r + q \tag{4.39}$$

   such that $Rex_i(\sigma) < 0$ and $x_i(\sigma) \to 0$ when $\sigma \to 0^+$. This is clear if one defines $x_{\underline{i}}(\sigma) := 1/\mu_{\underline{i}}(\sigma)$.

4. In the fourth part of the proof, we study the inverse problem to (4.39), i.e. the eigenvalue problem
$$\det\Big(B - xA + \sigma_{\underline{j}}(x)I\Big) = 0 \qquad \underline{j} = 1, \dots N. \tag{4.40}$$
The meaning of the notation is that in this case one fixes x and then finds σ.

More precisely, it will be studied the behavior of the $N-r$ eigenvalues $-\sigma_{\underline{j}}(x)$ such that $\sigma_{\underline{j}}(x) \to 0$ when $x \to 0$. From the analysis in [10] it follows that
$$F(x) = -A_{11}x - A_{21}^T b^{-1} A_{21} x^2 + o(x^2) \qquad x \to 0$$
where $F(x)$ denotes the projection of $B - xA$ on the generalized subspaces converging to the kernel of B. The blocks A_{11} and A_{21} are defined by (4.33). Let $F_0(x)$ be the projection of $F(x)$ on the kernel of A_{11} and $F_\perp(x)$ be the projection on the subspace orthogonal to $\ker A_{11}$.

Thus again the analysis in [10] guarantees that
$$F_0(x) = -(A_{21}^I)^T b^{-1} A_{21}^I x^2 + o(x^2) \qquad x \to 0$$
and
$$F_\perp(x) = -\tilde{A}_{11}x + o(x) \qquad x \to 0.$$
In the previous expression it has been used the notations introduced in (4.10):
$$\begin{pmatrix} A_{11}(u,\,\sigma) & A_{21}(u\,\sigma)^T \\ A_{21}(u\,\sigma) & A_{22}(u,\,u_x,\,\sigma) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \big(A_{21}^I\big)^T(u,\,\sigma) \\ 0 & \tilde{A}_{11}(u,\,\sigma) & \big(A_{21}^{II}\big)^T(u,\,\sigma) \\ A_{21}^I(u) & A_{21}^{II}(u,\,\sigma) & A_{22}(u,\,\sigma,\,u_x) \end{pmatrix}$$
From Kawashima condition it follows that the matrix $(A_{21}^I)^T b^{-1} A_{21}^I$ is positive definite: one can proceed in the same way as in the proof of Lemma (4.2).

Hence projecting on suitable subspaces (we refer again to [10] for the precise computations) it follows that the following expansions hold:
$$\sigma_{\underline{j}}(x) = \mu_{\underline{j}} x^2 + o(x) \qquad x \to 0, \qquad j = 1, \dots q \tag{4.41}$$
and
$$\sigma_{\underline{j}}(x) = \lambda_{\underline{j}} x + o(x) \qquad x \to 0 \qquad j = q+1, \dots N-r. \tag{4.42}$$
In the previous expression, $\mu_{\underline{j}} > 0$ is an eigenvalue of $(A_{21}^I)^T b^{-1} A_{21}^I$ and $\lambda_{\underline{j}} \neq 0$ is an eigenvalue of $\tilde{A}_{11}$.

5. As a fifth step, we analyze equation (4.42): for every fixed $\underline{j} = q+1, \dots N-r$, the expansion guarantees that the function $\sigma_{\underline{j}}(x)$ is invertible in a neighborhood of $x = 0$. The inverse function $x_{\underline{j}}(\sigma)$ satisfies (4.39) and because of (4.42) the condition that $Rex_{\underline{j}}(\sigma) < 0$ when $\sigma \in \mathbb{R}^+$ is satisfied if and only if $\lambda_{\underline{j}}$ is an eigenvalue of $\tilde{A}_{11}$ with negative real part. This implies that there are exactly n_{11} functions $x_{\underline{j}}(\sigma)$ satisfying (4.42) and such that $Rex_{\underline{j}}(\sigma) < 0$ when $\sigma \in \mathbb{R}^+$.

   Moreover, there are $N - r - q$ functions $x_{\underline{j}}(\sigma)$ satisfying (4.39) and (4.42).

6. In the sixth part of the proof it will be analyzed equation (4.41). From the previous considerations one deduces that there are exactly $2q$ functions $x_{\underline{j}}(\sigma)$ satisfying (4.39) and (4.41).

   Actually in this case the function $\sigma_{\underline{j}}(x)$ is not invertible in a neighborhood of $x = 0$. Nevertheless, there exists certainly an algebraic function $x_{\underline{i}}(\sigma)$ which satisfies (4.39) and therefore $\sigma_{\underline{j}}(x_{\underline{i}}(\sigma)) = \sigma$. From known results on the algebraic functions (see for example [34]) the following expansion hold:
$$x_{\underline{i}}(\sigma) = \sum_{n=1}^{+\infty} c_{\underline{i}\,n} \big(\sqrt[p(\underline{i}),\underline{i}]{\sigma}\big)^n. \tag{4.43}$$

In the previous expression, $\sqrt[p(\underline{i})]{\sigma}$ denotes one of the roots of σ of order $p(\underline{i})$. Inserting (4.43) into (4.41), one finds that the condition $\sigma_{\underline{j}}(x_{\underline{i}}(\sigma)) = \sigma$ implies that $p(\underline{i}) \equiv 2$ for every $\underline{i} = 1, \dots 2q$. Since there are exactly $2q$ functions $x_{\underline{j}}(\sigma)$ satisfying (4.39) and (4.41), one deduces that there are q functions with strictly positive real part and q functions with strictly negative real part. The possibility of a root with zero real part is excluded on the basis of the same considerations as in step 2.

Hence from steps 5 and 6 one obtains that there are exactly $n_{11} + q$ solutions $x_{\underline{i}}(\sigma)$ of (4.39) such that $Re x_{\underline{i}}(\sigma) < 0$ and $x_{\underline{i}}(\sigma) \to 0$ when $\sigma \to 0^+$. Thanks to the considerations in steps 1, 2 and 3 this concludes the proof of the lemma.

□

In order to introduce the following result, which is actually a corollary of Lemma 4.7, it is useful to introduce some further notations: let

$$V^s(u) := \text{span}\langle \Theta_{n_{11}-q+1}, \dots \Theta_{k-1} \rangle \tag{4.44}$$

be the space generated by the vectors that satisfy

$$\Big(A(\bar{u}, 0) - \mu_i(\bar{u}, 0) B(\bar{u}) \Big) \Theta_i(\bar{u}, 0) = 0 \qquad \mu_i(\bar{u}, 0) < 0. \tag{4.45}$$

Let

$$V^u(u) := \text{span}\langle \Xi_{k+1}, \dots \Xi_N \rangle \tag{4.46}$$

the subspace generated by the Ξ_i satisfying

$$\Big(A(u, 0) - \lambda_i E(u) \Big) \Xi_i = 0 \qquad \lambda_i > 0. \tag{4.47}$$

When $A(u, 0)$ is invertible the dimension of V^u is equal to $(N-n)$, where n is the number of negative eigenvalues of A defined as in (3.11). On the other side, when $A(u, 0)$ is singular the dimension of V^u is equal to $(N-k)$, where $(k-1)$ is the number of strictly negative eigenvalues of A as in (3.16). In this case, the following subspace is non trivial:

$$V^c(u) := \text{span}\langle \, \Theta_k \rangle, \;\; \text{where } A(u, 0)\, \Theta_k = 0. \tag{4.48}$$

The proof of the following result is analogous to that of Lemma 7.1 in [4]:

Lemma 4.8. *Let Hypotheses 2 and 3 hold. Then*

$$V^s \cap V^u = \{0\} \qquad V^s \cap V^c = \{0\} \qquad V^c \cap V^u = \{0\},$$

where V^s, V^c and V^u are defined by (4.44), (4.48) and (4.46) respectively.

4.2.3 The non characteristic case

This section deals with the characterization of the Riemann solver induced by the parabolic approximation (4.1) when the boundary is not characteristic. More precisely, it will be assumed that Hypothesis 8 holds, i.e. that all the eigenvalues of the matrix $E^{-1}(u) A(u, 0)$ are well separated from 0, in the sense defined by equation (3.11).

The basic ideas in the construction were already introduced in Section 3.2.2 considering the Riemann solver induced by the parabolic approximation (4.1) when the boundary is not characteristic and the the viscosity is invertible. The technical difficulties arising from the singularity of the matrix B are overcome making reference to Lemmata 4.4 and 4.7.

On the other side, the way the boundary datum is assigned in the parabolic IBVP (4.1) marks a difference between the two cases: indeed, when the viscosity is invertible one can prescribe a full boundary condition $\bar{u}_b$. Moreover, one finds that if $|\bar{u}_0 - \bar{u}_b|$ is small enough then it is always possible to construct a boundary Riemann solver connecting $\bar{u}_0$ and $\bar{u}_b$ (i.e. a vector $(s_1, \dots s_N)$ such that

(3.14) holds): in other words, the hyperbolic limit is uniquely determined. On the other side, when the viscosity is singular one cannot assign in general a full boundary conditions: in other words, the function ß does not coincide with the identity, but is given by Definition 2.1. The construction of the Riemann solver has hence to be coupled with such a non trivial boundary conditions. Nevertheless, thanks to a transversality Lemma which will be proved in Section 4.3, it turns out that also in the case of a singular viscosity the boundary Riemann solver induced in the hyperbolic limit is uniquely determined by the initial and boundary datum imposed on (4.1).

As it was already observed in the case of a non singular viscosity, the study of the boundary Riemann solver also provides a precise selection principle for the boundary datum $\bar{u}$ one imposes in the hyperbolic initial boundary value problem

$$\begin{cases} E(u)u_t + A(u,\,0)u_x = 0 & u \in \mathbb{R}^N \\ u(t,\,0) = \bar{u} & u(0,\,x) = \bar{u}_0. \end{cases} \tag{4.49}$$

The construction of the boundary Riemann solver is the following. The initial datum $\bar{u}_0$ is connected to the state $\bar{u}$ through waves of positive speed: more precisely, one has to consider travelling wave profiles satisfying

$$B(U)U'' = \Big(A(U,\,U') - \sigma E(U) \Big) U'$$

for some positive speed $\sigma > 0$. The construction of the corresponding curve of admissible states has been sketched in Section 4.2.1. One obtains therefore that the admissibility criterium induced by the parabolic approximation (4.1) on the boundary datum $\bar{u}$ in (4.49) is the following: there must be a vector $(s_{n+1}, \dots s_N)$ such that

$$\bar{u} = T^{n+1}_{s_{n+1}} \circ \dots T^N_{s_N} \bar{u}_0.$$

In order to face the possibility of a loss of boundary condition when passing from the parabolic approximation into the hyperbolic limit one has to study the system satisfied by the boundary layers, namely

$$\begin{cases} B(U)U'' = A(U,\,U')U' \\ U(0) = \bar{u}_b & \lim_{x \to +\infty} U(x) = \bar{u}. \end{cases}$$

More precisely, one looks for a characterization of the set of states $\bar{u}_b$ such that the previous system admits a solution. This is equivalent to study the stable manifold of system

$$B(U)U'' = A(U,\,U')U' \tag{4.50}$$

around the equilibrium point $(U = \bar{u},\, U' = 0)$. Thanks to the considerations in Section 4.2.2, an in particular to Lemma 4.7, this manifold is is parametrized by a function

$$\phi : \mathbb{R}^{n - n_{11} - q} \to \mathbb{R}^N \times \mathbb{R}^N$$

which is tangent in zero to the stable space (4.31). Denoting by

$$\begin{aligned} \pi : \mathbb{R}^N \times \mathbb{R}^N &\to \mathbb{R}^N \\ (u,\,p) &\mapsto u \end{aligned}$$

the projection on the second component, one has

$$D(\pi \circ \phi)\big|_{u=\bar{u},\, s_{n_{11}+q+1} = \dots = s_n = 0} = \Big(\Theta_{n_{11}+q+1}(\bar{u},\,0) | \dots | \Theta_n \Big), \tag{4.51}$$

where Θ_i satisfies (4.45).

Putting together all the previous considerations, one obtains the definition of a status $\bar{u}_b$ satisfying

$$\bar{u}_b = \psi(s_{n_{11}+q+1},\, \dots s_N,\, \bar{u}_0) := \pi\Big(\phi(s_{n_{11}+q+1}, \dots, s_n) \Big) \circ T^{n+1}_{s_{n+1}} \circ \dots T^N_{s_N} \bar{u}_0. \tag{4.52}$$

Thanks to (4.51) and to (4.28), it holds

$$D\psi\big|_{u=\bar u_0,\, s_{n_{11}+q+1}=\cdots=s_N=0} = \Big(\Theta_{n_{11}+q+1}|\dots|\Theta_n|\Xi_{n+1}|\dots|\Xi_N\Big) \tag{4.53}$$

where Θ_i satisfies (4.45) and Ξ_i

$$\Big(E(\bar u_0) - \lambda_i A(\bar u_0,\, 0)\Big)\Xi_i = 0 \qquad \lambda_i > 0.$$

Lemma 4.8 guarantees that the previous vectors are independent and hence that the application ψ is injective in a neighborhood of $(s_{n_{11}+q+1}, \dots, s_N) = \vec{0}$.

The previous construction has to be coupled with the non trivial boundary condition imposed on the parabolic IBVP (4.1). More precisely, one has to consider the composition map $ß \circ \psi$. Lemma 4.9, which will be introduced in Section 4.3, guarantees that the manifold ψ is transversal to the level lines of the function ß. In particular, this is enough to conclude that $ß \circ \psi$ is invertible in a small enough neighborhood of $(s_{n_{11}+q+1} \dots s_N) = \vec{0}$. In other words, there exists one and only one point $\bar u_b$ satisfying (4.52) and $ß(\bar u_b) = \bar g$.

Hence the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation is uniquely determined once the data $\bar g$ and $\bar u_0$ are known.

The considerations carried on so far are summarized in the following

Proposition 4.1. *Let Hypotheses 2, 3, 5, 6, 7 and 8 hold. Then the Riemann solver induced by the parabolic approximation*

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} & u^\varepsilon \in \mathbb{R}^N \\ ß(u^\varepsilon(t,\, 0)) \equiv \bar g \qquad u^\varepsilon(0,\, x) \equiv \bar u_0 \end{cases} \tag{4.54}$$

in the hyperbolic limit is implicitly defined by the relation

$$\bar g = ß \circ \psi(s_{n_{11}+q+1},\, \dots,\, s_N,\, \bar u_0) := ß \circ \pi\Big(\phi(s_{n_{11}+q+1}, \dots s_n)\Big) \circ T^{n+1}_{s_{n+1}} \circ \dots T^N_{s_N}\bar u_0 \tag{4.55}$$

where $T^N_{s_N} \dots T^{n+1}_{s_{n+1}}$ are the curves of admissible states described in Section 4.2.1 and $\pi(\phi)$ is the first component of the map that parametrizes the stable manifold of the system (4.50) *of stationary solutions. Moreover, the composition map $ß \circ \psi$ is locally invertible and hence if the data $\bar g$ and $\bar u_0$ are fixed there exists a unique Riemann solver satisfying* (4.55)*. The state*

$$\bar u = T^{n+1}_{s_{n+1}} \circ \dots T^N_{s_n}\bar u_0 \tag{4.56}$$

is the trace induced on the axis $x = 0$ in the hyperbolic limit. Hence the admissibility criterium induced by the parabolic approximation (4.54) *on the boundary datum $\bar u$ of problem*

$$\begin{cases} E(u)u_t + A(u,\, 0)u_x = 0 & u \in \mathbb{R}^N \\ u(t,\, 0) \equiv \bar u \qquad u(0,\, x) \equiv \bar u_0 \end{cases}$$

is that (4.56) *has to hold for some vector $(s_{n+1}, \dots s_N)$.*

4.2.4 The boundary characteristic case

This section deals with the characterization of the boundary Riemann solver limit of the parabolic approximation (4.1) in the case the viscosity matrix is singular and the boundary is characteristic. More precisely, it is assumed that Hypothesis 9 holds, i.e. that the k-th eigenvalue of $E^{-1}(u)A(u,\, u_x)$ is close to the speed of the boundary:

$$\lambda_k(u,\, u_x) \sim 0.$$

By strict hyperbolicity (Hypothesis 3) there exists a constant $c > 0$ such that

$$\lambda_1(u,\, u_x) < \dots \lambda_{k-1}(u,\, u_x) \le -c < 0 < c \le \lambda_{k+1}(u,\, u_x) < \dots < \lambda_N(u,\, u_x).$$

The difficulties in this case arise both from the singularity of the viscosity matrix and from the possibility of an interaction between the center and the stable component of the stationary solutions. Nevertheless, the main ideas involved in the construction were actually already introduced in Section 3.2.2 to overcome the problems coming from the fact that the boundary is characteristic and in Section 4.2.3 to face the difficulties originated from the singularity of the viscosity matrix: here, therefore, we will only sketch the key points of the construction.

The initial value $\bar{u}_0$ is connected to the state $\bar{u}_k$ through the curves of admissible states $T^N_{s_N}, \dots T^{k+1}_{s_{k+1}}$, the same which were introduced in [5] in the boundary free case and whose construction was described in Section 4.2.1:

$$\bar{u}_k = T^{k+1}_{k+1} \circ \dots T^N_{s_N} \bar{u}_0.$$

To define the k-th curve $\tilde{T}^k_{s_k}$ of admissible states in the boundary Riemann problem, we introduce as in Section 3.2.3 the following notations:

$$\bar{s}_k = \min\{s_k : \ \sigma(s_k) = 0\} \qquad \bar{u} = T^k_{\bar{s}_k} \qquad \underline{s}_k = \max\{s_k : \ \sigma(s_k) = 0\} \qquad \underline{u} = T^k_{\underline{s}_k}. \tag{4.57}$$

In the previous expression $T^k_{s_k}$ is the k-th curve of the admissible states in the boundary free case and $\sigma_k(s_k)$ is the corresponding curve of speeds. As far as $s \leq s_k$, then $\sigma_k(s_k) \geq 0$ and the curve of admissible states $\tilde{T}^k_{s_k}\bar{u}_k$ coincides with $T^k_{s_k}\bar{u}_k$.

Moreover, the state $\bar{u}$ is an admissible boundary datum for the hyperbolic problem

$$\begin{cases} E(u)u_t + A(u, 0)u_x = 0 \\ u(t, 0) \equiv \bar{u} \qquad u(0, x) \equiv \bar{u}_0 \end{cases}$$

if and only if

$$\bar{u} = T^k_{s_k} \circ T^{k+1}_{k+1} \circ \dots T^N_{s_N} \bar{u}_0$$

for some vector $(s_k, \dots s_N)$ such that $\sigma_k(s) > 0$ for every $s < s_k$, i.e. such that $s_k \leq \bar{s}_k$.

To extend the curve of admissible states $\tilde{T}^k_{s_k}$ for $s_k > \underline{s}_k$ one has to study the set of values $\bar{u}_b$ such that the following system admits a solution:

$$\begin{cases} B(U)U'' = A(U, U')U' \\ U(0) = \bar{u}_b \qquad \lim_{x \to +\infty} U(x) = \underline{u}. \end{cases}$$

Lemma 4.4 guarantees that the previous equation can be reduced to an explicit form like

$$\begin{cases} u_x = \Big(\bar{w}(u, \tilde{z}, 0), \ \tilde{w}(u, \tilde{z}, 0), \ \bar{z}(u, \tilde{z}, 0), \ \tilde{z}\Big)^T \\ \tilde{z}_x = f(u, \tilde{z}, 0) \\ U(0) = \bar{u}_b \qquad \lim_{x \to +\infty} U(x) = \underline{u}. \end{cases}$$

As in the case of a non invertible viscosity, it is not restrictive to consider only the orbits laying on the center-stable manifold. Moreover, the solution can be decomposed along its center and uniformly stable manifold. However, because of the singularity of the viscosity matrix, the dimension of the uniformly stable manifold is in general smaller than $(N + k - 1)$. More precisely, Lemma 4.7 guarantees that it is equal to $(N + k - 1 - n_{11} - q)$. Nevertheless, with computations analogous to those carried on in Appendix (A.4) one finds the coupled systems

$$\begin{cases} u_{kx} = \check{r}_k(u_k + u_s, V^s, v_k)v_k \\ v_{kx} = \check{\lambda}_k(u_k + u_s, V^s, v_k)v_k \end{cases} \qquad \text{and} \qquad \begin{cases} u_{sx} = \check{R}^s(u_k + u_s, V^s, v_k)V^s \\ V_{sx} = \check{\Lambda}_s(u_s + u_k, V^s, v_k)V^s, \end{cases} \tag{4.58}$$

which are associated to the limit conditions

$$\begin{cases} \lim_{x \to \infty} u_k(x) = \underline{u} \\ \lim_{x \to \infty} v_k(x) = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} \lim_{x \to \infty} u_s(x) = 0 \\ \lim_{x \to \infty} V^s(x) = 0. \end{cases} \tag{4.59}$$

In the previous expressions, $V^s \in \mathbb{R}^{k-1-n_{11}-q}$, $v_k \in \mathbb{R}$, $\check{\lambda}_k \in \mathbb{R}$ is close to 0 and the matrix $\check{\Lambda}^s$ has dimension $(k - 1 - n_{11} - q) \times (k - 1 - n_{11} - q)$ and is negatively definite. As in the case of a non singular viscosity, the difficulty in the study of the previous systems comes from the fact that they are coupled, in the sense that the coefficients of the first depend also on the solution of the second, and viceversa. Techniques analogous to those exploited in Section 3.2.3 allow to show that there exists a Lipschitz function

$$\begin{aligned} \phi : B_\nu(0) \subseteq \mathbb{R}^{k-1} \times [\underline{s}_k,\, \underline{s}_k + \nu[\; &\to \mathbb{R}^N \times \mathbb{R}^N \\ (s_1, \dots s_{k-1},\, s_k) \; &\mapsto \Big(\phi_1(s_1, \dots s_{k-1},\, s_k),\, \phi_2(s_1, \dots s_{k-1},\, s_k) \Big) \end{aligned} \tag{4.60}$$

such that the coupled systems (4.58) admits a solution if and only if $u_k(0) \in \phi_1\big(B_\nu(0) \times [\underline{s}_k,\, \underline{s}_k + \nu[\big)$ and $u_s(0) \in \phi_2\big(B_\nu(0) \times [\underline{s}_k,\, \underline{s}_k + \nu[\big)$. Moreover, this function satisfies

$$\begin{aligned} & D_{s_1 \dots s_{k-1}} \phi_1 \Big|_{s_{n_{11}+q+1} = \dots s_{k-1} = s_k = 0} = 0 && \frac{\partial}{\partial s_k} \phi_1 \Big|_{s_{n_{11}+q+1} = \dots s_{k-1} = s_k = 0} = \Theta_k \\ & D_{s_{n_{11}+q+1} \dots s_{k-1}} \phi_2 \Big|_{s_{n_{11}+q+1} = \dots s_{k-1} = s_k = 0} = (\Theta_{n_{11}+q+1} | \dots | \Theta_{k-1}) && \frac{\partial}{\partial s_k} \phi_2 \Big|_{s_{n_{11}+q+1} = \dots s_{k-1} = s_k = 0} = 0, \end{aligned} \tag{4.61}$$

where Θ_i satisfies

$$\Big(A(\underline{u},\, 0) - \mu_i(\underline{u},\, 0) B(\underline{u}) \Big) \Theta_i = 0 \qquad \mu_i(\underline{u},\, 0) \leq 0.$$

The proof of the existence and of the differential properties of ϕ is based on the same considerations that guarantee the validity of Lemma 3.4. Moreover, in order to underline the dependence of ϕ on the value $\underline{u}$ which is the limit assigned in (4.59) in the following we will use the notation $\phi(s_1, \dots, s_{k-1},\, s_k,\, \underline{u})$.

The construction of the boundary Riemann solver is then completed introducing the functions

$$\tilde{T}^k_{s_k}(\bar{u}_k, s_{n_{11}+q+1} \dots s_{k-1}) := \begin{cases} T^k_{s_k}(\bar{u}_k) & s_k \leq \underline{s}_k \\ \phi_1(s_{n_{11}+q+1}, \dots s_{k-1},\, s_k,\, T^k_{\underline{s}_k}(\bar{u}_k)) & s_k > \underline{s}_k \end{cases} \tag{4.62}$$

and

$$\tilde{\phi}_2(s_1 \dots s_{k-1},\, s_k,\, \bar{u}_k) := \begin{cases} \phi_2\big(s_{n_{11}+q+1} \dots s_{k-1},\, \underline{s}_k,\, T^k_{s_k}(\bar{u}_k)\big) + T^k_{s_k}(\bar{u}_k) & s_k \leq \underline{s}_k \\ \phi_2(s_{n_{11}+q+1}, \dots s_{k-1},\, s_k,\, \underline{u}) + \tilde{T}^k_{s_k}(\underline{u}_k) & s_k > \underline{s}_k. \end{cases} \tag{4.63}$$

Such a definition formalizes the fact that when $s \leq \underline{s}_k$ there is no interaction between the manifold of boundary layer profiles and the k-th curve of admissible states, while if $s > \underline{s}_k$ the construction is determined by function (4.60).

Putting all the previous considerations together one obtains a definition of a state

$$\bar{u}_b := \psi(s_{n_{11}+q}, \dots s_N) = \tilde{\phi}_2(s_{n_{11}+q+1}, \dots s_{k-1},\, s_k,\, \cdot) \circ \tilde{T}^k_{s_k}(s_{n_{11}+q+1}, \dots s_{k-1},\, \cdot) \circ T^{k+1}_{s_{k+1}} \circ \dots T^N_{s_N} \bar{u}_0. \tag{4.64}$$

From (4.28) and (4.61) it follows that

$$D\phi \Big|_{s_{n_{11}+q+1} = \dots = s_N = 0} = \Big(\Theta_{n_{11}+q+1} \,|\, \dots \,|\, \Theta_k \,|\Xi_{k+1} \,|\, \dots \,|\Xi_N \Big). \tag{4.65}$$

Just to fix the ideas, one can assume that $A(\bar{u}_0,\, 0)$ is singular: in this case the vectors Θ_i satisfy (4.45) when $i < k$ and Θ_k is defined by (4.48), while vectors Ξ_j satisfy (4.47).

Being the viscosity singular, in order to complete the construction of the boundary Riemann solver relation (4.64) has to be coupled to the non trivial boundary condition assigned through the function ß. This step was not dealt with explicitly in Section 4.2.4 because when B is invertible ß $\equiv I$. In the case of a singular viscosity, however, the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation (4.1) is implicitly defined by the relation

$$\bar{g} = \text{ß} \circ \psi(s_{n_{11}+q+1}, \dots s_N). \tag{4.66}$$

One can then show straightforwardly that the composition map $ß \circ \psi$ is locally invertible: the proof relies on the transversality Lemma 4.9, which will be introduced in Section 4.3. In particular, it turns out that, once the data $\bar{g}$ and $\bar{u}_0$ are assigned, the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation (4.1) is uniquely determined by formula (4.66).

The following proposition presents a synthesis of all the considerations carried on so far:

Proposition 4.2. *Let Hypotheses 2, 3, 5, 6, 7 and 9 hold. Then the Riemann solver induced by the parabolic approximation*

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon, \, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ ß(u^\varepsilon(t, \, 0)) \equiv \bar{g} \qquad u^\varepsilon(0, \, x) \equiv \bar{u}_0 \end{cases} \tag{4.67}$$

in the hyperbolic limit is implicitly defined by the relation

$$\bar{g} = ß \circ \psi(s_{n_{11}+q+1}, \, \dots, s_N) = ß \circ \tilde{\phi}_2(s_{n_{11}+q+1}, \dots s_{k-1}, \, s_k, \, \cdot) \circ \tilde{T}^k_{s_k}(s_{n_{11}+q+1}, \dots s_{k-1}, \, \cdot) \circ T^{k+1}_{s_{k+1}} \circ \dots T^N_{s_N} \bar{u}_0. \tag{4.68}$$

In the previous expression, $T^N_{s_N} \dots T^{k+1}_{s_{k+1}}$ are the curves of admissible states described in Section 4.2.1, while the maps $\tilde{T}^k_{s_k}$ and $\tilde{\psi}_2$ are defined by (4.62) *and* (4.63) *respectively. The composition map $ß \circ \psi$ turns out to be locally invertible: this means, in particular, that the boundary Riemann solver which is induced in the hyperbolic limit by the parabolic approximation* (4.1) *is completely and uniquely determined by the relation* (4.68).

The admissibility criterium induced by the parabolic approximation (4.67) *on the boundary datum $\bar{u}$ of problem*

$$\begin{cases} E(u)u_t + A(u, \, 0)u_x = 0 \qquad u \in \mathbb{R}^N \\ u(t, \, 0) \equiv \bar{u} \qquad u(0, \, x) \equiv \bar{u}_0 \end{cases}$$

is that it must hold

$$\bar{u} := \tilde{T}^k_{\bar{s}_k} \circ T^{n+1}_{s_{n+1}} \circ \dots T^N_{s_n} \bar{u}_0,$$

some vector $(\bar{s}_k, \, s_{k+1}, \dots s_N)$ such that $\sigma_k(s_k) > 0$ for any $s_k < \bar{s}_k$. The curve of speeds $\sigma_k(s_k)$ is associated to the curve $T^k_{s_k}$ of the admissible states for the boundary free Riemann problem, whose description was recalled in Section 4.2.1.

4.3 A transversality lemma

The aim of this section is first to state and prove Lemma 4.9. It is a transversality result and, as it was anticipated in Section 4.2.3 and 4.2.4, it guarantees that the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation (4.1) is uniquely determined by the data $\bar{g}$ and $\bar{u}_0$. In the second part of this section it is introduced a quite general way of assigning the boundary condition on

$$\begin{cases} E(u)u_t + A(u, \, u_x)u_x = B(u)u_{xx} \\ ß(u(t, \, 0)) = g(t) \qquad u(0, \, x) = u_0(x). \end{cases} \tag{4.69}$$

More precisely, it is described a set of admissible functions ß which includes, in particular, the map ß introduced with Definition 2.1. Such an extension preserves the well-posedness of the parabolic IBVP and the property that if the initial datum and the boundary condition imposed on the parabolic approximation are assigned as constants, then the boundary Riemann solver induced in the hyperbolic limit is uniquely determined: in particular, this is guaranteed by Lemma 4.9.

Before stating Lemma 4.9, some of the notations introduced in Section 2.2 have to be recalled: let A_{11}, E_{11} be the blocks of A, E defined by (4.33). Let $\vec{\zeta}_i(u) \in \mathbb{R}^{N-r}$ be an eigenvector of $E^{-1}_{11}(u)A_{11}(u)$ associated to a non positive eigenvalue $\eta_i \leq 0$ and let $Z_i(u) \in \mathbb{R}^N$ be defined by

$$Z_i := \begin{pmatrix} \vec{\zeta}_i \\ 0 \end{pmatrix} \tag{4.70}$$

and

$$\mathcal{Z}(u) := \text{span}\langle Z_1(u), \dots, Z_{n_{11}+q}(u) \rangle.$$

We define the complementary subspace

$$\mathcal{W}(u) := \text{span}\Big\langle \begin{pmatrix} 0 \\ \vec{e}_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vec{e}_r \end{pmatrix}, \begin{pmatrix} \vec{w}_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \vec{w}_{N-r-n_{11}-q} \\ 0 \end{pmatrix} \Big\rangle,$$

where $\vec{e}_i \in \mathbb{R}^r$ are the vectors of a basis in $\mathbb{R}^r$ and $\vec{w}_j \in \mathbb{R}^{N-r}$ are the eigenvectors of $E_{11}^{-1}A_{11}$ associated to strictly positive eigenvalues. Recall that by Definition 2.1 the function ß(u) is the component u_1 of the vector u in the decomposition

$$u = u_1 + u_2, \quad u_1 \in \mathcal{W}(\bar{u}_0), u_2 \in \mathcal{V}(u). \tag{4.71}$$

This definition of function ß relies on the observation that the components of the vector $u(t, 0)$ which can be controlled by the boundary data are only the ones with a strictly positive speed in the hyperbolic part of (4.69).

Moreover, let

$$\mathcal{V}(u) := \text{span}\langle \Theta_{n_{11}+q+1}(u,\, 0), \dots, \Theta_{k-1}(u,\, 0), \Theta_k(u,\, 0),\, \Xi_{k+1}, \dots \Xi_N(u,\, 0)\rangle, \tag{4.72}$$

when the matrix A is singular. The vectors Θ_i and Ξ_j are as in (4.45), (4.47) and (4.48). The subspace $\mathcal{V}(u)$ is defined as

$$\mathcal{V}(u) := \text{span}\langle \Theta_{n_{11}+q+1}(u,\, 0), \dots, \Theta_n(u,\, 0),\, \Xi_{n+1}, \dots \Xi_N(u,\, 0)\rangle,$$

when A is invertible. With n it is denoted the number of negative eigenvalues of A, according to (3.11). Recalling the results of (4.53), (4.65), $\mathcal{V}(u)$ is the space generated by the derivative of the Riemann solver at $(s_{n_{11}+q+1} \dots s_N) = \vec{0}$.

The following lemma guarantees that, even if $\mathcal{W}(u)$ and $\mathcal{V}(u)$ do not coincide in general, nevertheless they are transversal to the same subspace $\mathcal{Z}(u)$.

Lemma 4.9. *For every $u \in B_\delta(\bar{u}_0)$ it holds:*

$$\mathcal{V}(u) \oplus \mathcal{Z}(u) = \mathbb{R}^N \qquad \mathcal{W}(u) \oplus \mathcal{Z}(u) = \mathbb{R}^N$$

Proof. The second part of the statement is trivial.

In the proof of the first part for simplicity it will be considered only the case A singular, being the invertible case absolutely analogous. Moreover, the proof is organized into four steps:

1. First, we prove that if V is a non zero vector in $\text{span}\langle \Xi_{k+1}, \dots, \Xi_N\rangle$, then

$$\langle V,\, AV\rangle > 0. \tag{4.73}$$

The passages are analogous to those in the proof of Lemma 7.1 in [4], but for completeness we will repeat them.

If $V \in \text{span}\langle \Xi_{k+1}, \dots, \Xi_N\rangle$, then V belongs to the unstable manifold of the linear system

$$u_x = Au$$

and hence there exists a solution of

$$\begin{cases} u_x = Au \\ u(0) = V \qquad \lim_{x\to-\infty} u(x) = 0. \end{cases}$$

Thanks to the symmetry of A, such a solution satisfies

$$\frac{d}{dx}\langle u,\, Au\rangle = 2\langle u_x,\, Au\rangle = 2|Au|^2 > 0$$

and hence to conclude it is enough to observe that

$$\lim_{x\to-\infty}\langle u(x),\, Au(x)\rangle = 0.$$

2. if
$$Z = \sum_{i=1}^{n_{11}+q} x_i Z_i,$$
then
$$\langle Z,\, AZ \rangle \leq 0. \tag{4.74}$$
Indeed, the matrices A_{11}, E_{11} are symmetric and E_{11} is positive definite, hence they admit an orthogonal basis of eigenvectors $\zeta_1 \dots \zeta_{N-r}$ such that $\langle \zeta_j,\, A_{11}\zeta_i \rangle = \eta_i \delta_{i\,j} \langle \zeta_i,\, E_{11}\zeta_i \rangle$. In particular,
$$\langle Z,\, AZ \rangle = \sum_{i=1}^{n_{11}+q} \eta_i x_i^2 \langle \zeta_i, E_{11}\zeta_i \rangle \leq 0.$$

3. if
$$\Theta = \sum_{j=n_{11}+q+1}^{k} y_j \Theta_j,$$
then
$$\langle A\Theta,\, \Theta \rangle \leq 0. \tag{4.75}$$
Indeed, by Lemma 4.4 one deduces that the following system admits a solution:
$$\begin{cases} Bu_x = Au \\ u(0) = \Theta \qquad \lim_{x \to +\infty} u(x) = 0. \end{cases}$$
Hence, by considerations analogous to those performed in the first step, one concludes that (4.75) holds true.

4. it holds
$$\mathrm{span}\langle Z_1, \dots, Z_{n_{11}+q}, \Theta_{n_{11}+q+1}, \dots, \Theta_k \rangle \cap \mathrm{span}\langle \Xi_{k+1}, \dots, \Xi_N \rangle = \{0\}. \tag{4.76}$$
To prove it, it is enough to show that if
$$Z \in \mathrm{span}\langle Z_1, \dots, Z_{n_{11}+q} \rangle, \qquad \Theta \in \mathrm{span}\langle \Theta_{n_{11}+q+1}, \dots, \Theta_k \rangle, \qquad \Big(Z + \Theta \Big) \neq 0,$$
then
$$\Big(Z + \Theta \Big) \notin \mathrm{span}\langle \Xi_{k+1}, \dots, \Xi_N \rangle.$$
Since
$$\langle AZ,\, \Theta \rangle = \langle Z,\, A\Theta \rangle = \langle Z,\, B\Theta \rangle = 0$$
and
$$\langle Z + \Theta,\, AZ + A\Theta \rangle = \langle Z,\, AZ \rangle + 2\langle Z,\, A\Theta \rangle + \langle \Theta,\, A\Theta \rangle,$$
from (4.74), (4.73) and (4.75) it follows that (4.76) holds true.

5. to conclude, it is enough to show that for every $i = 1, \dots, (n_{11} + q)$
$$Z_i \notin \mathrm{span}\langle \Theta_{n_{11}+q+1}(u,\, 0), \dots, \Theta_k(u,\, 0) \rangle. \tag{4.77}$$
Suppose by contradiction that
$$Z_i = \sum_{j=n_{11}+q+1}^{k} c_j \Theta_j \tag{4.78}$$
for suitable numbers $c_{n_{11}+q+1}, \dots c_k$. Because of Lemma 4.5, for every j, Θ_j has the structure described by (4.22), where $\vec{\xi}_j$ is an eigenvector of $\underline{b}^{-1}\underline{a}$. The matrices $\underline{b}$ and $\underline{a}$ have dimension $(r - q)$ and are defined by (4.21). Considering the last $(r - q)$ lines of the equality (4.78) one obtains
$$0 = \sum_j c_j \vec{\xi}_j,$$
which implies $c_j = 0$ for every j, because the $\vec{\xi}_j$ are all independent. Hence (4.78) cannot hold and (4.77) is proved.

$\square$

It is now possible to introduces a way of assigning the boundary condition on the parabolic IBVP 4.69 which generalizes Definition 2.1:

Definition 4.1. Let $\text{ß} : \mathbb{R}^N \mapsto \mathbb{R}^{N-n_{11}-q}$ be a sufficiently smooth function such that

$$\mathrm{Rank}(\nabla \text{ß}(\bar{u}_0)) = N - n_{11} - q,$$

and

$$\mathrm{span}\Big\{\mathrm{Ker}(\nabla \text{ß}(\bar{u}_0)), \mathcal{W}(\bar{u}_0)\Big\} = \mathrm{span}\Big\{\mathrm{Ker}(\nabla \text{ß}(\bar{u}_0)), \mathcal{V}(\bar{u}_0)\Big\} = \mathbb{R}^N, \tag{4.79}$$

where $\bar{u}_0 = \lim_{x\to 0^+} u_0(x)$.

Then the boundary datum is assigned as follows:

$$\text{ß}(u(t, 0+)) = g(t), \tag{4.80}$$

with $g(t)$ a given function $g : [0, +\infty[\to \mathbb{R}^{N-n_{11}-q}$.

By Hypothesis 4, the state u is close to $\bar{u}_0$ and hence, thanks to Lemma 4.9, the function ß introduced with Definition 2.1 satisfies (4.79) and Definition 4.1.

One can also prove that, if the function ß satisfies Definition 4.1, then it is possible to construct a local in time solution of problem (4.69) if the data are sufficiently smooth.

Moreover, it turns out that the whole construction of the boundary Riemann solver preserves its validity if ß is supposed to satisfy, instead of Definition 2.1, the generalization given by Definition 4.1.

More precisely, the meaning of condition (4.79) is to guarantee that the $\mathbb{R}^{N-n_{11}-1}$ dimensional level sets of $\text{ß}(u)$ are transversal to $\mathcal{W}(u)$ and to $\mathcal{V}(u)$, for u sufficiently close to $\bar{u}_0$. It follows, in particular, that for all values $g(t)$ there is a unique point $u_b(t)$ in $\mathcal{V}(u(t, 0+))$ such that (4.80) holds. The subspace $V(\bar{u}_0)$ is generated by the columns of the jacobian $D\psi$, computed in $(s_{n_{11}+q+1} \dots s_N) = \vec{0}$: the map ψ is defined by (4.52) when the boundary is non characteristic and by (4.64) in the boundary characteristic case. From the previous considerations it follows, in particular, that if ß satisfies (4.79), then the composition map $\text{ß} \circ \psi$ is locally invertible: in other words, if $|\text{ß}(\bar{u}_0) - \bar{g}|$ is sufficiently small, then the boundary Riemann solver induced in the hyperbolic limit by the parabolic approximation

$$\begin{cases} E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx} \\ \text{ß}(u(t,\, 0)) = \bar{g} \qquad u(0,\, x) = \bar{u}_0. \end{cases}$$

is uniquely and completely determined.

A Appendix

A.1 Proof of (3.8) and (3.9)

The center manifold theorem ensures that a center manifold can be parametrized by a map

$$\begin{aligned} \psi : V^c \qquad\qquad\qquad\qquad & \to \mathbb{R}^N \times \mathbb{R}^n \times \mathbb{R} \\ \Big(u,\, v_i r_i(u^+,\, 0),\, \sigma_i\Big) & \mapsto \Big(u,\, v(u,\, v_i,\, \sigma_i),\, \sigma_i\Big) \end{aligned}$$

Hence, the manifold is characterized by the relation

$$v = v_i r_i(u^+,\, 0) + \sum_{i \neq j} \psi_j(u,\, v_i,\, \sigma_i) r_i(u^+,\, 0), \tag{A.1}$$

where $r_1(u^+,\, 0) \dots r_N(u^+,\, 0)$ are the eigenvectors of $E^{-1}(u^+)A(u^+,\, 0)$ and they constitute a basis. Since all the equilibrium points $(u,\, 0,\, \sigma_i)$ lay on the center manifold, it turns out that when $v_i = 0$, then

$$\psi_j(u,\, 0,\, \sigma_i) = 0 \quad \forall j,\; u,\; \sigma_i.$$

Hence for all j, $\psi_j(u,\, v_i,\, \sigma_i) = v_i \phi_j(u,\, v_i,\, \sigma_i)$ for a suitable regular function ϕ_j. The relation (A.1) can be therefore rewritten as

$$v = v_i\Big(r_i(u^+,\, 0) + \sum_{j\neq i} \phi_j(u,\, v_i,\, \sigma_i) r_j(u^+,\, 0)\Big) := v_i \tilde{r}_i(u,\, v_i,\, \sigma_i).$$

Because of the tangency condition of the center manifold to the center space, it holds

$$\tilde{r}_i(u^+,\, 0,\, \sigma_i) = r_i(u^+,\, 0).$$

Inserting the expression found for v into (3.7), one gets

$$v_{ix} B(u)\tilde{r}_i + v_i^2 B(u) D\tilde{r}_i \tilde{r}_i + v_i v_{ix} B(u)\tilde{r}_{iv} = v_i\Big(A(u,\, v_i\tilde{r}_i) - \sigma_i E(u)\Big)\tilde{r}_i.$$

Considering the scalar product of the previous expression with $\tilde{r}_i$ one obtains

$$v_{ix}\Big(\langle \tilde{r}_i,\, B(u)\tilde{r}_i\rangle + v_i \langle \tilde{r}_i,\, B(u)\tilde{r}_{iv}\rangle\Big) = v_i\Big(\langle \tilde{r}_i,\, \Big(A(u,\, v_i\tilde{r}_i) - \sigma_i E(u)\Big)\tilde{r}_i\rangle - v_i\langle \tilde{r}_i,\, B(u)D\tilde{r}_i\tilde{r}_i\rangle\Big).$$

Hence setting

$$c_i(u,\, v_i,\, \sigma_i) := \langle \tilde{r}_i,\, B(u)\tilde{r}_i\rangle + v_i\langle \tilde{r}_i,\, B(u)\tilde{r}_{iv}\rangle \qquad a_i(u,\, v_i,\, \sigma_i) := \langle \tilde{r}_i,\, \Big(A(u,\, v_i\tilde{r}_i) - \sigma_i E(u)\Big)\tilde{r}_i\rangle - v_i\langle \tilde{r}_i,\, B(u)D\tilde{r}_i\tilde{r}_i\rangle$$

one can define

$$\phi_i(u,\, v_i,\, \sigma_i) := \frac{a_i(u,\, v_i,\, \sigma_i)}{c_i(u,\, v_i,\, \sigma_i)}.$$

The fraction is well defined since $c_i(u^+,\, 0,\, \lambda_i(u^+,\, 0)) \geq c_B(u^+) > 0$ and hence c_i is strictly positive in a small neighborhood. Hence

$$\begin{aligned}
\frac{\partial \phi_i}{\partial \sigma_i}\Big|_{\big(u^+,\, 0,\, \lambda_i(u^+,\, 0)\big)} &= -\frac{a_i}{c_i^2}\frac{\partial c_i}{\partial \sigma_i} + \frac{1}{c_i}\frac{\partial a_i}{\partial \sigma_i} \\
&= \frac{1}{c_i}\Big(\langle \tilde{r}_{i\sigma},\, \Big(A(u^+,\, 0) - \lambda_i E(u^+)\Big) r_i(u^+,\, 0)\rangle - \langle r_i(u^+,\, 0),\, E(u^+) r_i(u^+,\, 0)\rangle\Big) \\
&\quad + \frac{1}{c_i}\Big(\langle r_i(u^+,\, 0),\, \Big(A(u^+,\, 0) - \lambda_i E(u^+)\Big)\tilde{r}_{i\sigma}\rangle\Big) \\
&= -c_E(u^+) < 0.
\end{aligned}$$

In the previous computations, we have exploited the symmetry of $A(u^+,\, 0)$ and $E(u^+)$ and hence the fact that

$$\langle r_i(u^+,\, 0),\, \Big(A(u^+,\, 0) - \lambda_i E(u^+)\Big)\tilde{r}_{i\sigma}\rangle = \langle\Big(A(u^+,\, 0) - \lambda_i E(u^+)\Big) r_i(u^+,\, 0),\, \tilde{r}_{i\sigma}\rangle = 0.$$

Hence the system (3.7) reduced on the center manifold takes the form (3.8), where ϕ_i satisfies (3.9).

A.2 Proof of Lemma 3.3

Let

$$\|(u,\, v,\, \sigma)\| := \|u\|_\infty + \|v\|_\infty + \nu\|\sigma\|_\infty$$

a norm defined on $\mathcal{C}^0\big([0,\, s],\, \mathbb{R}^3\big)$: such a norm is equivalent to the standard one. The exact value of ν will be determined in the following. Then, let

$$X \subseteq \mathcal{C}^0\big([0,\, s],\, \mathbb{R}^3\big) \qquad X := \{(u(\tau),\, v(\tau),\, \sigma(\tau)) : \|(u - \bar{u}_k,\, v,\, \sigma)\| \leq \tilde{\delta}\}$$

for a sufficiently small $\tilde{\delta}$, whose exact value will be determined in the following. Since X is a close subspace of a Banach space, then it is complete with respect to the norm defined above. Let

$$T : X \to \mathcal{C}^0\big([0,\, s],\, \mathbb{R}^3\big)$$

a map defined by

$$\begin{cases} T_1(u,\,v,\,\sigma)(\tau) = \bar{u}_k + \displaystyle\int_0^\tau \tilde{r}_k(u(\xi),\,v(\xi),\,\sigma(\xi))d\xi \\ T_2(u,\,v,\,\sigma)(\tau) = f_k(\tau,\,u,\,v,\,\sigma) - \text{mon}f_k(\tau,\,u,\,v,\,\sigma) \\ T_3(u,\,v,\,\sigma)(\tau) = \dfrac{d}{d\tau}\text{mon}f_k(\tau,\,u,\,v,\,\sigma). \end{cases}$$

In the following we will show that $T(u,\,v,\,\sigma) \in X$ and that it is a contraction, s we will be able to conclude that it admits one fixed point, which is the unique solution of (3.19) belonging to X. First of all, we will recall some formulas that will be useful in the following: one can check that

$$\|\text{mon}f\|_\infty \leq \|f\|_\infty \qquad \left\|\frac{d}{d\tau}\text{mon}f\right\|_\infty \leq \left\|\frac{d}{d\tau}f\right\|_\infty \qquad \forall f \in \mathcal{C}^0([0,\,s],\,\mathbb{R})$$
$$|\text{mon}f - \text{mon}g|(\tau) \leq \text{mon}|f-g|(\tau) \qquad \forall\, f,\, g \in \mathcal{C}^0([0,\,s],\,\mathbb{R}).$$

and that

$$\left\|\frac{d}{d\tau}\text{mon}f - \frac{d}{d\tau}\text{mon}g\right\|_\infty \leq \left\|\frac{d}{d\tau}f - \frac{d}{d\tau}g\right\|_\infty$$

Moreover, it is useful to recall that in Section 3.2.1 f_k was defined by

$$f_k(\tau,\,v,\,\sigma) := \int_0^\tau \Big(\phi_k(u,\,v,\,\sigma) + \sigma_k(\xi)\Big)d\xi,$$

where ϕ_k satisfies (3.8). Since

$$\frac{\partial \phi_k}{\partial \sigma_k}\bigg|_{(u_k,\,0,\,\lambda_k(u,\,0))} = -1,$$

then one can chose ν in such a way that

$$\left|\frac{\partial(\phi_k + \sigma_k)}{\partial \sigma_k}\bigg|_{(u_k,\,0,\,\lambda_k(u,\,0))}\right| \leq \nu \tag{A.2}$$

in a small enough neighborhood of $(u_k,\,0,\,\lambda_k(u_k,\,0))$.

Thanks to the previous formulas, one can check by straightforward computations that when s and $\tilde{\delta}$ are small enough, $T(u,\,v,\,\sigma) \in X$ for any $(u,\,v,\,\sigma) \in X$. Moreover, for any $\tau \in [0,\,s]$

$$\begin{aligned}
\big|T_1(u^1,\,v^1,\,\sigma^1)(\tau) - T_1(u^2,\,v^2,\,\sigma^2)(\tau)\big| &\leq \mathcal{O}(1)\tau\big(\|u^1-u^2\|_\infty + \|v^1-v^2\|_\infty + \|\sigma^1-\sigma^2\|_\infty\big) \\
&\leq \mathcal{O}(1)\frac{s}{\nu}\|(u^1,\,v^1,\,\sigma^1) - (u^2,\,v^2,\,\sigma^2)\| \\
\big|T_2(u^1,\,v^1,\,\sigma^1)(\tau) - T_2(u^2,\,v^2,\,\sigma^2)(\tau)\big| &\leq 2|f_k(u^1,\,v^1,\,\sigma^1)(\tau) - f_k(u^2,\,v^2,\,\sigma^2)(\tau)| \\
&\leq \mathcal{O}(1)\tau\big(\|u^1-u^2\|_\infty + \|v^1-v^2\|_\infty + \|\sigma^1-\sigma^2\|_\infty\big) \\
&\leq \mathcal{O}(1)\frac{s}{\nu}\|(u^1,\,v^1,\,\sigma^1) - (u^2,\,v^2,\,\sigma^2)\| \\
\big|T_3(u^1,\,v^1,\,\sigma^1)(\tau) - T_3(u^2,\,v^2,\,\sigma^2)(\tau)\big| &\leq \|df_k(\tau,\,u^1,\,v^1,\,\sigma^1)/d\tau - df_k(\tau,\,u^2,\,v^2,\,\sigma^2)/d\tau\| \\
&\quad + \|d\,\text{mon}\,f_k(\tau,\,u^1,\,v^1,\,\sigma^1)/d\tau - d\,\text{mon}\,f_k(\tau,\,u^2,\,v^2,\,\sigma^2)/d\tau\| \\
&\leq 2\|df_k(u^1,\,v^1,\,\sigma^1)/d\tau - df_k(u^2,\,v^2,\,\sigma^2)/d\tau\|_\infty \\
&\leq 2\|\phi_k(u^1,\,v^1,\,\sigma^1) + \sigma^1 - \phi_k(u^2,\,v^2,\,\sigma^2) - \sigma^2\|_\infty \\
&\leq \mathcal{O}(1)\big(\|u^1-u^2\|_\infty + \|v^1-v^2\|_\infty + \nu\|\sigma^1-\sigma^2\|_\infty\big)
\end{aligned}$$

and hence the map T is a contraction in X.

As concerns the regularity of the fixed point maps $(u(\tau),\,v(\tau),\,\sigma(\tau))$, the first two maps are certainly Lipschitz being the integral of bounded functions. The Lipschitz regularity of the map $\sigma(\tau)$ can be proved observing that for every fixed h, it holds

$$\Big(\text{mon}f_k\Big)(\cdot + h) = \Big(\text{mon}f_k(\cdot + h)\Big)$$

and hence

$$\left|\frac{d\text{mon}f_k}{d\tau}(\tau+h)-\frac{d\text{mon}f_k}{d\tau}(\tau)\right|\le\left|\frac{df_k}{d\tau}(\tau+h)-\frac{df_k}{d\tau}(\tau)\right|\le\nu|h|$$

thanks to (A.2).

A.3 Proof of (3.21)

System (3.20) is equivalent to

$$\begin{cases} U_x = p \\ p_x = B^{-1}(p)A(U,\, p)p. \end{cases} \tag{A.3}$$

Linearizing around the equilibrium point $(\underline{u},\, 0)$, one finds that the Jacobian of the application is represented by the $2N \times 2N$ matrix

$$\begin{pmatrix} 0 & I_N \\ 0 & B^{-1}(\underline{u})A(\underline{u},\, 0) \end{pmatrix}$$

By construction the matrix $E^{-1}(\underline{u})A(\underline{u},\, 0)$ has $(k-1)$ strictly negative eigenvalues, the k-th eigenvalue equal to zero and the remaining $(N-k)$ eigenvalues are strictly positive. Lemma 3.1 ensures then that the same holds for the matrix $B^{-1}(\underline{u})A(\underline{u},\, 0)$:

$$\lambda_1(\underline{u},\, 0) < \dots < \lambda_{k-1}(\underline{u},\, 0) < 0 = \lambda_k(\underline{u},\, 0) < \lambda_{k+1}(\underline{u},\, 0) < \dots \lambda_N(\underline{u},\, 0).$$

Let $\vec{\xi_i}$ be the eigenvector associated to $\lambda_i(\underline{u},\, 0) < 0$, $i = 1, \dots k-1$ and $\vec{\xi_k}$ be the eigenvector of $A(\underline{u},\, 0)$ associated to the eigenvalue $\lambda_k = 0$. Then the stable space V^s of system (A.3) linearized around the equilibrium point $(\underline{u},\, 0)$ is

$$V^s = \{\sum_{i=1}^{k-1} s_i \vec{\xi_i} :\ s_i \in \mathbb{R}\}$$

while the center space V^c is

$$V^c = \{(w,\, s\vec{\xi_k}) :\ w \in \mathbb{R}^N,\, s \in \mathbb{R}\}.$$

The center-stable manifold is parametrized by the direct sum $V^s \oplus V^c$: moreover, if one fixes in $\mathbb{R}^N$ the basis defined by the eigenvectors of $B^{-1}(\bar{u})A(\bar{u},\, 0)$ and denotes by $(p_1, \dots p_N)$ the corresponding coordinates of p, the center-stable manifold is characterized by the relations

$$p_{k+i} = \psi_{k+i}(u,\, p_1 \dots p_k), \quad i = 1 \dots N-k.$$

Since the equilibrium points $(u,\, 0)$ certainly lay on the center-stable manifold, $\psi_{k+i}(u,\, 0 \dots 0) = 0$ for all $i = 1 \dots N-k$. Therefore there exists a k-dimensional vector $\varphi_{k+i}(u,\, p_1 \dots p_k)$ such that

$$\psi_{k+i}(u,\, p_1 \dots p_k) = \langle \varphi_{k+i}(u,\, p_1 \dots p_k),\, (p_1 \dots p_k) \rangle.$$

Hence the center-stable manifold is characterized by the relations

$$p = \left(\vec{\xi_1} | \dots | \vec{\xi_k}\right)\begin{pmatrix} p_1 \\ \dots \\ p_k \end{pmatrix} + \left(\vec{\xi}_{k+1} | \dots | \vec{\xi_N}\right)\begin{pmatrix} \varphi^t_{k+1}(u,\, p_1 \dots p_k) \\ \hline \dots \\ \hline \varphi^t_N(u,\, p_1 \dots p_k) \end{pmatrix}\begin{pmatrix} p_1 \\ \dots \\ p_k \end{pmatrix} \tag{A.4}$$

We will denote by $R^\sharp(u,\, p_1 \dots p_k) \in \mathbb{M}^{N \times k}$ the generalized eigenvector

$$R^\sharp(u,\, p_1 \dots p_k) = \left(\vec{\xi_1} | \dots | \vec{\xi_k}\right) + \left(\vec{\xi}_{k+1} | \dots | \vec{\xi_N}\right)\begin{pmatrix} \varphi^t_{k+1}(u,\, p_1 \dots p_k) \\ \hline \dots \\ \hline \varphi^t_N(u,\, p_1 \dots p_k) \end{pmatrix}.$$

It satisfies

$$R^\sharp(\bar{u},\, 0 \dots 0) = \left(\vec{\xi_1} | \dots | \vec{\xi_k}\right)$$

because of the tangency condition of the center-stable manifold to $V^s \oplus V^c$.

In the following we will denote the vector $(p_1, \dots p_k)$ with V^{cs}: it turns out that the center stable manifold is characterized by the relation $p = R^\sharp(u, V^{cs})V^{cs}$. Inserting this expression into system (A.3), one finds

$$\Big((D_u R^\sharp) R^\sharp V^{cs} + (D_v R^\sharp) V^{cs}_x\Big) V^{cs} + R^\sharp V^{cs}_x = B^{-1} A R^\sharp V^{cs}. \tag{A.5}$$

Moreover, one can check that the matrix

$$D(u, V^{cs}) = (d_{ih}) \qquad d_{ih} = \sum_{j=1}^{k} \frac{\partial R^\sharp_{ij}}{\partial V_h} V_j$$

satisfies

$$D(u, 0) = 0 \qquad \Big((D_v R^\sharp) V^{cs}_x\Big) V^{cs} = D(u, V^{cs}) V^{cs}_x.$$

Hence (A.5) rewrites as

$$\Big(R^\sharp + D(u, V^s)\Big) V^{cs}_x = \Big(B^{-1} A R^\sharp - (D_u R^\sharp) R^\sharp V^{cs}\Big) V^{cs}. \tag{A.6}$$

Since $R^\sharp(u, V^{cs}) + D(u, V^{cs})$ coincides in $u = \underline{u}$, $V^{cs} = 0$ with the matrix whose columns are $\vec{\xi}_1, \dots \vec{\xi}_k$, then columns of $R^\sharp(u, V^{cs}) + D(u, V^{cs})$ are independent in a small enough neighborhood. Let $L^\sharp(u, V^{cs})$ the matrix whose rows are the vectors of the dual basis, then multiplying (A.6) by $L^\sharp(u, V^{cs})$ one finds

$$V^{cs}_x = \Big(L^\sharp B^{-1} A R^\sharp - L^\sharp (D_u R^\sharp) R^\sharp V^{cs}\Big) V^{cs}.$$

The proof of (3.21) is then completed setting

$$\Lambda^{cs}(u, V^{cs}) := L^\sharp B^{-1} A R^\sharp - L^\sharp (D_u R^\sharp) R^\sharp V^{cs}.$$

It follows straightforwardly that $\Lambda^{cs}(\underline{u}, 0)$ is the diagonal matrix with eigenvalues $\lambda_1(\underline{u}, 0) \dots \lambda_k(\underline{u}, 0) = 0$.

A.4 Proof of (3.22)

The $N+k$-dimensional system (3.21) linearized around the equilibrium point $(\underline{u}, 0 \dots 0)$ is represented by the $(N+k) \times (N+k)$ Jacobian matrix

$$\begin{pmatrix} 0 & R^{cs}(\underline{u}, 0) \\ 0 & \Lambda^{cs}(\bar{u}, 0 \dots 0). \end{pmatrix}.$$

Since $\Lambda^{cs}(\bar{u}, 0 \dots 0)$ is the diagonal matrix with eigenvalues $\lambda_1(\underline{u}, 0) \dots \lambda_k(\underline{u}, 0) = 0$, the center and the stable spaces are respectively

$$\mathcal{V}^c = \{(u, v_k \vec{e}_k) : \ u \in \mathbb{R}^N, \ v_k \in \mathbb{R}\} \qquad \mathcal{V}^s = \{(0, \sum_{j=1}^{k-1} v_j \vec{e}_j) : \ v_1 \dots v_{k-1} \in \mathbb{R}\},$$

where $\vec{e}_1 \dots \vec{e}_k$ are the vectors of the canonical basis in $\mathbb{R}^k$. Moreover, the manifold

$$E := \{(u, 0) : \ |u - \underline{u}| < \hat{\delta}\}$$

is entirely constituted by equilibria, hence there exits a *uniformly stable manifold*, which is parametrized by $E \times \mathcal{V}^s$, which contains all the orbits that when $x \to +\infty$ converge with exponential speed to some point is E. The existence of the uniformly stable manifold is implied by Hadamard-Perron theorem: it is made reference to [29] for a general presentation of Hadamard-Perron theorem and to [3] for the applications to the study of the uniformly stable manifold.

In the following, system (3.21) will be decomposed along its center and its uniformly stable manifold, in a sense that will be made more precise soon. For simplicity, the remaining part of the construction is organized into three steps:

1. *construction of the uniformly stable manifold of system* (3.21)
   The uniformly stable manifold is parametrized by a map

$$\begin{aligned} \psi : E \times \mathcal{V}^s & \to \mathbb{R}^N \times \mathbb{R}^k \\ (u,\, v_1 \dots v_{k-1}) & \mapsto \Big(\psi_1(u,\, V^s),\, \psi_2(u,\, V^s) \Big), \end{aligned}$$

   where $V^s = (v_1 \dots v_{k-1})^T$. If π_s denotes the projection on the stable space, then the map ψ satisfies

$$\pi_s \Big(\psi(u,\, V^s) \Big) = V^s \qquad \psi(u,\, 0) = (u,\, 0)$$

   and ψ_2 is tangent to V^s in $(\underline{u},\, 0)$.
   Let $\bar{R}^s \in \mathbb{M}^{k \times (k-1)}$ be the matrix whose columns are $\vec{e}_1 \dots \vec{e}_{k-1}$. Then on the center uniformly stable manifold it holds

$$V^{cs} = \psi_2(u,\, V^s) = \bar{R}^s V^s + \vec{e}_k \phi_k(u,\, V^s).$$

   Since $\psi(u,\, 0) = (u,\, 0)$, then

$$\phi_k(u,\, V^s) = \langle \varphi_k(u,\, V^s),\, V^s \rangle$$

   for a suitable vector function $\varphi_k \in \mathbb{R}^{k-1}$. Hence on the uniformly stable manifold it holds

$$V^{cs} = \psi_2(u,\, V^s) = \Big(\bar{R}^s + \vec{e}_k \varphi_k(u,\, V^s) \Big) := \hat{R}^s(u,\, V^s) V^s.$$

   Because of the tangency condition,

$$\hat{R}^s(\underline{u},\, 0) = \bar{R}^s.$$

   In the following, it will be useful to have an explicit expression for the product $\Lambda^{cs} \hat{R}^s V^s$. To compute it, one can observe that considering system (3.21) reduced on the uniformly stable manifold one finds

$$\Big((D_u \hat{R}^s) R^{cs} \hat{R}^s V^s \Big) V^s + \Big(D(u,\, V^s) + \hat{R}^s \Big) V^s_x = \Lambda^{cs} \hat{R}^s V^s, \tag{A.7}$$

   where the matrix $D(u,\, V^s) \in \mathbb{M}^{k \times (k-1)}$ is defined by

$$D = (d_{ih}) \qquad d_{ih} = \sum_{j=1}^{k-1} \frac{\partial \hat{R}^s_{ij}}{\partial V^s_h} V^s_j$$

   and satisfies

$$\Big((D_v \hat{R}^s) V^s_x \Big) V^s = D(u,\, V^s) V^s_x \qquad D(u,\, 0) = 0.$$

   Moreover, the columns of the matrix $D(u,\, V^s) + \hat{R}^s(u,\, V^s)$ are all independent one from each other in a small enough neighborhood of $(\underline{u},\, 0)$: that's because in this equilibrium point they coincide with $\vec{e}_1 \dots \vec{e}_{k-1}$. Let $L^s(u,\, V^s) \in \mathbb{M}^{(k-1) \times k}$ be the matrix satisfying

$$L^s(u,\, V^s) \big(D(u,\, V^s) + \hat{R}^s(u,\, V^s) \big) = I_{k-1}.$$

   Then multiplying (A.7) by $L^s(u,\, V^s)$ one finds

$$V^s_x = L^s \Big(\Lambda^{cs} \hat{R}^s - (D_u \hat{R}^s) R^{cs} \hat{R}^s V^s \Big) V^s := \hat{\Lambda}^s(u,\, V^s,\, V^{cs}) V^s, \tag{A.8}$$

   where $\hat{\Lambda}(u,\, V^s) \in \mathbb{M}^{(k-1) \times (k-1)}$ coincides in $(\underline{u},\, 0)$ with the diagonal matrix whose eigenvalues are $\lambda_1(\underline{u},\, 0) \dots \lambda_{k-1}(\underline{u},\, 0)$. Hence substituting (A.8) in (A.7) one finds

$$\Lambda^{cs} \hat{R}^s V^s = \Big((D_u \hat{R}^s) R^{cs} \hat{R}^s V^s \Big) V^s + \Big(D(u,\, V^s) + \hat{R}^s \Big) \hat{\Lambda}^s V^s. \tag{A.9}$$

2. *construction of the center manifold of system* (3.21)
The center manifold is parametrized by V^c and making the considerations analogous (but easier) to those performed in Appendix A.3, it turns out that it is characterized by the relation

$$V^{cs} = v_k \hat{r}_k(u,\, v_k)$$

for a suitable vector function $\hat{r}_k$ coinciding with $(\underline{u},\, 0)$ with $\vec{e}_k$. Then with the same computations as before one finds that

$$\Lambda^{cs}\hat{r}_k v_k = \Big(\hat{r}_k + v_k \hat{r}_{kv}\Big)\hat{\lambda}_k(u,\, v_k)v_k + (D_u\hat{r}_k)R^{cs}\hat{r}_k v_k^2, \tag{A.10}$$

where

$$\hat{\lambda}_k(u,\, v_k,\, V^{cs}) = \hat{l}_k(u,\, v_k)\Big(\Lambda^{cs}\hat{r}_k - (D_u\hat{r}_k)R^{cs}\hat{r}_k v_k\Big)$$

coincides with 0 in $(\underline{u},\, 0)$. In the previous expression, the vector $\hat{l}_k(u,\, v_k)$ is defined by the relation

$$\hat{l}_k(u,\, v_k)\Big(\hat{r}_k + v_k\hat{r}_{kv}\Big) = 1.$$

3. *decomposition of system* (3.21) *with respect to the stable and the center component*
The decomposition of system (3.21) along its center and its uniformly stable manifold is introduced imposing

$$V^{cs} = \hat{r}_k(u,\, v_k)v_k + \hat{R}^s(u,\, V^s)V^s,$$

where $\hat{r}_k$ and $\hat{R}^s$ are the matrices introduced before. Inserting this decomposition into the second line of system (3.21) and exploiting formulas (A.7) and (A.10), one obtains

$$\begin{aligned}&\Big(\hat{r}_k + \hat{r}_{kv}v_k\Big)v_{kx} + \Big(\hat{R}^s + D(u,\, V^s)\Big)V^s_x + \Big((D_u\hat{r}_k)R^{cs}\hat{R}^s\Big)V^s v_k + \Big((D_u\hat{R}^s)R^{cs}\hat{r}_k\Big)V^s v_k\\ &= \Big(\hat{r}_k + \hat{r}_{kv}v_k\Big)\hat{\lambda}_k v_k + \Big(\hat{R}^s + D(u,\, V^s)\Big)\hat{\Lambda}^s V^s.\end{aligned} \tag{A.11}$$

When $u = \underline{u}$, $v_k = 0$ and $V^s = 0$, then

$$\hat{r}_k + \hat{r}_{kv}v_k = \vec{e}_k$$

and $\hat{R}^s + D(u,\, V^s)$ coincides with the matrix whose columns are the vectors $\vec{e}_1 \dots \vec{e}_{k-1}$. Hence $\hat{r}_k + \hat{r}_{kv}v_k$ and the columns of $\hat{R}^s + D(u,\, V^s)$ are all independent in a small enough neighborhood. Let $\hat{\ell}_1 \dots \hat{\ell}_k$ be the dual base and $\hat{L}^s \in \mathbb{M}^{(k-1)\times k}$ be the matrix whose rows are $e\hat{l}l_1 \dots \hat{\ell}_{k-1}$. Then multiplying (A.11) on the left by $\hat{L}^s$ and $\hat{\ell}_k$ one obtains respectively

$$V^s_x = \Big(\hat{\Lambda}^s - \hat{L}^s(D_u\hat{r}_k)R^{cs}\hat{R}^s v_k - \hat{L}^s(D_u\hat{R}^s)R^{cs}\hat{r}_k v_k\Big)V^s := \breve{\Lambda}^s(u,\, V^s,\, v_k)V^s$$

and

$$v_{kx} = \Big(\hat{\lambda}_k - \hat{\ell}_k(D_u\hat{r}_k)R^{cs}\hat{R}^s V^s - \hat{\ell}^k(D_u\hat{R}^s)R^{cs}\hat{r}_k V^s\Big)v_k := \breve{\lambda}_k(u,\, V^s,\, v_k)v_k,$$

with $\breve{\Lambda}^s(u,\, V^s,\, v_k) \in \mathbb{M}^{(k-1)\times(k-1)}$, $\breve{\lambda}_k(u,\, V^s,\, v_k) \in \mathbb{R}$ and

$$\breve{\Lambda}^s(\underline{u},\, 0,\, 0) = \mathrm{diag}\Big(\lambda_1(\underline{u},\, 0), \dots \lambda_{k-1}(\underline{u},\, 0)\Big) \qquad \breve{\lambda}_k(\underline{u},\, 0,\, 0) = 0.$$

Finally, the proof of (3.22) is completed observing that the first line of system (3.21) can be rewritten as

$$u_x = R^{cs}\Big(\hat{r}_k v_k + \hat{R}^s V^s\Big) = \breve{r}_k v_k + \breve{R}^s V^s$$

provided that

$$\breve{r}_k(u,\, v_k,\, V^s) := R^{cs}(u,\, v_k,\, V^s)\hat{r}_k(u,\, v_k) \qquad \breve{R}^s(u,\, v_k,\, V^s) := R^{cs}(u,\, v_k,\, V^s)\hat{R}^s(u,\, V^s).$$

One can then obtain exactly (3.22) setting

$$u_{kx} = \breve{r}_k v_k \qquad u_{sx} = \breve{R}^s V^s \qquad u_k + u_s = u.$$

A.5 Proof of (4.27)

The construction is actually analogous to that performed in Appendix A.1, but there are some more technicalities due to the singularity of the matrix B: in the following, however, we will only sketch the most important points.

With the same passages as in Appendix A.1 one gets that the center manifold of system (4.2) is characterized by the relation

$$\tilde{z} = v_i \vec{\xi_\sharp} i(u,\, v_i,\, \sigma_i)$$

for a suitable vector function $\vec{\xi}_i^\sharp$ satisfying $\vec{\xi}_i^\sharp(u^+,\, 0,\, \lambda_i(u^+,\, 0)) = \vec{\xi}_i(u^+,\, 0)$. Let

$$\Xi_i(u,\, v_i,\, \sigma_i) := \Big(\bar{w}(u,\, v_i \vec{\xi}_i^\sharp,\, \sigma_i),\ \tilde{w}(u,\, v_i \vec{\xi}_i^\sharp,\, \sigma_i),\ \bar{z}(u,\, v_i \vec{\xi}_i^\sharp,\, \sigma_i),\ v_i \vec{\xi}_i^\sharp \Big)^T$$

Since $\bar{w}(u,\, 0,\, \sigma) = \tilde{w}(u,\, 0,\, \sigma) = \bar{z}(u,\, 0,\, \sigma) = 0$, then

$$\Xi_i(u,\, v_i,\, \sigma_i) = v_i \Xi_i^\sharp(u,\, v_i,\, \sigma_i)$$

for a suitable vector function $\Xi_i^\sharp \in \mathbb{R}^N$. Moreover, eventually performing a suitable change of variable, one can suppose that $|\Xi_i^\sharp(u,\, v_i,\, \sigma_i)| = 1$. Inserting the expression

$$u_x = v_i \Xi_i^\sharp(u,\, v_i,\, \sigma_i)$$

into system

$$B(u) u_{xx} = A(u,\, u_x,\, \sigma) u_x$$

one finds that

$$c_i(u,\, v_i,\, \sigma_i) v_{ix} = a_i(u,\, v_i,\, \sigma_i) v_i,$$

where

$$c_i(u,\, v_i,\, \sigma_i) = \langle \Xi_i^\sharp(u,\, v_i,\, \sigma_i),\, B(u) \Xi_i^\sharp(u,\, v_i,\, \sigma_i) \rangle \qquad a_i(u,\, v_i,\, \sigma_i) = \langle \Xi_i^\sharp(u,\, v_i,\, \sigma_i),\, A(u,\, v_i \Xi_i^\sharp,\, \sigma_i) \Xi_i^\sharp(u,\, v_i,\, \sigma_i) \rangle.$$

With the same considerations as in Appendix A.1 one deduce that $c_i(u,\, v_i,\, \sigma_i) > 0$ and that, setting

$$\phi_i(u,\, v_i,\, \sigma_i) := \frac{a_i(u,\, v_i,\, \sigma_i)}{c_i(u,\, v_i,\, \sigma_i)},$$

it holds

$$\frac{\partial \phi_i}{\partial \sigma_i} \bigg|_{u = u^+,\, v_i = 0,\, \sigma_i = \lambda_i(u^+,\, 0)} < 0.$$

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