RENORMALIZATION FOR AUTONOMOUS NEARLY INCOMPRESSIBLE BV VECTOR FIELDS IN 2D

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Abstract. Given a bounded autonomous vector field $b\colon \mathbb{R}^d \to \mathbb{R}^d$, we study the uniqueness of bounded solutions to the initial value problem for the related transport equation

$$\partial_t u + b \cdot \nabla u = 0.$$

We are interested in the case where b is of class BV and it is nearly incompressible. Assuming that the ambient space has dimension $d = 2$, we prove uniqueness of weak solutions to the transport equation. The starting point of the present work is the result which has been obtained in [7] (where the *steady* case is treated). Our proof is based on splitting the equation onto a suitable partition of the plane: this technique was introduced in [3], using the results on the structure of level sets of Lipschitz maps obtained in [1]. Furthermore, in order to construct the partition, we use Ambrosio's superposition principle [4].

Keywords: transport equation, continuity equation, renormalization, disintegration of measures, Lipschitz functions, Superposition principle.

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1. Introduction and notation

In this paper we consider the *continuity equation*

$$\partial_t u + \operatorname{div}(ub) = 0 \tag{1.1}$$

and the *transport equation*

$$\partial_t u + b \cdot \nabla u = 0, \tag{1.2}$$

for a scalar field $u\colon I \times \mathbb{R}^d \to \mathbb{R}$ (where $I = (0,T)$, $T > 0$) with a vector field $b\colon I \times \mathbb{R}^d \to \mathbb{R}^d$. We study the initial value problems for these equations with the same initial condition

$$u(0,\cdot) = \overline{u}(\cdot), \tag{1.3}$$

where $\bar{u}\colon \mathbb{R}^d \to \mathbb{R}$ is a given scalar field.

Our aim is to investigate uniqueness of weak solutions to (1.1), (1.3) (and to (1.2), (1.3)) under weak regularity assumptions on the vector field b.

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When $b \in L^\infty(I \times \mathbb{R}^d)$ then (1.1) is understood in the standard sense of distributions: $u \in L^\infty(I \times \mathbb{R}^d)$ is called a *weak solution* of the continuity equation if (1.1) holds in $\mathscr{D}'(I \times \mathbb{R}^d)$. One can prove (see e.g. [12]) that, if u is a weak solution of (1.1), then there exists a map $\widetilde{u} \in L^\infty([0,T] \times \mathbb{R}^d)$ such that $u(t,\cdot) = \widetilde{u}(t,\cdot)$ for a.e. $t \in I$ and $t \mapsto \widetilde{u}(t,\cdot)$ is weakly* continuous from $[0,T]$ into $L^\infty(\mathbb{R}^d)$. This allows us to prescribe an initial condition (1.3) for a weak solution u of the continuity equation in the following sense: we say that $u(0,\cdot) = \bar{u}(\cdot)$ holds if $\widetilde{u}(0,\cdot) = \bar{u}(\cdot)$.

Definition of weak solutions of the transport equation (1.2) is slightly more delicate. If the divergence of b is absolutely continuous with respect to the Lebesgue measure then (1.2) can be written as

$$\partial_t u + \operatorname{div}(ub) - u \operatorname{div} b = 0,$$

and the latter equation can be understood in the sense of distributions (see e.g. [13] for the details). We are interested in the case when $\operatorname{div} b$ is not absolutely continuous. In this case the notion of weak solution of (1.2) can be defined for the class of *nearly incompressible vector fields*.

Definition 1.1. A bounded, locally integrable vector field $b \colon I \times \mathbb{R}^d \to \mathbb{R}^d$ is called *nearly incompressible* if there exists a function $\rho \colon I \times \Omega \to \mathbb{R}$ (called *density* of b) such that $\ln(\rho) \in L^\infty(I \times \Omega)$ and

$$\partial_t \rho + \operatorname{div}(\rho b) = 0 \qquad \text{in } \mathscr{D}'(I \times \Omega). \tag{1.4}$$

Nearly incompressible vector fields were introduced in connection with the hyperbolic conservation laws, namely, the Keyfitz-Kranzer system [16]. See e.g. [12] for the details. Using mollification one can prove that if $\operatorname{div} b \in L^\infty(I \times \mathbb{R}^d)$ then b is nearly incompressible. The converse implication does not hold, so near incompressibility can be considered as a weaker version of the assumption $\operatorname{div} b \in L^\infty(I \times \mathbb{R}^d)$.

Definition 1.2. Let b be a nearly incompressible vector field with density ρ. We say that a function $u \in L^\infty(I \times \mathbb{R}^d)$ is a *(ρ–)weak solution* of (1.2) if

$$(\rho u)_t + \operatorname{div}(\rho u b) = 0 \quad \text{in } \mathscr{D}'(I \times \mathbb{R}^2).$$

Thanks to Definition 1.2 one can prescribe the initial condition for a ρ–weak solution of the transport equation similarly to the case of the continuity equation, which we mentioned above (see [12] for the details).

Existence of weak solutions to initial value problem for transport equation with a nearly incompressible vector field can be proved by a standard regularization argument [12]. The problem of *uniqueness* of weak solutions is much more delicate. The theory of uniqueness in the non-smooth framework has started with the seminal paper of R.J. DiPerna and P.-L. Lions [13] where uniqueness was obtained as a corollary of so-called *renormalization property* for the vector fields with Sobolev regularity. Thanks to Definition 1.2 the renormalization property can be defined also for nearly incompressible vector fields:

Definition 1.3. We say that a nearly incompressible vector field b with density ρ has the *renormalization property* if for every ρ–weak solution $u \in$

$L^\infty(I \times \mathbb{R}^2)$ of (1.2) and any function $\beta \in C^1(\mathbb{R})$ the function $\beta(u)$ also is a ρ-weak solution of (1.2), i.e. it satisfies

$$\partial_t (\rho\beta(u)) + \operatorname{div}(\rho\beta(u)b) = 0 \quad \text{in } \mathscr{D}'(I \times \mathbb{R}^2).$$

Nearly incompressible vector fields are related to a conjecture, made by A. Bressan in [10]. In particular, it has been proved in [5] that Bressan's conjecture would follow from the following one:

Conjecture 1.4. *Any bounded, nearly incompressible vector field $b \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^d)$ has the renormalization property in the sense of Definition 1.3.*

The renormalization property can also be generalized for the systems of transport equations. Moreover, if η is another density of the nearly incompressible vector field b and b has the renormalization property with the density ρ, then any ρ–weak solution of (1.2) is also an η–weak solution and vice versa. In other words, the property of being a ρ–weak solution does not depend on the choice of the density ρ provided that renormalization holds. We refer to [12] for the details.

If the functions ρ, u and b were smooth, renormalization property would be an easy corollary of the chain rule. Out of the smooth setting, the validity of this property is a key step to get uniqueness of weak solutions. Indeed, if we for simplicity consider $\mathbb{T}^d$ instead of $\mathbb{R}^d$, then integrating the equation above over the torus we get

$$\partial_t \int_{\mathbb{T}^d} \rho\beta(u)\,dx = 0.$$

So if $\bar{u} = 0$ then for $\beta(y) = y^2$ we get

$$\int_{\mathbb{T}^d} \rho(t,x)u^2(t,x)\,dx = 0$$

for a.e. t which implies $u(t,\cdot) = 0$ for a.e. t.

The problem of uniqueness of solutions is thus shifted to prove the renormalization property for b: in [13] the authors proved that renormalization property holds under Sobolev regularity assumptions; some years later, L. Ambrosio [4] improved this result, showing that renormalization holds for vector fields which are of class BV (locally in space) and have absolutely continuous divergence.

Another approach giving explicit compactness estimates has been introduced in [11], and further developed in [9, 15]: see also the references therein.

In the two dimensional autonomous case the problem of uniqueness is addressed in the papers [3], [1] and [7]. Indeed, in two dimensions and for divergence-free autonomous vector fields, renormalization theorems are available even under mild assumptions, because of the underlying Hamiltonian structure. In [3], the authors characterize the autonomous, divergence-free vector fields b on the plane such that the Cauchy problem for the continuity equation (1.1) admits a unique bounded weak solution for every bounded initial datum (1.3). The characterization they present relies on the so called *Weak Sard Property*, which is a (weaker) measure theoretic version of Sard's Lemma. Since the problem admits a Hamiltonian potential, uniqueness is proved following a strategy based on splitting the equation on

the level sets of this function, reducing thus to a one-dimensional problem. This approach requires a preliminary study on the structure of level sets of Lipschitz maps defined on $\mathbb{R}^2$, which is carried out in the paper [1].
Finally, in [7] the *steady nearly incompressible* case is treated: these vector fields constitute a proper subset of nearly incompressible ones but the results obtained in [7] are the starting points of this work. Furthermore, we mention that the problem of renormalization is also related to the problem of locality of divergence operator and to the chain rule problem (see again [7]).

The main result of this paper is a partial answer to the Conjecture 1.4:

Main Theorem. *Every bounded, autonomous, nearly incompressible* BV *vector field on the two dimensional torus* $\mathbb{T}^2$ *has the renormalization property.*

1.1. **Structure of the paper.** The proof of the Main Theorem can be divided into two parts.

The *first part* (presented in Sections 2-5) is based on a local argument, which is a generalization of the argument from the case when the density ρ is steady [7]. In this case, since $\operatorname{div}(\rho b) = 0$, there exists a Lipschitz *Hamiltonian* $H\colon \mathbb{T}^2 \to \mathbb{R}$ such that

$$\rho b = \nabla^\perp H,$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. This allows us to split an equation of the form

$$\operatorname{div}(ub) = \mu, \qquad u\colon \mathbb{T}^2 \to \mathbb{R} \tag{1.5}$$

where μ is a measure on $\mathbb{T}^2$, into a equivalent family of equations along the level sets of H, similar to [7]. This is done in Section 3, where we also recall the main results of [1, 3] and adapt them to our setting. In Section 5.3 we establish so-called Weak Sard Property for the Hamiltonian H.

In the general nearly incompressible case it is not possible to construct the Hamiltonian H directly as in the case of steady density. So in the *second part* we reduce the problem to the steady case using the following argument. Suppose that a nonnegative bounded function ϱ solves the continuity equation

$$\varrho_t + \operatorname{div}(\varrho b) = 0,$$

$t \mapsto \varrho(t, \cdot)$ is weak* continuous and for some open set Ω and $t_{1,2} \in [0, T]$ we have $\varrho(t_1, \cdot) = \varrho(t_2, \cdot) = 0$ a.e. on Ω. Integrating the continuity equation with respect to time on $[t_1, t_2]$ it is easy to see that

$$r(x) := \int_{t_1}^{t_2} \varrho(t, x)\, dt$$

solves

$$\operatorname{div}(rb) = 0$$

in $\mathscr{D}'(\Omega)$. Therefore in Ω one can construct a *local Hamiltonian* H_Ω such that

$$rb = \nabla^\perp H_\Omega$$

in Ω. Then we are in a position to apply the results of the first part.

It is not obvious that a nontrivial function ϱ with the properties stated above exists. Moreover, a single function of this kind can vanish on a large set and therefore may not provide all the required information. In this paper we construct a *countable family* of the functions ϱ from the (non-steady) density ρ using Ambrosio's superposition principle (Sections 2 and 2.2). We prove that the level sets of the corresponding local Hamiltonians agree if they intersect, and cover the set

$$M^c := \mathbb{T}^2 \setminus M,$$

where $M := \{b = 0\}$. By "gluing" together these level sets in Sections 7 and 7.3 we construct a partition of M^c into an uncountable disjoint family $\{F_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}\setminus\{+\infty\}}$ of simple (possibly closed) curves $F_{\mathfrak{a}}$ which can be parametrized in a *canonical* way by the solutions of the ODE $\dot\gamma = b(\gamma)$.

In Section 6 we prove that the divergence is *local* a sense that the measure μ in (1.5) vanishes on the set M (Proposition 6.2).

Finally, using locality of the divergence, we prove that either the curves $F_{\mathfrak{a}}$, $\mathfrak{a} \neq +\infty$, are periodic or the domain of their canonical parametrization is the whole real line (Section 8).

Using Disintegration Theorem we reduce the equation (1.5) to an equivalent family of equations along the curves from the family $\{F_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}\setminus\{+\infty\}}$, (Proposition 9.6 of Section 9) and, passing to injective Lipschitz parametrizations, we obtain a one-dimensional problem which can be solved explicitly, concluding in Section 9.1 the proof of the **Main Theorem** (Theorem 9.8).

1.2. **Notation.** Throughout the paper, we use the following notation:

- (X, d) is a metric space;
- $\mathbb{1}_E$ is the characteristic function of the set $E \subset X$, defined as $\mathbb{1}_E(x) = 1$ if $x \in E$ and $\mathbb{1}_E(x) = 0$ otherwise;
- $\mathrm{dist}(x, E)$ is the distance of x from the set E, defined as the infimum of $d(x, y)$ as y varies in E;
- $\mathrm{dist}(E_1, E_2)$ is the distance between the sets E_1 and E_2, defined as the infimum of the distances $d(x_1, x_2)$, for all $x_1 \in E_1$, $x_2 \in E_2$;
- Ω denotes in general an open set in $\mathbb{R}^d$;
- $B(x, r)$ or, equivalently, $B_r(x)$ is the open ball in $\mathbb{R}^d$ with radius r and centre x; $B(r)$ is the open ball in $\mathbb{R}^d$ with radius r and centre 0;
- $⨍_E f\,d\mu$ denotes the *average* of the function f over the set E with respect to the positive measure μ, that is
$$⨍_E f\,d\mu := \frac{1}{\mu(E)} \int_E f\,d\mu,$$
- $|\mu|$ is the total variation of a measure μ;
- μ^{sing} the singular component of μ with respect to the Lebesgue measure;
- $\mathscr{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and $\mathscr{H}^k$ is the k-dimensional Hausdorff measure;
- $\mathrm{Lip}(X)$ is the space of real-valued Lipschitz functions; $\mathrm{Lip}_c(X)$ is the space of real-valued compactly supported Lipschitz functions;
- $C^\infty_c(\Omega)$ is the space of smooth compactly supported functions, also called *test functions*;
- $\mathrm{BV}(\Omega)$ set of functions with bounded variation;
- $\mathscr{D}'(\Omega)$ is the space of distributions on the open set Ω;

- $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the two dimensional torus;
- $\Gamma := C([0,T];\mathbb{T}^2)$ will denote the set of continuous curves in $\mathbb{T}^2$;
- $\dot{\Gamma} := \{\gamma \in \Gamma : \gamma(t) = \gamma(0),\ \forall t \in [0,T]\}$;
- $\widetilde{\Gamma} := \Gamma \setminus \dot{\Gamma}$;
- $e_t \colon \Gamma \to \mathbb{T}^2$ is the *evaluation map* at time t, i.e. $e_t(\gamma) = \gamma(t)$.

Moreover, if $A \subset \mathbb{T}^2$ is a measurable set,

- $\Gamma_A := \{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0\}$;
- $\widetilde{\Gamma}_A := \widetilde{\Gamma} \cap \Gamma_A$;
- $\dot{\Gamma}_A := \dot{\Gamma} \cap \Gamma_A$.
- for every $s \in [0,T]$, we set

$$\begin{aligned}\Gamma_A^s &:= \{\gamma \in \Gamma : \gamma(s) \in A\},\\ \widetilde{\Gamma}_A^s &:= \Big\{\gamma \in \widetilde{\Gamma} : \gamma(s) \in A\Big\},\\ \dot{\Gamma}_A^s &:= \Big\{\gamma \in \dot{\Gamma} :\ \gamma(s) \in A\Big\}.\end{aligned}$$

If $E \subseteq \mathbb{R}^2$, we denote by

$$\begin{aligned}\mathrm{Conn}(E) &:= \Big\{C \subset E : C \text{ is a connected component of } E\Big\},\\ \mathrm{Conn}^\star(E) &:= \Big\{C \in \mathrm{Conn}(E) : \mathscr{H}^1(C) > 0\Big\},\end{aligned}$$

and

$$E^\star := \bigcup_{C \in \mathrm{Conn}^\star(E)} C.$$

When the measure is not specified, it is assumed to be the Lebesgue measure, and we often write

$$\int f(x)\,dx$$

for the integral of f with respect to $\mathscr{L}^d$.

Let μ be a Radon measure on a metric space X. Let Y be a metric space Y and a let $f \colon X \to Y$ be a Borel function. We denote by $f_\# \mu$ the *image measure* of μ under the map f. In particular, for any $\varphi \in C_c(Y)$ we have

$$\int_X \varphi(f(x))\,d\mu(x) = \int_Y \varphi(y)\,d(f_\#\mu)(y).$$

Let ν be a Radon measure on Y such that $f_\#|\mu| \ll \nu$. According to the Disintegration Theorem (Theorem 2.28 of [6] or for the most general statement Section 452 of [14]) there exists a unique measurable family of Radon measures $\{\mu_y\}_{y\in Y}$ such that for ν-a.e. $y \in Y$ the measure μ_y is concentrated on the level set $f^{-1}(y)$ and

$$\mu = \int_y \mu_y\,d\nu(y),$$

that is, for any $\phi \in C_c(X)$

$$\int_X \varphi(x)\,d\mu(x) = \int_Y \left(\int_X \varphi(x)\,d\mu_y(x)\right)\,d\nu(y).$$

The family $\{\mu_y\}_{y\in Y}$ is called the *disintegration of* μ *with respect to* f *(and* ν*).*

2. Setting of the problem

2.1. **Ambrosio's Superposition Principle.** In [4], L. Ambrosio proved the *Superposition Principle*. Since we will use it later on in this section, we present here the statement. Let us consider the continuity equation in the form

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(b\mu_t) = 0, \\ \mu_0 = \overline{\mu}, \end{cases} \tag{2.1}$$

where $[0,T] \ni t \mapsto \mu_t$ is a measure valued function and $b\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded, Borel vector field. A solution to (2.1) has to be understood in distributional sense.

We have the following

Theorem 2.1 (Superposition Principle)**.** *Let $b\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded, Borel vector field and let $[0,T] \ni t \mapsto \mu_t$ be a positive, locally finite, measure-valued solution of the continuity equation (2.1). Then there exists a family of probability measures $\{\eta_x\}_{x\in\mathbb{R}^d}$ on Γ such that*

$$\mu_t = \int e_{t\#}\eta_x d\bar{\mu}(x),$$

for any $t \in (0,T)$ and $(e_0)_\#\,\eta_x = \delta_x$. Moreover, η_x is concentrated on absolutely continuous integral solutions of the ODE starting from x, for $\overline{\mu}$-a.e. $x \in \mathbb{R}^d$.

2.2. **Partition and curves.** Let $b\colon \mathbb{T}^2 \to \mathbb{R}^2$ be an *autonomous*, nearly incompressible vector field, with $b \in \mathrm{BV}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$; we assume b is defined everywhere and Borel. Let us consider the countable covering $\mathscr{B}$ of $\mathbb{T}^2$ given by

$$\mathscr{B} := \left\{ B(x,r) : x \in \mathbb{Q}^2, r \in \mathbb{Q}^+ \right\}.$$

For each ball $B \in \mathscr{B}$, we are interested to the trajectories of b which cross B, staying inside B for a positive amount of time. We therefore define the following sets:

$$\mathsf{T}_B := \left\{ \gamma \in \Gamma_B : \gamma(t) = \gamma(0) + \int_0^t b(\gamma(\tau))\,d\tau,\, \gamma(0) \notin B, \gamma(T) \notin B \right\}.$$

where we have set

$$\Gamma_B := \left\{ \gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in B\}) > 0 \right\}.$$

Remark 2.2. It is fairly easy to see that

$$\bigcup_{B\in\mathscr{B}} T_B = \widetilde{\Gamma}.$$

Indeed, for every curve which is moving there exists a point $\gamma(t) \neq \gamma(0), \gamma(T)$, so that one has just to choose a ball in $\mathscr{B}$ containing $\gamma(t)$ but not $\gamma(0), \gamma(T)$.

By Definition 1.1, there exists a function $\rho\colon [0,T] \times \mathbb{T}^2 \to \mathbb{R}$ which satisfies continuity equation (1.4) in $\mathscr{D}'((0,T) \times \mathbb{T}^2)$. Therefore, by Ambrosio's Superposition Principle 2.1, there exists a measure η on Γ, concentrated on the set of trajectories of b, such that

$$\rho(t,\cdot)\mathscr{L}^2 = (e_t)_\#\eta, \tag{2.2}$$

where we recall that $e_t \colon \Gamma \to \mathbb{T}^2$ is the evaluation map $\gamma \mapsto \gamma(t)$. For a fixed ball $B \in \mathscr{B}$, we consider the measure $\eta_B := \eta \llcorner \mathsf{T}_B$ and we define ρ_B by $\rho_B(t,\cdot)\mathscr{L}^2 = (e_t)_\# \eta_B$. Then we set

$$r_B(x) := \int_0^T \rho_B(t,x)dt, \qquad x \in B. \tag{2.3}$$

Lemma 2.3. *It holds* $\mathrm{div}(r_B b) = 0$ *in* $\mathscr{D}'(B)$.

Proof. For any $\phi \in C_c^\infty(B)$ we have

$$\begin{aligned}
\int_B r_B b(x) \cdot \nabla\phi(x) dx &= \int_B \int_0^T \rho_B(t,x) b(x) \cdot \nabla\phi(x)\, dt\, dx \\
&= \int_0^T \int_{\mathsf{T}_B} b(\gamma(t)) \cdot (\nabla\phi)(\gamma(t))\, d\eta_B\, dt \\
&= \int_0^T \int_{\mathsf{T}_B} \dot\gamma(t) \cdot (\nabla\phi)(\gamma(t))\, d\eta_B\, dt \\
&= \int_0^T \int_{\mathsf{T}_B} \frac{d}{dt}\phi(\gamma(t))\, d\eta_B\, dt \\
&= \int_{\mathsf{T}_B} \Big[\phi(\gamma(T)) - \phi(\gamma(0))\Big]\, d\eta_B = 0.
\end{aligned}$$

because for η_B-a.e. $\gamma \in \mathsf{T}_B$, $\gamma(0) \notin B$, $\gamma(T) \notin B$. ☐

3. Recent results for uniqueness in the two dimensional case

We recall here some facts about uniqueness of bounded solutions for the continuity equation in the two dimensional case, following in particular [1, 3].

3.1. Structure of level sets of Lipschitz functions.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open set and let $f \colon \Omega \to \mathbb{R}$ be a Lipschitz function. For any $r \in \mathbb{R}$, we denote by $E_r := f^{-1}(r)$ the corresponding level set.

Theorem 3.1 ([1, Thm. 2.5])**.** *Suppose that* $f \colon \Omega \to \mathbb{R}$ *is a compactly supported Lipschitz function. For any* $r \in \mathbb{R}$, *let* $E_r := f^{-1}(r)$. *Then the following statements hold for* $\mathscr{L}^1$*-a.e.* $r \in H(\Omega)$*:*

1. $\mathscr{H}^1(E_r) < \infty$ *and* E_r *is countable* $\mathscr{H}^1$*-rectifiable;*
2. *for* $\mathscr{H}^1$*-a.e.* $x \in E_r$ *the function* f *is differentiable at* x *with* $\nabla f(x) \neq 0$*;*
3. $\mathrm{Conn}^\star(E_r)$ *is countable and every* $C \in \mathrm{Conn}^\star(E_r)$ *is a closed simple curve;*
4. $\mathscr{H}^1(E_r \setminus E_r^\star) = 0$.

For brevity, we will say that the level set E_r is *regular with respect to* Ω if it satisfies conditions (1)-(2)-(3)-(4) (or it is empty). In this way, the theorem above can be stated by saying that for a.e. $r \in \mathbb{R}$ the level sets E_r are regular with respect to Ω.

3.2. **Disintegration of Lebesgue measure with respect to Hamiltonians.** From Lemma 2.3 we have $\operatorname{div}(rb) = 0$ in B; since B is simply connected, there exists a Lipschitz potential $H_B\colon B \to \mathbb{R}$ such that

$$\nabla^\perp H_B(x) = r_B(x) b(x), \qquad \text{for } \mathscr{L}^2\text{-a.e. } x \in B.$$

Using Theorem 3.1 on the Lipschitz function H_B, we can define the negligible set N_1 such that E_h is regular in B whenever $h \notin N_1$; moreover, let N_2 denote the negligible set on which the measure $((H_B)_\# \mathscr{L}^2)^{\mathrm{sing}}$ is concentrated, where $((H_B)_\# \mathscr{L}^2)^{\mathrm{sing}}$ is the singular part of $((H_B)_\# \mathscr{L}^2)$ with respect to $\mathscr{L}^1$. Then we set

$$N := N_1 \cup N_2 \qquad \text{and} \qquad E := \cup_{h \notin N} E_h^* \tag{3.1}$$

Therefore we can associate to B a triple (H_B, N, E). For any $x \in E$ let C_x denote the connected component of E such that $x \in C_x$. By definition of E for any $x \in E$ the corresponding connected component C_x has strictly positive length.

Let us fix an arbitrary ball $B \in \mathscr{B}$. For brevity let H denote the corresponding Hamiltonian H_B.

Lemma 3.2 ([3, Lemma 2.8]). *There exist Borel families of measures σ_h, κ_h, $h \in \mathbb{R}$, such that*

$$\mathscr{L}^2 \llcorner B = \int \left(c_h \mathscr{H}^1 \llcorner E_h + \sigma_h \right) dh + \int \kappa_h \, d\zeta(h), \tag{3.2}$$

where

1. *$c_h \in L^1(\mathscr{H}^1 \llcorner E_h^\star)$, $c_h > 0$ a.e.; moreover, by Coarea formula, we have $c_h = 1/|\nabla H|$ a.e. (w.r.t. $\mathscr{H}^1 \llcorner E_h^\star$);*
2. *σ_h is concentrated on $E_h^\star \cap \{\nabla H = 0\}$;*
3. *κ_h is concentrated on $E_h^\star \cap \{\nabla H = 0\}$;*
4. *$\zeta := H_\# \mathscr{L}^2 \llcorner (B \setminus E^\star)$ is concentrated on N (hence $\zeta \perp \mathscr{L}^1$).*

Remark 3.3. Using Coarea formula, we can show

$$\mathscr{H}^1(E_h \cap \{\nabla H = 0\}) = 0$$

for $\mathscr{L}^1$-a.e. $h \notin N$. Therefore $\sigma_h \perp \mathscr{H}^1$ for $\mathscr{L}^1$-a.e. $h \notin N$.

Remark 3.4. Thanks to (3.2) we always can add to N, if necessary, an $\mathscr{L}^1$-negligible set so that for any $h \notin N$ for $\mathscr{H}^1$-a.e. $x \in E_h^\star$ we have $r(x) > 0$, $b(x) \neq 0$ and $r(x)b(x) = \nabla^\perp H(x)$.

Remark 3.5. The measure σ_h is actually concentrated on $E_h \cap \{b \neq 0, r_B = 0\}$. This can be proved using minor modifications of the proof of [7, Theorem 8.2]: indeed, we have that, being b of class BV and hence approximately differentiable a.e., $H_\# \mathscr{L}^2 \llcorner \{b = 0\} \perp \mathscr{L}^1$: by comparing two disintegrations of $\mathscr{L}^2 \llcorner \{b = 0\}$ we conclude that σ_h is concentrated on $\{b \neq 0\}$ for a.e. h.

3.3. **Reduction of the equation on the level sets.** Our goal is now to study the equation $\operatorname{div}(ub) = \mu$, where u is a bounded Borel function on $\mathbb{T}^2$ and μ is a Radon measure on $\mathbb{T}^2$, inside a ball from the collection $\mathscr{B}$.

Lemma 3.6. *Suppose that μ is a Radon measure on $\mathbb{T}^2$ and $u \in L^\infty(\mathbb{T}^2)$. Then equation*

$$\operatorname{div}(ub) = \mu \tag{3.3}$$

holds in $\mathscr{D}'(B)$ *if and only if:*

- *the disintegration of* μ *with respect to* H *has the form*

$$\mu = \int \mu_h \, dh + \int \nu_h \, d\zeta(h), \tag{3.4}$$

where ζ *is defined in Point* (4) *of Lemma 3.2;*

- *for* $\mathscr{L}^1$*-a.e.* h

$$\operatorname{div}\left(uc_h b \mathscr{H}^1 \llcorner E_h\right) + \operatorname{div}(ub\sigma_h) = \mu_h; \tag{3.5}$$

- *for* ζ*-a.e.* h

$$\operatorname{div}(ub\kappa_h) = \nu_h. \tag{3.6}$$

Proof. Let λ^s be a measure on $\mathbb{R}$ such that $H_\#|\mu| \ll \mathscr{L}^1 + \zeta + \lambda^s$, where ζ is defined as in Lemma 3.2 and $\lambda^s \perp \mathscr{L}^1 + \zeta$. Applying the Disintegration Theorem, we have that

$$\mu = \int \mu_h dh + \int \nu_h d\zeta(h) + \int \lambda_h d\lambda^s(h), \tag{3.7}$$

with μ_h, ν_h, λ_h concentrated on $\{H = h\}$. Writing equation (3.3) in distribution form we get

$$\int_{\mathbb{T}^2} u(b \cdot \nabla \phi) \, dx + \int \phi d\mu = 0, \qquad \forall \phi \in C_c^\infty(B).$$

By an elementary approximation argument, it is clear that we can use as test functions ϕ Lipschitz with compact support.

Using the disintegration of Lebesgue measure (3.2) and the disintegration (3.7) we thus obtain

$$\begin{aligned} \int \bigg[\int_{\mathbb{T}^2} uc_h(b \cdot \nabla \phi) \, d\mathscr{H}^1 \llcorner E_h + \int_{\mathbb{T}^2} u(b \cdot \nabla \phi) \, d\sigma_h\bigg] \, dh \\ + \int \int_{\mathbb{T}^2} u(b \cdot \nabla \phi) \, d\kappa_h \, d\zeta(h) + \int \int_{\mathbb{T}^2} \phi \, d\mu_h \, dh \\ + \int \int_{\mathbb{T}^2} \phi \, d\nu_h \, d\zeta(h) + \int \int_{\mathbb{T}^2} \phi \, d\lambda_h \, d\lambda^s(h) = 0, \end{aligned} \tag{3.8}$$

for every $\phi \in \mathrm{Lip}_c(B)$. In particular we can take

$$\phi = \psi(H(x))\varphi(x), \qquad \psi \in C^\infty(\mathbb{R}), \ \varphi \in C_c^\infty(B),$$

so that we can rewrite (3.8) as

$$\begin{aligned} \int \psi(h) \bigg[\int_{\mathbb{T}^2} uc_h(b \cdot \nabla \varphi) \, d\mathscr{H}^1 \llcorner E_h + \int_{\mathbb{T}^2} u(b \cdot \nabla \varphi) \, d\sigma_h\bigg) \, dh \\ + \int \psi(h) \int_{\mathbb{T}^2} u(b \cdot \nabla \varphi) \, d\kappa_h \, d\zeta(h) + \int \psi(h) \int_{\mathbb{T}^2} \varphi \, d\mu_h \, dh \\ + \int \psi(h) \int_{\mathbb{T}^2} \varphi \, d\nu_h \, d\zeta(h) + \int \psi(h) \int_{\mathbb{T}^2} \varphi \, d\lambda_h \, d\lambda^s(h) = 0, \end{aligned}$$

because

$$b \cdot \nabla \phi = \psi(H(x)) b \cdot \nabla \varphi(x)$$

for $\mathscr{L}^2$-a.e. $x \in \mathbb{T}^2$.

Since the equalities above hold for all $\psi \in C^\infty(\mathbb{R})$ we have

$$\int \left[\int_{\mathbb{T}^2} uc_h(b\cdot\nabla\varphi)\, d\mathscr{H}^1 \llcorner E_h + \int_{\mathbb{T}^2} u(b\cdot\nabla\varphi)\, d\sigma_h\right] dh + \int\int_{\mathbb{T}^2} \varphi\, d\mu_h\, dh = 0,$$

$$\int \left[\int_{\mathbb{T}^2} u(b\cdot\nabla\varphi)\, d\kappa_h + \int_{\mathbb{T}^2} \varphi\, d\nu_h\right] d\zeta(h) = 0,$$

$$\int\int_{\mathbb{T}^2} \varphi\, d\lambda_h\, d\lambda^s(h) = 0,$$

which give, respectively, (3.5), (3.6) and (3.4).

□

3.4. **Reduction on connected components of level sets.** If $K \subset \mathbb{R}^d$ is a compact then, in general, not any connected component C of K can be separated from $K \setminus C$ by a smooth function. However, it can be separated by a sequence of such functions:

Lemma 3.7 ([1, Section 2.8])**.** *If $K \subset \mathbb{R}^d$ is compact then for any connected component C of K there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ such that*

(1) $0 \le \phi_n \le 1$ on $\mathbb{R}^d$ and $\phi_n \in \{0,1\}$ on K for all $n \in \mathbb{N}$;
(2) for any $x \in C$, we have $\phi_n(x) = 1$ for every $n \in \mathbb{N}$;
(3) for any $x \in K \setminus C$, we have $\phi_n(x) \to 0$ as $n \to +\infty$;
(4) for any $n \in \mathbb{N}$, we have $\operatorname{supp} \nabla\phi_n \cap K = \emptyset$.

With the aid of this lemma we can now study the equation (3.5) on the nontrivial connected components of the level sets. In view of Lemma 3.6 in what follows we always assume that $h \notin N$ (see (3.1)).

Lemma 3.8. *The equation (3.5) holds iff*

- *for any nontrivial connected component C of E_h it holds*
$$\operatorname{div}\left(uc_h b\mathscr{H}^1 \llcorner C\right) + \operatorname{div}(ub\sigma_h \llcorner C) = \mu_h \llcorner C; \tag{3.9}$$
- *it holds*
$$\operatorname{div}(ub\sigma_h \llcorner (E_h \setminus E_h^\star)) = \mu_h \llcorner (E_h \setminus E_h^\star). \tag{3.10}$$

Proof. For any Borel set $A \subset \mathbb{T}^2$ we introduce the following functional

$$\Lambda_A(\psi) := \int_A uc_h(b\cdot\nabla\psi)\, d\mathscr{H}^1 \llcorner E_h + \int_A u(b\cdot\nabla\psi)\, d\sigma_h + \int_A \psi\, d\mu_h,$$

for all $\psi \in C_c^\infty(B)$.

Now fix a connected component C of E_h and take a sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ given by Lemma 3.7 (applied with $K := E_h$). By assumption, we have that $\Lambda(\psi\phi_n) = 0$ for every $\psi \in C_c^\infty(B)$ and for every n. Let us pass to the limit as $n \to \infty$.

On one hand we have

$$\int \psi\phi_n\, d\mu_h = \int_C \psi\, d\mu + \int_{E_h\setminus C} \psi\phi_n\, d\mu \to \int_C \psi\, d\mu$$

because the second term converges to 0 since $\phi_n \to 0$ pointwise on $E_h \setminus C$.

On the other hand $\nabla(\psi\phi_n) = \psi\nabla\phi_n + \phi_n\nabla\psi$. In the terms with $\phi_n\nabla\psi$ we pass to the limit as above. The terms with the product $\psi\nabla\phi_n$ identically vanish thanks to the condition (4) on ϕ_n in Lemma 3.7. Therefore, we have that for every $\psi \in C_c^\infty(B)$

$$\Lambda_{E_h}(\psi\phi_n) \to \int_C uc_h(b\cdot\nabla\psi)\,d\mathscr{H}^1 + \int_C u(b\cdot\nabla\psi)\,d\sigma_h + \int_C \psi\,d\mu_h = \Lambda_C(\psi),$$

as $n \to +\infty$. Since $\Lambda_{E_h}(\psi\phi_n) = 0$ for every n, we deduce that $\Lambda_C(\psi) = 0$ and this gives (3.9).

In order to get (3.10), it is enough to observe that $E_h^\star$ is a countable union of connected component C, therefore (from the previous step) we deduce that

$$\int_{E_h^\star} uc_h(b\cdot\nabla\psi)\,d\mathscr{H}^1 + \int_{E_h^\star} u(b\cdot\nabla\psi)\,d\sigma_h + \int_{E_h^\star} \psi\,d\mu_h = 0, \qquad \forall\psi\in C_c^\infty(B).$$

Hence

$$\Lambda_{E_h\setminus E_h^\star} := \int_{E_h^\star\setminus E_h} uc_h(b\cdot\nabla\psi)\,d\mathscr{H}^1 + \int_{E_h^\star\setminus E_h} u(b\cdot\nabla\psi)\,d\sigma_h + \int_{E_h^\star\setminus E_h} \psi\,d\mu_h = 0,$$

for every $\psi \in C_c^\infty(B)$. Remembering that $\mathscr{H}^1(E_h^\star \setminus E_h) = 0$ by Theorem 3.1 we get (3.10) and this concludes the proof.

The converse implication can be easily obtained by summing the equations (3.9) and (3.10). □

Lemma 3.9. *Equation* (3.9) *holds iff*

$$\operatorname{div}\left(uc_h b\mathscr{H}^1 \llcorner C\right) = \mu_h \llcorner C, \tag{3.11a}$$

$$\operatorname{div}(ub\sigma_h \llcorner C) = 0. \tag{3.11b}$$

The proof would be fairly easy in the case γ is straight line: roughly speaking, σ_h is concentrated on a $\mathscr{L}^1$-negligible set S, and the set of C^1-functions which have 0-derivative on S is dense in C^0 in the set of Lipschitz functions. The only technicality here is to repeat this argument on a curve. Before presenting the formal proof of Lemma 3.9 we would like to discuss the parametric version of the equation (3.11a).

Let $\gamma\colon I \to \mathbb{T}^2$ be an injective Lipschitz parametrization of C, where $I = \mathbb{R}/\ell\mathbb{Z}$ or $I = (0,\ell)$ for some $\ell > 0$ is the domain of γ. In view of Remark 3.4) we can assume that the directions of b and $\nabla^\perp H$ agree $\mathscr{H}^1$-a.e. on C. So there exists a constant $\varpi \in \{+1,-1\}$ such that

$$\frac{b(\gamma(s))}{|b(\gamma(s))|} = \varpi \frac{\gamma'(s)}{|\gamma'(s)|} \tag{3.12}$$

for a.e. $s \in I$. We will say that γ is an *admissible parametrization* of C if $\varpi = +1$. In the rest of the text we will consider only admissible parametrizations of the connected components C.

Lemma 3.10. *Equation* (3.11a) *holds iff for any admissible parametrization γ of C*

$$\partial_s(\hat{u}\hat{c}_h|\hat{b}|) = \hat{\mu}_h \tag{3.13}$$

where $\gamma_\#\hat{\mu}_h = \mu_h \llcorner C$, $\hat{u} = u\circ\gamma$, $\hat{c}_h = c_h\circ\gamma$ and $\hat{b} = b\circ\gamma$.

In the proof of Lemma 3.10 we will use the following result:

Lemma 3.11 ([1, Section 7])**.** *Let $a \in L^1(I)$ and μ a Radon measure on I, where $I = \mathbb{R}/\ell\mathbb{Z}$ or $I = (0,\ell)$ for some $\ell > 0$. Suppose that $\gamma\colon I \to \Omega$ is an injective Lipschitz function such that $\gamma' \neq 0$ a.e. on I and $\gamma(0,\ell) \subset \Omega$. Consider the functional*

$$\Lambda(\phi) := \int_I \phi' a\, dt + \int_I \phi\, d\mu, \qquad \forall \phi \in \mathrm{Lip}_c(I).$$

If $\Lambda(\varphi \circ \gamma) = 0$ for any $\varphi \in C_c^\infty(\Omega)$ then $\Lambda(\phi) = 0$ for any $\phi \in \mathrm{Lip}_c(I)$.

Proof of Lemma 3.10. Let us recall a corollary from Area formula: if $\gamma\colon I \to \mathbb{T}^2$ is an injective Lipschitz parametrization of C then

$$\mathscr{H}^1 \llcorner C = \gamma_\# \left(|\gamma'| \mathscr{L}^1 \right).$$

Using this formula the distributional version of (3.11a),

$$\int_C u c_h b \cdot \nabla \phi d\mathscr{H}^1 \llcorner C + \int_C \phi d\mu_h = 0, \quad \forall \phi \in C_c^\infty(B),$$

can be written as

$$\int_I u(\gamma(s)) c_h(\gamma(s)) b(\gamma(s)) \cdot (\nabla\phi)(\gamma(s)) |\gamma'(s)|\, ds + \int_I \phi(\gamma(s)) d\hat{\mu}_h(s) = 0$$

where $\hat{\mu}_h$ is defined by $\hat{\mu}_h := \left(\gamma^{-1}\right)_\# \mu_h$.

Using (3.12) we can write the equation above as

$$\int_I u(\gamma(s)) c_h(\gamma(s)) \gamma'(s) (\nabla\phi)(\gamma(s)) |b(\gamma(s))|\, ds + \int_I \phi(\gamma(s)) d\hat{\mu}_h(s) = 0,$$

which reads as

$$\int_I u(\gamma(s)) c_h(\gamma(s)) \partial_s \phi(\gamma(s)) |b(\gamma(s))|\, ds + \int_I \phi(\gamma(s)) d\hat{\mu}_h(s) = 0.$$

Since the equation above holds for any $\phi \in C_c^\infty(B)$ it remains to apply Lemma 3.11. □

Proof of Lemma 3.9. Let us write $\Lambda_C(\phi) = M(\phi) + N(\phi)$, where

$$M(\phi) := \int_C u c_h (b \cdot \nabla \phi)\, d\mathscr{H}^1 + \int_C \phi\, d\mu_h$$

and

$$N(\phi) := \int_C u b \cdot \nabla \phi d\sigma_h$$

for every $\phi \in C_c^\infty(B)$.

Fix a test function ϕ: the idea of the proof is to "perturb" ϕ in such a way that $N(\phi)$ becomes arbitrary small and $M(\phi)$ remains almost unchanged. Since $\Lambda(\phi) = 0$ we will obtain that $|M(\phi)| < \varepsilon$ and this will imply that $M(\phi) = N(\phi) = 0$.

By Remark 3.3, we have $\sigma_h \perp \mathscr{H}^1 \llcorner C$ therefore there exists a $\mathscr{H}^1$-negligible set $S \subset C$ such that σ_h is concentrated on S. Moreover, by inner regularity, for every $n \in \mathbb{N}$, we can find a compact $K \subset S$ such that

$$\sigma_h(S \setminus K) < \frac{1}{n}.$$

Using the fact that $\mathscr{H}^1(K) = 0$, for every $n \in \mathbb{N}$, we can find countably many open balls $\{B_{r_j}(z_j)\}_{j\in\mathbb{N}}$ which cover K and whose radii r_j satisfy

$$\sum_{j\in\mathbb{N}} r_j < \frac{1}{n}.$$

Furthermore, by compactness, we can extract from $\{B_{r_j}(z_j)\}_{l\in\mathbb{N}}$ a finite subcovering, $\{B_{r_j}(z_j)\}$ with $j = 1, \dots, \nu$ where $\nu = \nu(n) \in \mathbb{N}$ (we stress that ν depends on n).

For every $j \in \{1, \dots, \nu\}$, let $P_i^{j,n}$ denote the projection of $B_{r_j}(z_j)$ onto the x_i-axis, with $i = 1, 2$. We have $P_i^{j,n}$ is an open interval and therefore we can find a smooth function $\psi_i^{j,n} \colon \mathbb{R} \to \mathbb{R}$ such that

$$\psi_i^{j,n}(\xi) = \begin{cases} 0 & \xi \in P_i^{j,n}, \\ 1 & \operatorname{dist}(\xi, \partial P_i^{j,n}) > 2r_i, \end{cases}$$

and $0 \le \psi_i^{j,n} \le 1$ for every $\xi \in \mathbb{R}$. Now we consider the product $\psi^{j,n} := \psi_1^{j,n}\psi_2^{j,n}\cdots\psi_\nu^{j,n}$ and we define the functions $\chi^{j,n} \colon \mathbb{R} \to \mathbb{R}$ as

$$\chi^{j,n}(\xi) := \int_0^\xi \psi^{j,n}(w)\, dw$$

for $j = 1, 2$ and $n \in \mathbb{N}$. Now we set $\chi^n(x) := (\chi^{1,n}(x), \chi^{2,n}(x))$ and $\phi_n := \phi \circ \chi^n$. Since $\|\chi^n - \mathrm{id}\|_\infty \le 4\sum_i r_i \le \frac{4}{n}$ we deduce that $\phi_n \to \phi$ uniformly in C because

$$|\phi_n(x) - \phi(x)| \le \|\nabla\phi\|_\infty \|\chi^n - \mathrm{id}\|_\infty \to 0$$

as $n \to +\infty$.

Let us now take an admissible parametrization of C, $\gamma \colon I \to \mathbb{R}$, and let us introduce the functions $\hat{\phi}_n := \phi_n \circ \gamma$. Using for instance the density of C^1 functions in $L^1(I)$, we can actually show that $\partial_s\hat{\phi}_n \rightharpoonup^\star \partial_s\hat{\phi}$ in weak* topology of L^∞. Passing to the parametrization as in the proof of Lemma 3.10 we get

$$\int_C uc_h(b \cdot \nabla\phi_n)\, d\mathscr{H}^1 = \int_I \hat{u}\hat{c}_h\hat{b}\, \partial_s\hat{\phi}_n\, ds,$$

where we denote by $\hat{\cdot}$ the composition with γ.

Using weak* convergence, we obtain that

$$\int_C uc_h(b \cdot \nabla\phi_n)\, d\mathscr{H}^1 \to \int_C uc_h(b \cdot \nabla\phi)\, d\mathscr{H}^1.$$

On the other hand, by uniform convergence, we immediately get

$$\int \phi_n\, d\mu_h \to \int \phi\, d\mu_h,$$

as $n \to +\infty$. In particular, we have that $M(\phi_n) \to M(\phi)$.

Now observe that $\nabla\phi_n = 0$ on K by construction, hence we get

$$N(\phi_n) \le \int_{S\setminus K} |ub||\nabla\phi_n| d\sigma_h \le \|ub\|_\infty \|\nabla\phi\|_\infty \frac{1}{n} \to 0$$

and this implies that $N(\phi) = 0$. Therefore, $0 = \Lambda(\phi) = M(\phi)$, which concludes the proof. □

We note, in particular, that from (3.11b), being $b \in \mathrm{BV}$ and taking $u \equiv 1$ in (3.3), we have that $\operatorname{div}(b\sigma_h \llcorner E_h) = 0$ for a.e. h.

Let

$$F := \{b \neq 0, r_B = 0\} \cap E. \tag{3.14}$$

By Remark 3.5, σ_h is concentrated on $F \cap E_h$ hence we have

$$\operatorname{div}(\mathbb{1}_F b\sigma_h) = 0, \qquad \text{for } \mathscr{L}^1\text{-a.e. } h. \tag{3.15}$$

This important piece of information is very useful to prove the following

Lemma 3.12. *We have* $\operatorname{div}(\mathbb{1}_F b) = 0$ *in* $\mathscr{D}'(B)$.

Proof. For every test function $\phi \in C_c^\infty(B)$, we have

$$\int_F (b(x)\nabla\phi(x))\,dx = \int\int_{F\cap E_h} (b(x)\cdot\nabla\phi(x))\,d\sigma_h(x)dh.$$

Using Remark 3.5 and (3.15), we get that

$$\int_{F\cap E_h} (b(x)\nabla\phi(x))\,d\sigma_h(x) = 0$$

and then we conclude. □

4. Level sets and trajectories

In this section, we assume that H_B is defined on all $\mathbb{T}^2$ (using standard theorems for the extension of Lipschitz maps).

4.1. **Trajectories.** We now present some lemmas which relate the trajectories $\gamma \in \mathsf{T}_B$ to the level sets of the Hamiltonian. The first result we prove is that η-a.e. γ is contained in a level set.

Lemma 4.1. *Let* $B \in \mathscr{B}$, $t_1, t_2 \in [0,T]$ *and set* $\mathsf{T} := \{\gamma : \gamma((t_1,t_2)) \subset B\}$. *Then* η*-a.e.* $\gamma \in \mathsf{T}$ *we have* $(t_1,t_2) \ni t \mapsto H(\gamma(t))$ *is a constant function.*

Proof. Let $(\varrho_\varepsilon)_\varepsilon$ be the standard family of convolution kernels in $\mathbb{R}^2$. We set $H_\varepsilon(x) := H \star \varrho_\varepsilon(x)$ for any $x \in B$.

For every $t \in [t_1,t_2]$ define

$$I(t) := \int_{\mathsf{T}} |H(\gamma(t)) - H(\gamma(0))| d\eta(\gamma)$$

and we will prove $I \equiv 0$.

First note that I is positive because the integrand is non-negative and η is positive. On the other hand,

$$\begin{aligned} I(t) \le \underbrace{\int_{\mathsf{T}} |H(\gamma(t)) - H_\varepsilon(\gamma(t))| d\eta(\gamma)}_{I_1^\varepsilon} &+ \underbrace{\int_{\mathsf{T}} |H_\varepsilon(\gamma(t)) - H_\varepsilon(\gamma(0))| d\eta(\gamma)}_{I_2^\varepsilon} \\ &+ \underbrace{\int_{\mathsf{T}} |H_\varepsilon(\gamma(0)) - H(\gamma(0))| d\eta(\gamma)}_{I_3^\varepsilon}. \end{aligned}$$

Now for a.e. $x \in \mathbb{T}^2$ we have $H_\varepsilon(x) \to H(x)$: hence

$$\int_{\mathsf{T}} |H_\varepsilon(\gamma(t)) - H(\gamma(t))|\,d\eta(\gamma) \le \int_B |H_\varepsilon(x) - H(x)| \rho(t,x) dx \to 0$$

as $\varepsilon \to 0$. Therefore, we can infer that

$$I_1^\varepsilon \to 0, \qquad I_3^\varepsilon \to 0$$

as $\varepsilon \downarrow 0$.

Let us study I_2^ε. We have

$$\begin{aligned}
I_2^\varepsilon(t) &\le \int_{\mathsf{T}} \int_{t_1}^t |\partial_s H_\varepsilon(\gamma(s))|\, ds\, d\eta(\gamma) \\
&= \int_{\mathsf{T}} \int_{t_1}^t |\nabla H_\varepsilon(\gamma(s)) \cdot b(\gamma(s))|\, ds\, d\eta(\gamma) \\
&= \int_{t_1}^t \int |\nabla H_\varepsilon(x) \cdot b(x)|\, d(e_t \# \eta \llcorner \mathsf{T})(x)\, ds \\
&\le \int_0^T \int |\nabla H_\varepsilon(x) \cdot b(x)| \rho_{\mathsf{T}}(t,x)\, dx\, ds \\
&= \int |\nabla H_\varepsilon(x) \cdot b(x)| r_{\mathsf{T}}(x)\, dx \to \int |\nabla H(x) \cdot b(x)| r_{\mathsf{T}}(x)\, dx = 0
\end{aligned}$$

where we have used $\nabla H_\varepsilon(x) \to \nabla H(x)$ for a.e. x. In the end, we have that $I_2^\varepsilon \to 0$ as $\varepsilon \downarrow 0$ and this concludes the proof. □

We now show that Lemma 4.1 can be improved, showing indeed that η_B-a.e. γ is contained in a *regular* level set of H.

Lemma 4.2. *Up to a η_B negligible set, the image of every $\gamma \in \mathsf{T}_B$ is contained in a connected component of a* regular *level set of H_B.*

Proof. Using Lemma 4.1, we remove η_B-negligible set of trajectories along which H_B is not constant. Set $E^c := B \setminus E$ and consider the set

$$\mathscr{P} := \{\gamma \in \mathsf{T}_B : \gamma((0,T)) \cap B \subset E^c\}.$$

It is enough to show that $\eta(\mathscr{P}) = 0$: this means that for η-a.e. γ the image $\gamma(0,T)$ is not contained in the complement of E and thus we must have (in the ball) $\gamma(0,T) \subset E$ for η-a.e. $\gamma \in \mathsf{T}_B$ (this follows remembering that a.e. γ is contained in a level set).

By Coarea formula, $|\nabla H| \mathscr{L}^2 \llcorner E^c = 0$, i.e.

$$\int \mathbb{1}_{E^c}(x) |\nabla H(x)|\, dx = 0.$$

Since $\nabla H = r_B b^\perp$ in B and $r_B \ge 0$ (since $\rho_B > 0$), we have

$$\begin{aligned}
0 &= \int \mathbb{1}_{E^c}(x) |r_B(x) b(x)|\, dx \\
&= \int \mathbb{1}_{E^c}(x) r_B(x) |b(x)|\, dx \\
&= \int \int_0^T \mathbb{1}_{E^c}(x) \rho_B(t,x) |b(x)|\, dx\, dt.
\end{aligned}$$

Using (2.2) we have

$$0 = \int_0^T \int \mathbb{1}_{E^c}(\gamma(t)) |b(\gamma(t))|\, d\eta(\gamma)\, dt = \int_0^T \int_{\mathscr{P}} |b(\gamma(t))|\, d\eta(\gamma)\, dt$$

which implies (by Fubini) that for η-a.e. $\gamma \in \mathscr{P}$ we have

$$\int_0^T |b(\gamma(t))| \, dt = 0.$$

This gives $|b(\gamma(t))| = 0$ for a.e. $t \in [0, T]$ and this contradicts the definition of T_B. Hence $\eta(\mathscr{P}) = 0$. □

5. Matching property and Weak Sard Property of Hamiltonians

5.1. **Matching properties I.** As we have seen at the beginning of Section 3.2, to every Hamiltonian H we can associate a triple (H, N, E) where N is the set given by Theorem 3.1 and $E = \cup_{h \notin N} E_h^\star$.

Suppose now we have another triple $(\widetilde{H}, \widetilde{N}, \widetilde{E})$; we ask whether, given $x \in E \cap \widetilde{E}$ it is true that $C_x = \widetilde{C}_x$. This is essentially the definition of matching property; moreover, we will prove the "Matching Lemma", which states that gradients of H and $\widetilde{H}$ being parallel (in a simply connected set) is a sufficient condition for matching.

5.2. **Matching of two Hamiltonians.** Let us consider two Lipschitz Hamiltonians H_1 and H_2, defined on the same simply connected set A; according to Theorem 3.1, we have two negligible sets N_1 and N_2 such that the level sets E_h^1 and $E_{h'}^2$ of H_1 and H_2 are regular for $h \notin N_1$ and $h' \notin N_2$. We set $E_1 := \cup_{h \notin N_1} E_h^1$ and $E_2 := \cup_{h' \notin N_2} E_{h'}^2$.

Definition 5.1. The Hamiltonians H_1 and H_2 *match* in an open subset $A' \subset A$ if $C_x^1 = C_x^2$ for $\mathscr{L}^2$-a.e. $x \in A' \cap E_1 \cap E_2$, where C_x^i denotes the connected component in A' of the level sets $H_i^{-1}(H_i(x))$ which contains x.

We now state and prove the following

Lemma 5.2 (Matching lemma)**.** *Let H_1, H_2 be defined as above. If $\nabla H_1 \parallel \nabla H_2$ a.e. on $A' \subset A$ open, then the Hamiltonians H_1 and H_2 match in A'.*

Proof. Let $b_1 := \nabla^\perp H_1$. Then $\operatorname{div} b_1 = 0$. Let us prove that

$$\operatorname{div}(H_2 b_1) = 0 \tag{5.1}$$

in the sense of distributions. Indeed, we have for every $\varphi \in \mathrm{Lip}_c(A')$

$$\int H_2 (b_1 \cdot \nabla\varphi) \, dx = \int [b_1 \cdot \nabla(H_2 \varphi) - \varphi (b_1 \cdot \nabla H_2)] \, dx.$$

The first term is zero because $\operatorname{div} b_1 = 0$ (and φH_2 can be used as test function since it is Lipschitz); the second term is also zero because $\nabla H_2 \parallel \nabla H_1$ a.e. on A', hence $b_1 \perp \nabla H_2$ a.e. on E.

From (5.1), using [7, Theorem 4.1 and 6.1], we obtain that there exists a $\mathscr{L}^1$ negligible set N such that H_2 is constant on every non trivial connected components $C \cap A'$ of the level sets of H_1 which do not correspond to values in N. By disintegration, we have that the sets of points $x \in A' \cap E_1$ such that $H_1(x) \notin N$ are a negligible set and therefore we can infer that for a.e. $x \in A' \cap E_1$, H_2 is constant along the connected components in A' of the level sets of H_1. By repeating the same argument for H_2 we get the claim. □

5.3. **The Weak Sard property.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz function and let S be the critical set of f, defined as the set of all $x \in \mathbb{R}^2$ where f is not differentiable or $\nabla f(x) = 0$. We are interested in the following property: *the push-forward according to f of the restriction of $\mathscr{L}^2$ to S is singular with respect to $\mathscr{L}^1$, that is*

$$f_\# \left(\mathscr{L}^2 \llcorner S\right) \perp \mathscr{L}^1.$$

This property clearly implies the following *Weak Sard Property*, which is used in [3, Section 2.13]:

$$f_\# \left(\mathscr{L}^2 \llcorner (S \cap E^\star)\right) \perp \mathscr{L}^1,$$

where the set $E^\star$ is the union of all connected components with positive length of all level sets of f. We point out that the relevance of the Weak Sard Property in the framework of transport and continuity equation is explained is [3, Theorem 4.7].

Now we give the following

Definition 5.3. We set

$$\widetilde{r}_B := r_B + \mathbb{1}_F,$$

where we recall that r_B is the function defined in (2.3) and F is the set defined in (3.14).

By linearity of divergence, by Lemma 2.3 and Lemma 3.12, we have

$$\operatorname{div}(\widetilde{r}_B b) = 0$$

in $\mathscr{D}'(B)$. Therefore, we conclude that there exists a Lipschitz potential $\widetilde{H}$ such that $\nabla \widetilde{H}^\perp = \widetilde{r}_B b$.

Moreover, we observe that $\nabla H \parallel \nabla \widetilde{H}$ a.e. in B: therefore we can apply Matching Lemma 5.2 to get that the regular level sets of H and of $\widetilde{H}$ agree. In particular, we obtain $E = \widetilde{E} \mod \mathscr{L}^2$, directly from the definition of $\widetilde{H}$. We note also that the function $\widetilde{H}$ has the Weak Sard property: indeed, directly from the construction, we have $\nabla \widetilde{H} \neq 0$ on E hence, since $E = \widetilde{E}$ mod $\mathscr{L}^2$, it follows that $\mathscr{L}^2(\widetilde{E} \cap \widetilde{S}) = 0$.

Finally, disintegrating $\mathscr{L}^2 \llcorner E$ with respect to H we get

$$\mathscr{L}^2 \llcorner E = \int_{\mathbb{R}} (c_h \mathscr{H}^1 \llcorner E_h + \sigma_h)\, dh,$$

while using the Hamiltonian $\widetilde{H}$

$$\mathscr{L}^2 \llcorner E = \int_{\mathbb{R}} \widetilde{c}_h \mathscr{H}^1 \llcorner \widetilde{E}_h \, dh.$$

In particular, it follows that $\sigma_h = 0$ for a.e. h, which means that $H = \widetilde{H}$ (up to additive constants) and H has the Weak Sard Property.

We collect this result in the following

Lemma 5.4. *The Hamiltonian H_B has the weak Sard property.*

6. Locality of the divergence

In this section we prove that the if $\operatorname{div}(ub)$ is a measure, then it is 0 on the set

$$M := \Big\{x \in \mathbb{T}^2 : b(x) = 0,\, x \in \mathcal{D}_b \text{ and } \nabla^{\mathrm{appr}} b(x) = 0\Big\}, \tag{6.1}$$

where $\mathcal{D}_b$ is the set of approximate differentiability points and $\nabla^{\mathrm{appr}} b$ is the approximate differential, according to Definition [6, Def. 3.70]. For shortness, we will call this property *locality of the divergence.*

Remark 6.1. We remark that $M = \{b = 0\} \mod \mathscr{L}^2$. This can be proved using the following result (see [2, Prop. 4.2]): a bounded, Borel vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ whose divergence and curl are measures is approximately differentiable a.e.. Furthermore, using locality property stated in [6, Prop. 3.73 - Rem. 3.93], we have that at every Lebesgue point x of the set $\{b = 0\}$ at which b is approximate differentiable we also have $\nabla^{\mathrm{appr}} b(x) = 0$.

The main result of this section is the following

Proposition 6.2. *Let $u \in L^\infty(\mathbb{R}^d)$ and suppose that $\operatorname{div}(ub) = \lambda$ in the sense of distributions, where λ is a Radon measure on $\mathbb{R}^d$. Then $|\lambda| \llcorner M = 0$.*

The proof is based on Besicovitch-Vitali covering Lemma ([6, Thm. 2.19]) and uses some basic facts about the trace properties of L^∞ vector fields whose divergence is a measure (we refer to [12]). In particular, we recall the following Theorem (for the proof, see [12, Prop 7.10]):

Theorem 6.3 (Fubini's Theorem for traces)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and $B \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$ be a vector field whose distributional divergence $\operatorname{div} B =: \mu$ is a Radon measure with locally finite variation in Ω. Let $F \in C^1(\Omega)$. Then for a.e. $t \in \mathbb{R}$ we have*

$$Tr(B, \partial\{F > t\}) = B \cdot \nu \qquad \mathscr{H}^{d-1}\text{-a.e. on } \Omega \cap \{F > t\}, \tag{6.2}$$

where ν denotes the exterior unit normal to $\{F > t\}$ and the distribution $Tr(B, \partial\Omega')$ is defined by

$$\langle Tr(B, \partial\Omega'), \phi\rangle := \int_{\Omega'} \phi \, d\mu + \int_{\Omega'} \nabla\phi \cdot B \, dx, \qquad \forall \phi \in C^\infty_c(\Omega).$$

for every open subset $\Omega' \subset \Omega$ with C^1 boundary.

Furthermore, we will use the following elementary

Lemma 6.4. *Let $G\colon \mathbb{R}^d \to \mathbb{R}$ be a bounded, Borel function. For every $r > 0$ there exists a set of positive measure of real numbers $s = s(r) \in [r, 2r]$ such that*

$$\int_{\partial B_{s(r)}} |G(x)| \, d\mathscr{H}^{d-1}(x) \le \frac{1}{r} \int_{B_{2r}} |G(y)| \, dy.$$

We can now prove Proposition 6.2.

Proof. Let $S \subset M$ be an arbitrary bounded subset. By regularity of λ, there exists an open set $O \supset S$ such that $|\lambda|(O \setminus S) < \varepsilon$. Hence, for any $x \in S$

there exists ρ_x such that $B(x,r) \subset O$ for every $r < \rho_x$, which means that the covering

$$\mathscr{F} := \{B(x,r) : x \in S, r < \rho_x\}$$

is a fine covering of S. Moreover, taking only balls $B(x,r)$ whose radii r satisfy (6.2) with $F(\cdot) := |\cdot|^2$, we still have a fine covering of S.

Hence we can apply Besicovitch-Vitali covering Lemma ([6, Thm. 2.19]): there exists a disjoint (countable) family $\mathscr{F}' := \{B_i\}_{i\in\mathbb{N}}$ such that

$$|\lambda| \left(S \setminus \bigcup_i B_i \right) = 0.$$

On the other hand, since $\bigcup_i B_i \subset O$ by construction, we have

$$|\lambda| \left(\bigcup_i B_i \setminus S \right) < \varepsilon.$$

Let now $x \in M$ be arbitrarily fixed. We write for brevity $B_r = B_r(x)$; by (6.2) with $F(\cdot) := |\cdot|^2$, we get

$$|\lambda(B_r)| = \left| \int_{\partial B_r} ub \cdot \nu d\mathscr{H}^{d-1} \right| \leq C \int_{\partial B_r} |b| d\mathscr{H}^{d-1}.$$

From Lemma 6.4, we have

$$C \int_{\partial B_r} |b| d\mathscr{H}^{d-1} \leq \frac{C}{r} \int_{B_{2r}} |b(x)| dx = o(r^d)$$

because, by definition of M, we have $\fint_{B_r} |b| = o(r)$ for every $x \in M$. Therefore, we can conclude

$$|\lambda(B_r)| = o(r^d). \tag{6.3}$$

Using (6.3), we have

$$\lambda \left(\bigcup_i B_i \right) = \sum_i \lambda(B_i) = o(1)\mathscr{L}^2 \left(\bigcup_i B_i \right).$$

Hence

$$\lambda(S) = \lambda \left(\bigcup_i B_i \right) - \lambda \left(\bigcup_i B_i \setminus S \right) \to 0$$

as $r \downarrow 0$ and this gives that $\lambda \llcorner M = 0$. □

6.1. **Comparison between $\mathscr{L}^2$ and η.** We present here two general lemmas which relate the Lebesgue measure $\mathscr{L}^2$ and the measure η and are based on nearly incompressibility of the vector field b.

Lemma 6.5. *Let $A \subset \mathbb{T}^2$ be a measurable set. Then $\mathscr{L}^2(A) = 0$ if and only if $\eta(\Gamma_A) = 0$ where*

$$\Gamma_A := \left\{ \gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0 \right\}.$$

Proof. Let us prove first that $\mathscr{L}^2(A) = 0$ implies $\eta(\Gamma_A) = 0$. We denote by ρ_A the density such that $\rho_A(t,\cdot)\mathscr{L}^2 = e_{t\#}(\eta \llcorner \Gamma_A)$ and $r_A(x) := \int_0^T \rho_A(t,x)\,dt$. We have, using Fubini,

$$
\begin{aligned}
0 = \mathscr{L}^2(A) = r_A \mathscr{L}^2(A) &= \int_0^T \int_\Gamma \mathbb{1}_A(x)\rho_A(t,x)\,dx\,dt \\
&= \int_0^T \int_\Gamma \mathbb{1}_A(\gamma(t))\,d\eta(\gamma)\,dt \\
&= \int_\Gamma \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\
&= \int_{\Gamma_A} \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\
&= \int_{\Gamma_A} \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\})\,d\eta(\gamma),
\end{aligned}
$$

hence, $\mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) = 0$ for η-a.e. $\gamma \in \Gamma_A$.

For the opposite direction, using that ρ is uniformly bounded from below by $1/C$, we get

$$
\begin{aligned}
\frac{T}{C}\mathscr{L}^2(A) = \frac{T}{C}\int \mathbb{1}_A(x)\,dx = \frac{1}{C}\int_0^T \int \mathbb{1}_A(x)\,dx\,dt &\\
\leq \int_0^T \int \mathbb{1}_A(x)\rho(t,x)\,dx\,dt &\\
= \int_0^T \int_\Gamma \mathbb{1}_A(\gamma(t))\,d\eta(\gamma)\,dt &\\
= \int_\Gamma \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) &\\
= \int_{\Gamma_A} \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) &\\
= \int_{\Gamma_A} \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\})\,d\eta(\gamma) = 0. &\qquad \square
\end{aligned}
$$

Lemma 6.6. *We have $\mathscr{L}^2(A) = 0$ if and only if $\eta(\Gamma_A^s) = 0$ for every $s \in [0,T]$.*

Proof. For direct implication

$$
\begin{aligned}
0 = \mathscr{L}^2(A) &= \int \mathbb{1}_A(x)\rho(s,x)\,dx \\
&= \int_\Gamma \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) \\
&= \int_{\Gamma_A^s} \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) = \eta(\Gamma_A^s).
\end{aligned}
$$

For the opposite direction,

$$\begin{aligned}\frac{1}{C}\mathscr{L}^2(A) &= \frac{1}{C}\int \mathbb{1}_A(x)\,dx \\ &\leq \int \mathbb{1}_A(x)\rho(s,x)\,dx \\ &= \int_\Gamma \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) \\ &= \int_{\Gamma_A^s} \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) = \eta(\Gamma_A^s) = 0.\end{aligned}$$

□

We now recall the set M, defined in (6.1) as

$$M := \left\{x \in \mathbb{T}^2 : b(x) = 0,\ x \in \mathcal{D}_b \text{ and } \nabla^{\text{appr}} b(x) = 0\right\},$$

and we consider the sets

$$\widetilde{\Gamma}_M := \widetilde{\Gamma} \cap \Gamma_M$$

and

$$\widetilde{\Gamma}_A^s := \left\{\gamma \in \widetilde{\Gamma} : \gamma(s) \in A\right\}.$$

Using Proposition 6.2, we can show the following

Lemma 6.7. *Let M be the set defined in (6.1) and for every fixed $s \in [0,T]$ let $\widetilde{\Gamma}_M^s := \{\gamma \in \widetilde{\Gamma} \colon \gamma(s) \in M\}$. Then:*

- $\eta(\widetilde{\Gamma}_M^s) = 0$ *for a.e.* $s \in [0,T]$*;*
- $\eta(\widetilde{\Gamma}_M) = 0$.

Proof. Let us denote by $\eta_M^s := \eta \llcorner \widetilde{\Gamma}_M^s$ and consider the Borel function

$$\rho_M(t,\cdot)\mathscr{L}^2 = e_{t\#}\eta_M^s.$$

It is easy to see that ρ_M solves continuity equation

$$\partial_t \rho_M + \operatorname{div}(\rho_M b) = 0.$$

Integrating in time on $[0,t]$ we get

$$\operatorname{div}\left(b\int_0^t \rho_M(\tau,\cdot)d\tau\right) = (\rho_M(t,\cdot) - \rho_M(0,\cdot))\mathscr{L}^2.$$

In particular, thanks to Proposition 6.2, we have that

$$(\rho_M(t,\cdot) - \rho_M(0,\cdot))\mathscr{L}^2 \llcorner M = 0, \tag{6.4}$$

hence $\rho_M(t,\cdot) = \rho_M(0,\cdot)$, for a.e. x. Furthermore, integrating in space the continuity equation (6.1) we get the conservation of mass:

$$\frac{d}{dt}\int_{\mathbb{T}^2} \rho_M(t,x)\,dx = 0. \tag{6.5}$$

Therefore, using (6.4) and (6.5), we have

$$\begin{aligned}\int_{\mathbb{T}^2\setminus M}\rho_M(t,x)dx &= \int_{\mathbb{T}^2}\rho_M(t,x)dx - \int_M \rho_M(t,x)dx = \\ &= \int_{\mathbb{T}^2}\rho_M(s,x)dx - \int_M \rho_M(s,x)dx = \int_{\mathbb{T}^2\setminus M}\rho_M(s,x)dx = \\ &= \int \mathbb{1}_{\mathbb{T}^2\setminus M}(\gamma(s))d\eta_M(\gamma) = 0,\end{aligned}$$

which gives us $\rho_M(t,\cdot) = 0$ a.e. on $\mathbb{T}^2 \setminus M$. Hence

$$0 = \int_0^T \int_{\mathbb{T}^2\setminus M}\rho_M(t,x)\,dx = \int_0^T \int 1_{\mathbb{T}^2\setminus M}(\gamma(t))\,d\eta_M(\gamma)\,dt$$

and this implies that $\eta^s_M(\widetilde{\Gamma}^s_M) = 0$ for $s \in [0,T]$, since $\gamma \in \widetilde{\Gamma}$ are not constant functions (by definition) and $b = 0$ on M.

Now the second part easily follows from the first one by a Fubini-like argument: indeed, we set

$$I := \int_0^T \eta(\widetilde{\Gamma}^s_M)\,ds = 0.$$

Since $\eta(\widetilde{\Gamma}^s_M) = \int_{\widetilde{\Gamma}} 1_M(\gamma(s))\,d\eta(\gamma)$ and using Fubini's theorem we get

$$I = \int_{\widetilde{\Gamma}}\int_0^T 1_M(\gamma(s))\,ds\,d\eta(\gamma) = 0$$

i.e. $\mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in M\}) = 0$ for η-a.e. $\gamma \in \widetilde{\Gamma}_M$ and this concludes the proof. □

6.2. **Matching properties II.** Let us now consider two balls $B_1, B_2 \in \mathscr{B}$ and suppose that $B_1 \cap B_2 \neq \emptyset$ so that the intersection $B_1 \cap B_2$ is a non empty, open, simply connected set.

Since $\nabla H_1 \parallel \nabla H_2$ (since they are both parallel to b) we can apply Matching Lemma 5.2 and we thus obtain that H_1 and H_2 match: in other words, for a.e. $x \in (B_1 \cap B_2) \cap (E_1 \cap E_2)$ we have that $C^1_x = C^2_x$.

Now, for every fixed ball $\hat{B} \in \mathscr{B}$ we can consider all the balls $B_i \in \mathscr{B}$ such that $\hat{B} \cap B_i \neq \emptyset$: for each of these balls, we take the corresponding $\mathscr{L}^2$ negligible set $N_i \subset \hat{B}$ given by Matching Lemma 5.2 ($x \in N_i$ if $C^i_x \neq \hat{C}_x$, where $\hat{C}_x$ denotes the connected component inside $\hat{B}$ of the level set $H^{-1}_{\hat{B}}(H_{\hat{B}}(x))$). Since H is Lipschitz, we have that $Z_i := H(N_i) \subset \mathbb{R}$ are $\mathscr{L}^1$ null set for every i, hence also $Z := \cup_i Z_i$ is $\mathscr{L}^1$ negligible. This leads us to the final definition of *globally regular level sets*:

Definition 6.8. Let $h \in H(B)$: we say that E_h is a *globally regular* (or simply *regular*) level set if E_h is regular with respect to B and $h \notin Z$.

7. The labeling function

7.1. **Measurable selection of connected components.** We now recall the following

Definition 7.1. A Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *monotone* if the level sets $\{f = t\} := \{x \in \mathbb{R}^n : f(x) = t\}$ are connected for every $t \in \mathbb{R}$.

In [8] the authors proved that a real Lipschitz function of many variables with compact support can be decomposed into a sum of monotone functions: more precisely, we state

Theorem 7.2 ([8, Thm. 1]). *Let $f \in Lip_c(\mathbb{R}^n)$. Then there exists a countable family $\{f_i\}_{i\in\mathbb{N}}$ of functions in $\mathrm{Lip}_c(\mathbb{R}^n)$ such that $f = \sum_i f_i$ and each f_i is monotone. Moreover, there is a pairwise disjoint partition $\{A_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^n$ such that ∇f_i is concentrated on A_i.*

Now we fix a ball $B \in \mathscr{B}$ and the corresponding hamiltonian $H = H_B$. Then by Theorem 7.2 there exist countably many monotone Lipschitz functions $H_i \colon B \to \mathbb{R}$ such that $H = \sum_{i=1}^{\infty} H_i$ and for $i \neq j$ we have

$$\mathscr{L}^2(\{\nabla H_i \neq 0\} \cap \{\nabla H_j \neq 0\}) = 0.$$

In particular, we deduce that $E = \bigcup_{i\in I} E_i \mod \mathscr{L}^2$, where E_i are the regular level sets of the Hamiltonian H_i; therefore, we can define for every $x \in \mathbb{T}^2$ the map

$$k(x) := \sum_i i \mathbb{1}_{E_i}(x) \tag{7.1}$$

which is Borel thanks to [1, Appendix 6, Proposition 6.1].

7.2. **Construction of the labeling function.** We now turn to the construction of a suitable "labeling" function f which assigns to a point $x \in \mathbb{T}^2$ the label of the maximal extension of the level set of H passing through x. First we define the set where the labels take values:

$$\mathfrak{A} := \mathbb{N} \times \mathbb{R} \times \mathbb{N} \cup \{(+\infty, +\infty, +\infty)\}.$$

Then we introduce on $\mathfrak{A}$ an ordering as follows: if

$$\mathfrak{a}_1 := (n_1, h_1, k_1) \quad \text{and} \quad \mathfrak{a}_2 := (n_2, h_2, k_2),$$

then

$$\mathfrak{a}_1 < \mathfrak{a}_2 \iff \begin{cases} \text{either } [n_1 < n_2], \\ \text{or } [n_1 = n_2 \text{ and } h_1 < h_2], \\ \text{or } [n_1 = n_2 \text{ and } h_1 = h_2 \text{ and } k_1 < k_2]. \end{cases}$$

Notice that this is the standard lexicographic ordering on the product of ordered sets.

We construct f as pointwise limit of a sequence of Borel functions $f_n \colon \mathbb{T}^2 \to \mathfrak{A}$ which we define inductively. We set $f_0 \equiv +\infty$, where for brevity we write $+\infty := (+\infty, +\infty, +\infty)$. Then we define an auxiliary function $\widetilde{f}_{n+1}$ with support inside of B_{n+1}. More precisely, if $x \in B_{n+1}$ we call C_x the connected component of the level set $H_{n+1}^{-1}(H_{n+1}(x))$ which contains x and define

$$Y_x := \{y \in C_x : f_n(y) \neq +\infty\}.$$

Then set

$$\xi(x) := \begin{cases} \min_{Y_x} f_n(y) & \text{if } Y_x \neq \emptyset, \\ +\infty & \text{otherwise.} \end{cases}$$

Now let

$$\widetilde{f}_{n+1}(x) = \begin{cases} \xi(x) & \text{if } \xi(x) \neq +\infty, \\ +\infty & \text{if } \xi(x) = +\infty,\ x \notin E_{n+1}, \\ (n+1, H_{n+1}(x), k(x)) & \text{if } \xi(x) = +\infty, x \in E_{n+1}, \end{cases}$$

where E_{n+1} is the set of points which belong to regular level sets of H_{n+1} and $k(\cdot)$ is the function defined in (7.1). Using this auxiliary function, we define for every $x \in \mathbb{T}^2$

$$f_{n+1}(x) := \begin{cases} \widetilde{f}_{n+1}(x) & \text{if } x \in B_{n+1}, \\ \min\left\{f_n(x), \widetilde{f}_{n+1}(f_n^{-1}(f_n(x)))\right\} & \text{if } x \in B_1 \cup \ldots \cup B_n, \\ +\infty & \text{otherwise.} \end{cases}$$

The definition of this function takes into a account two possible situations:

a) different level sets (i.e. different values of H) could join only later in the construction (see Figure 1a). Therefore, at each step we define f_{n+1} not only inside the new ball but we also update the values outside by minimization (in order to make the sequence monotonically decreasing);
b) different connected component of the same level set could have disjoint extensions and this is the reason why we include also the function $k(\cdot)$, which roughly speaking corresponds to the number of the connected component of the level set (see Figure 1b).

The function f_n converges because it is fairly easy to see that it is monotonically decreasing and therefore we can define

$$f(x) := \lim_{n \to +\infty} f_n(x).$$

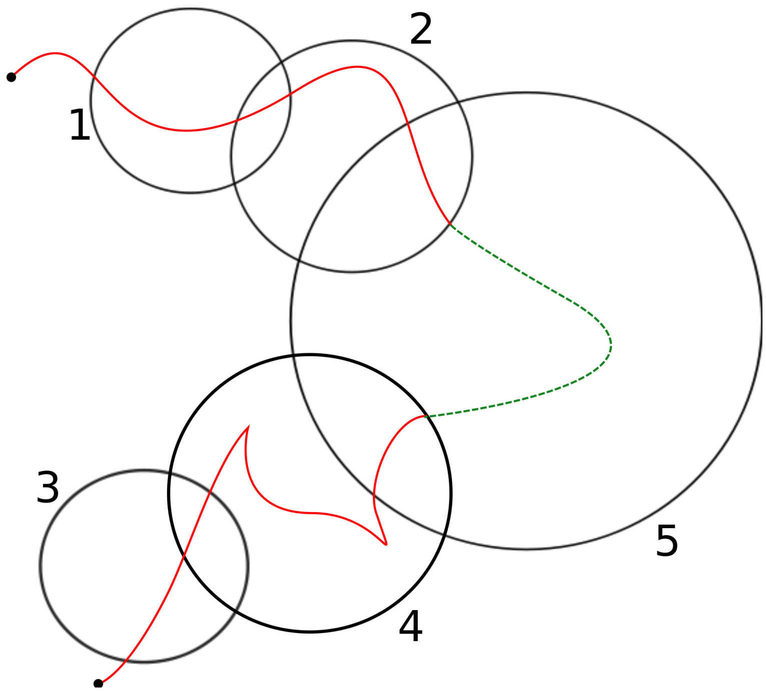
7.3. **Properties of the level sets of the labeling function.** We now prove some properties of the level sets of the function f constructed in the paragraph above. We denote $F_{\mathfrak{a}} = f^{-1}(\mathfrak{a})$ for every $\mathfrak{a} \in f(\mathbb{T}^2) \subset \mathfrak{A}$.

7.4. **Level sets are closed curves or simple Lipschitz curves.** We have the following

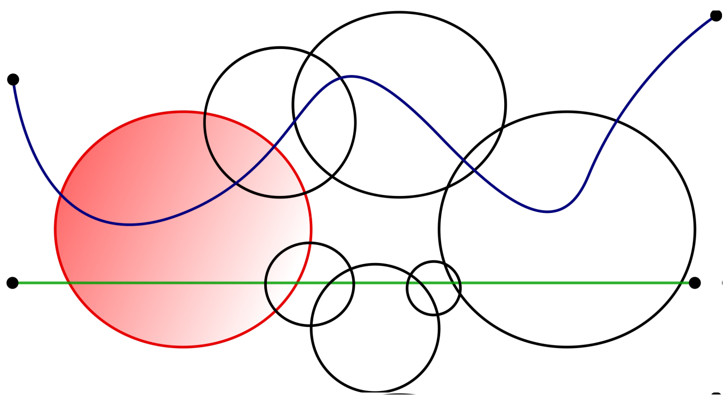
Lemma 7.3. *For any $\mathfrak{a} \in \mathfrak{A} \setminus \{+\infty\}$, the level set $F_{\mathfrak{a}}$ is either a closed curve or a simple Lipschitz curve. As a consequence, for every $B \in \mathscr{B}$ the set $F_{\mathfrak{a}} \cap B$ has at most countably many connected components.*

Proof. Fix $\mathfrak{a} := (m, h, k) \in f(\mathbb{T}^2) \setminus \{+\infty\}$ and suppose that $f^{-1}(\mathfrak{a})$ is not a closed curve. By the construction of f, $F_{\mathfrak{a}}$ must intersect the ball B_m: in particular, $F_{\mathfrak{a}} \cap B_m$ coincides with the k-th connected component of the level set $\{H_{B_m} = h\}$. Call this connected component Σ_m. We now distinguish two cases: either $\Sigma_m = F_{\mathfrak{a}}$ (in this case the lemma is proved) or $F_{\mathfrak{a}}$ is strictly *bigger* than C.

Suppose thus that $\Sigma_m \subset F_{\mathfrak{a}}$: by definition of f, $F_{\mathfrak{a}}$ cannot intersect B_i with $i < m$ in a regular level set of H_i (since f is defined taking minima). On the other hand, $F_{\mathfrak{a}}$ intersects some ball B_j with $j > m$ (since the balls of $\mathscr{B}$ cover all $\mathbb{T}^2$). In particular, let us consider B_k where $k := \min\{s \in \mathbb{N} : B_s \cap B_m \cap F_{\mathfrak{a}} \neq \emptyset\}$. The intersection $F_{\mathfrak{a}} \cap B_k$ must coincide with a regular



(a) Different level sets can join later: in the balls 1-2 and 3-4 the red curve has two different “labels”: when we turn to consider ball with number 5 we join these two pieces with the dashed green curve. The construction of the function f takes into account this situation.



(b) Different connected components of the same level set can have disjoint extensions: inside the red-shaded ball, the blue curve and the green one are two different connected components of the same level set of the corresponding Hamiltonian. As the picture shows, they have disjoint extensions.

Figure 1. Level sets of the Hamiltonians and of the function f.

level set of H_k and it is therefore a Lipschitz curve (we remark that this curve cannot be closed, otherwise we would be able to find a smaller ball $B' \in \mathscr{B}$ such that $F_{\mathfrak{a}} \cap B'$ contains a triod, which contradicts the regularity of the level set). If we iterate countably many times this procedure, we end up with a covering of F by balls B_i and thus our problem is shifted to prove that if we "glue" two Lipschitz curves we obtain a Lipschitz curve.

Therefore, let us consider $\Sigma_k := F_{\mathfrak{a}} \cap B_k$ (where k is defined above). Σ_k and Σ_m have finite length (because they are a connected component of a regular level set). Take *natural* parametrizations $\gamma_k \colon I_k \to \Sigma_k$ and $\gamma_m \colon I_m \to \Sigma_m$, where $I_k = (\alpha_k, \beta_k)$ and $I_m = (\alpha_m, \beta_m)$ are bounded intervals (because the curves have finite length). Let $t_k \in I_k$ be such that $\gamma_k(t_k) \in \partial B_m$: up to a translation, we can suppose that $t_k = \alpha_m$. Hence we can glue together the parametrizations, obtaining a function $\gamma_{m+1} \colon I_{m+1} \to \mathbb{T}^2$ where $I_{m+1} = (\alpha_k, \beta_m)$ and

$$x \mapsto \gamma_{m+1}(x) := \begin{cases} \gamma_1(t) & \text{if } t \in \widetilde{I}_k := (\alpha_k, t_k) \\ \gamma_m(t) & \text{if } t \in \widetilde{I}_m := [\alpha_m, \beta_m). \end{cases}$$

The function γ_{m+1} is injective, Lipschitz and its image is clearly $\Sigma_k \cup \Sigma_m$: injectivity is trivial and we just have to prove Lipschitz estimate for t, s belonging to different intervals. More precisely, let $t \in \widetilde{I}_k$ and $s \in \widetilde{I}_m$: then

$$\begin{aligned} |\gamma_{m+1}(t) - \gamma_{m+1}(s)| &= |\gamma_k(t) - \gamma_m(s)| \\ &\le |\gamma_k(t) - \gamma_k(t_k)| + |\gamma_k(t_k) - \gamma_m(s)| \\ &\le L_k|t - t_k| + L_m|t_k - s| \\ &\le L|t - s|, \end{aligned}$$

where L is the maximum of the Lipschitz constants L_k, L_m.

Then the second part easily follows noticing that any family of disjoint open intervals in $\mathbb{R}$ is at most countable. □

Remark 7.4. From inspection of the previous proof, one sees that the level set $F_{\mathfrak{a}}$ (when it is not a closed curve) can be parametrized by an injective Lipschitz function $\gamma_{\mathfrak{a}}$; moreover, we can choose the parametrization to respect the direction and modulus of b, i.e. $\dot{\gamma_{\mathfrak{a}}} = b(\gamma_{\mathfrak{a}})$. From now onwards, we will always assume that $F_{\mathfrak{a}}$ are parametrized in this way and we will refer to this parametrization as the *canonical parametrization.*

Remark 7.5. Thanks to Lemma 7.3, we can assign (in a unique way) to every connected component of $F_{\mathfrak{a}} \cap B$ a rational number $q \in \mathbb{Q}$ (respecting the canonical parametrization of $F_{\mathfrak{a}}$).

We have thus proved that for every $\mathfrak{a} \in \mathfrak{A}$, there exists a Lipschitz, injective parametrization $\gamma_{\mathfrak{a}} \colon I_{\mathfrak{a}} \to \mathbb{T}^2$ of $F_{\mathfrak{a}}$, where $I_{\mathfrak{a}}$ is either an open interval or $\mathbb{R}/(L_{\mathfrak{a}}\mathbb{Z})$ for some $L_{\mathfrak{a}} > 0$. From now onwards, we denote by $\mathfrak{A}_{\mathrm{NP}}$ the set of labels of non-periodic curves, i.e.

$$\mathfrak{A}_{\mathrm{NP}} := \Big\{ \mathfrak{a} \in f(\mathbb{T}^2) : \, I_{\mathfrak{a}} = (\alpha_{\mathfrak{a}}, \beta_{\mathfrak{a}}), \, \alpha_{\mathfrak{a}}, \beta_{\mathfrak{a}} \in \mathbb{R} \cup \{\pm\infty\} \Big\}.$$

We are now ready to prove a lemma about the relation between level sets $F_{\mathfrak{a}}$ and the trajectories of b.

Lemma 7.6. *There exists a η-negligible set $N \subset \Gamma$ such that for every $\gamma \in \Gamma \setminus N$ the function $f \circ \gamma \colon (0,T) \to \mathfrak{A}$ is constant.*

Proof. Applying Lemmas 4.1 and 4.2 countably many times, we can construct a subset $N \subset \Gamma$, with $\eta(N) = 0$, with the following property: for every $\gamma \in \Gamma \setminus N$, for every $B \in \mathscr{B}$, if $\gamma \in \mathsf{T}_B$ then $\gamma((0,T)) \cap B$ is contained in a regular level set of H_B. For any $\gamma \in \Gamma \setminus N$, set $C := \gamma((0,T))$.

Fix $\varepsilon > 0$: by compactness and connectedness, we can cover the set $C_\varepsilon := \gamma([\varepsilon, T-\varepsilon])$ with finitely many balls $B_1, \dots, B_n \in \mathscr{B}$ such that for every i, there exists $j \neq i$ such that $B_i \cap B_j \neq \emptyset$ (otherwise C would be disconnected). In particular, we observe that by construction, for every $i = 1, \dots, n$, the set $C_\varepsilon \cap B_i$ coincides with (a connected component of) the image of some $\tilde{\gamma}^i_\varepsilon \in \mathsf{T}_{B_i}$ (within the ball B_i).

Now by the construction of N, H_i is constant along connected components of $\tilde{\gamma}^i_\varepsilon(0,T) \cap B_i$ and let h_i be the value attained by H_i; on the other hand, f is constant along connected component of E_{h_i} hence f is constant along $C_\varepsilon \cap B$. Since the balls do intersect, the function f must be constant on C. □

Thus we have proved that for η-a.e. γ there exists $\mathfrak{a} \in \mathfrak{A}$ such that $\gamma \subset \gamma_{\mathfrak{a}}$, meaning that γ is a parametrization of some part of $\gamma_{\mathfrak{a}}$. We wonder how γ and $\gamma_{\mathfrak{a}}$ are related when $\mathfrak{a} \in \mathfrak{A}_{\mathrm{NP}}$. The answer is given by the following

Proposition 7.7. *Let N be the set given by Lemma 7.6 and let $\gamma \in \widetilde{\Gamma} \setminus N$ be fixed. Suppose that $f \circ \gamma \equiv \mathfrak{a}$ where $\mathfrak{a} \in \mathfrak{A}_{\mathrm{NP}}$. Then γ coincides with $\gamma_{\mathfrak{a}}$ up to a translation in time (where $\gamma_{\mathfrak{a}}$ is the canonical parametrization restricted to some time interval).*

In order to prove Proposition 7.7, we need the following auxiliary

Lemma 7.8. *Let $\gamma \colon I \to \mathbb{T}^2$ be a solution of the ordinary differential equation*

$$\dot{\gamma}(t) = b(\gamma(t)), \qquad t \in I \subset \mathbb{R},$$

where $I = [0,T]$ and $\frac{1}{|b|} \in L^1_{\mathrm{loc}}(\mathscr{H}^1 \llcorner \gamma(I))$. Assume that there exists a injective curve $\hat{\gamma}$ defined on I such that $\gamma(I) \subset \hat{\gamma}(I)$ and that $\dot{\hat{\gamma}} = b(\hat{\gamma})$. Then for any $t \in I$

$$\int_{\gamma((t_0,t))} \frac{d\mathscr{H}^1(w)}{|b(w)|} = (t - t_0) - \mathscr{L}^1(\{t \in [0,T] : \gamma'(t) = 0\}).$$

Proof. For fixed $t > 0$, observe that

$$\begin{aligned}\int_{\gamma((t_0,t))} \frac{d\mathscr{H}^1(w)}{|b(w)|} &\overset{(1)}{=} \int_{\gamma((t_0,t))} \frac{\mathbb{1}_{\{b\neq 0\}}(w)\, d\mathscr{H}^1(w)}{|b(w)|} \\ &\overset{(2)}{=} \int_{\{t\in[0,T] :\, \gamma'(t)\neq 0\}} \frac{|\gamma'(\tau)|}{|b(\gamma(\tau))|} d\tau \\ &= t - \mathscr{L}^1(\{t \in [0,T] : \gamma'(t) = 0\}),\end{aligned}$$

where

(1) follows by definition;
(2) is the Area formula, i.e. $\mathscr{H}^1 \llcorner C = \gamma_\#(|\gamma'|\mathscr{L}^1)$, where $C = \gamma((0,T))$, which can be applied because there exists $\hat{\gamma}$ by hypothesis.

This concludes the proof. □

Now we can prove Proposition 7.7.

Proof. Let $\overline{s} \in I_{\mathfrak{a}}$ such that $\gamma_{\mathfrak{a}}(\overline{s}) = \gamma(0)$. By Lemma 7.8, we have

$$\int_{\gamma((t_0,t))} \frac{d\mathscr{H}^1(w)}{|b(w)|} = t - \mathscr{L}^1([0,T] \cap \gamma^{-1}(\{b=0\})). \tag{7.2}$$

By Lemma 6.7 and the fact that $\mathscr{L}^2(\{b=0\} \setminus M) = 0$, where M is defined in (6.1), we know that for η-a.e. $\gamma \in \widetilde{\Gamma}$,

$$\mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in \{b=0\}\}) = 0,$$

hence (7.2) is actually

$$\int_{\gamma((t_0,t))} \frac{d\mathscr{H}^1(w)}{|b(w)|} = t. \tag{7.3}$$

On the other hand, applying again Lemma 7.8 to $\gamma_{\mathfrak{a}}$, which is injective, we get

$$\int_{\gamma_{\mathfrak{a}}(\overline{s}, t+\overline{s})} \frac{d\mathscr{H}^1(w)}{|b(w)|} = (t+\overline{s}) - \overline{s} = t. \tag{7.4}$$

Since, by definition, $\gamma_{\mathfrak{a}}(\overline{s}) = \gamma(0)$, comparing (7.3) and (7.4) and using the fact that $|b| > 0$ $\mathscr{H}^1$-a.e. on γ, we deduce that

$$\gamma(t) = \gamma_{\mathfrak{a}}(t+\overline{s})$$

which means that γ and $\gamma_{\mathfrak{a}}|_{[0,T]}$ coincide up to a translation in time. □

7.5. **Connected components of level sets of H and f.** We now want to exploit the connections between the level sets $F_{\mathfrak{a}}$ and the level sets of the Hamiltonians H. In particular, we prove that, inside of a ball, there is a bijection between the connected components of these level sets.

Let $B \in \mathscr{B}$ be fixed and consider the Hamiltonian $H = H_B$ and the triple associated to it (H, N, E) as in Section 3.2. For any $\vartheta \notin N$ and $l \in \mathbb{N}$ let $C_{\vartheta,l}$ denote the l-th connected component of E_ϑ (which can be empty set for some values of l).

Thanks to Lemma 7.3 and, in particular, to Remark 7.4, we can denote by $C_{\mathfrak{a},q}$ the q-th connected component of $F_{\mathfrak{a}} \cap B$, where $q \in \mathbb{Q}$. By the construction of f, for any $\vartheta \in H(E)$ and any $l \in \mathbb{N}$ such that $C_{\vartheta,l} \neq \emptyset$, there exists a unique $\mathfrak{a} \in \mathfrak{A}$ such that $C_{\vartheta,l} \subset F_{\mathfrak{a}}$; hence, due to connectedness, there exists unique $q \in \mathbb{Q}$ such that $C_{\vartheta,l} = C_{\mathfrak{a},q}$. Now fix $l \in \mathbb{N}$ and $q \in \mathbb{Q}$ and set

$$\Theta_l := \{\vartheta \in H(E) : C_{\vartheta,l} \neq \emptyset\}.$$

Then for any $\vartheta \in \Theta_l$ we define

$$A_{l,q}(\vartheta) := \mathfrak{a},$$

where $\mathfrak{a} \in \mathfrak{A}$ is the unique label such that $C_{\vartheta,l} = C_{\mathfrak{a},q}$. By construction, the function $A_{l,q}$ is injective.

Lemma 7.9. *We have that*

$$\{f \neq +\infty\} = \{b \neq 0\} \mod \mathscr{L}^2.$$

Proof. Let us call $E := \bigcup_{B\in\mathscr{B}} E_B$. On one hand, it is easy to see that

$$\{b \neq 0\} = E \mod \mathscr{L}^2. \tag{7.5}$$

Indeed, thanks to Remark 2.2 and to Lemma 6.7, if $b(x) \neq 0$ then $x \in E_B$ for some $B \in \mathscr{B}$; on the other hand, by Weak Sard Property we have $\mathscr{L}^2(E_B \cap \{b = 0\}) = 0$ for every $B \in \mathscr{B}$, hence we have (7.5). Now we show that

$$\{f \neq +\infty\} = E \mod \mathscr{L}^2. \tag{7.6}$$

If $x \in \mathbb{T}^2 \setminus E$, then x belongs to a non regular level set for H_B for every $B \in \mathscr{B}$: in particular, for every $n \in \mathbb{N}$, $f_n(x) = +\infty$ hence, passing to the limit, $f(x) = +\infty$. The other inclusion is also easy: if $x \in E_B$ for some $B \in \mathscr{B}$ then necessarily it has a label and hence $f(x) \neq +\infty$. The lemma now follows from (7.5) and (7.6). □

8. Disintegration with the labeling function

Applying Disintegration Theorem we get

$$\mathscr{L}^2 = \int_{\mathfrak{a}\neq+\infty} \Lambda_{\mathfrak{a}}\, d\xi(\mathfrak{a}) + \mathscr{L}^2 \llcorner \{f = +\infty\}, \tag{8.1}$$

where $\xi = f_\# \mathscr{L}^2 \llcorner \{f \neq +\infty\}$ and $\Lambda_{\mathfrak{a}}$ are concentrated on $F_{\mathfrak{a}}$. In the same way, for any Radon measure μ we write

$$\mu = \int_{\mathfrak{a}\neq+\infty} \mu_{\mathfrak{a}}\, d\xi(\mathfrak{a}) + \int_{\mathfrak{a}\neq+\infty} \nu_{\mathfrak{a}}\, d\sigma(\mathfrak{a}) + \mu \llcorner \{f = +\infty\} \tag{8.2}$$

where we denote by

$$\sigma = \left[f_\#(|\mu| \llcorner \{f \neq +\infty\})\right]^{\text{sing}} \tag{8.3}$$

the singular component (with respect to ξ) of the measure $f_\#(|\mu| \llcorner \{f \neq +\infty\})$.

8.1. Comparison of disintegrations of $\mathscr{L}^2$ and of μ.

In Section 7.5 we have built a bijection that allows us to relate disintegrations (8.1) and (8.2) with the disintegration w.r.t. the level sets of the Hamiltonian H_B in each ball $B \in \mathscr{B}$.

Lemma 8.1. *Let $B \in \mathscr{B}$ be fixed. For any $l \in \mathbb{N}$ and for any $q \in \mathbb{Q}$, we have $(A_{l,q})_\# (\mathscr{L}^1 \llcorner \Theta_l) \ll \xi$.*

Proof. Let $E \subset \mathfrak{A}$ such that $\xi(E) = 0$, i.e. $\mathscr{L}^2(f^{-1}(E)) = 0$. Hence also $\mathscr{L}^2(f^{-1}(E) \cap B) = 0$ and, since H_B is Lipschitz, $\mathscr{L}^1(H_B(f^{-1}(E) \cap B)) = 0$. The claim now follows because we have $A_{l,q}^{-1}(E) \subset H_B\left(f^{-1}(E) \cap B\right)$. □

In particular, by Lemma 8.1, applying Radon-Nikodým Theorem, we get that, for every $l \in \mathbb{N}$ and $q \in \mathbb{Q}$, there exists a function $g_{l,q} \in L^1(\mathfrak{A}, \xi)$ such that

$$(A_{l,q})_\# (\mathscr{L}^1 \llcorner \Theta_l) = g_{l,q}\xi.$$

Let us now set

$$G := G_l^q := \Big\{x \in B : \exists \vartheta \in \Theta_l,\ \exists \mathfrak{a} \in A_{l,q}(\Theta_l) \text{ such that } x \in C_{\vartheta,l} \cap C_{\mathfrak{a},q}\Big\}.$$

This allows us to compare the two disintegrations of μ on G: indeed, on the one hand we have, by (3.4) and $\nu_h \llcorner G = 0$,

$$\begin{aligned}\mu \llcorner G = \int \mu_h \llcorner G \, dh &= \int_{\Theta_l} \mu_h \, dh \\ &= \int_{\Theta_l} \mu_{A_{l,q}^{-1}(A_{l,q}(h))} \, dh \\ &= \int_{A_{l,q}(\Theta_l)} \mu_{A_{l,q}^{-1}(\mathfrak{a})} g_{l,q}(\mathfrak{a}) \, d\xi(\mathfrak{a}).\end{aligned} \tag{8.4}$$

On the other hand, from (8.2) we have

$$\mu \llcorner G = \int_{A_{l,q}(\Theta_l)} \mu_{\mathfrak{a}} \llcorner G \, d\xi(\mathfrak{a}) + \int_{A_{l,q}(\Theta_l)} \nu_{\mathfrak{a}} \llcorner G \, d\sigma(\mathfrak{a}). \tag{8.5}$$

Comparing (8.4) and (8.5) we deduce that $\sigma = 0$ on $A_{l,q}(\Theta_l)$ and that ξ-a.e. $\mathfrak{a} \in A_{l,q}(\Theta_l)$ we have

$$\mu_{\mathfrak{a}} = g_{l,q}(\mathfrak{a}) \mu_{A_{l,q}^{-1}(\mathfrak{a})},$$

which is, since $A_{l,q}^{-1}(\mathfrak{a}) = h$,

$$\mu_{\mathfrak{a}} = g_{l,q}(\mathfrak{a}) \mu_h.$$

This means that $\mu_{\mathfrak{a}}$ and μ_h on G coincide (up to the density $g_{l,q}$).

For Lebesgue measure, using (3.2) and arguing in the same way, we get

$$\widetilde{g}_{l,q}(h) \Lambda_{A_{l,q}(h)} \llcorner G = c_h \mathscr{H}^1 \llcorner (E_h \cap G).$$

for some density $\widetilde{g}_{l,q}$. We conclude that

$$\Lambda_{\mathfrak{a}} = c_{\mathfrak{a}} \mathscr{H}^1 \llcorner F_{\mathfrak{a}}$$

for some function $c_{\mathfrak{a}} \in L^1(\mathscr{H}^1 \llcorner F_{\mathfrak{a}})$ and

$$c_{\mathfrak{a}} \mathscr{H}^1 \llcorner (F_{\mathfrak{a}} \cap G) = c_h \mathscr{H}^1 \llcorner (E_h \cap G),$$

where $\mathfrak{a} = A_{l,q}(h)$. Hence we have

$$\mathscr{L}^2 \llcorner \{f \neq +\infty\} = \int_{\mathfrak{a} \neq +\infty} c_{\mathfrak{a}} \mathscr{H}^1 \llcorner F_{\mathfrak{a}} \, d\xi(\mathfrak{a}).$$

The following lemma is elementary, we prove it for completeness.

Lemma 8.2. *Let* $\gamma_1 \colon \left[0, \frac{T}{2}\right] \to \mathbb{T}^2$, $\gamma_2 \colon [0,T] \to \mathbb{T}^2$ *be Lipschitz functions such that*

$$\begin{aligned}\gamma_1'(t) &= b(\gamma_1(t)), \qquad a.e.\ t \in \left[0, \frac{T}{2}\right], \\ \gamma_2'(t) &= b(\gamma_2(t)), \qquad a.e.\ t \in [0,T].\end{aligned}$$

Assume that $\gamma_1(\cdot)$ *is injective and that*

$$\gamma_2([0,T]) \subset \gamma_1\left(\left[0, \frac{T}{2}\right]\right).$$

Then

$$\mathscr{L}^1\big(\{t \in [0,T] : \gamma_2'(t) = 0\}\big) \geq \frac{T}{2}.$$

Proof. We have

$$
\begin{aligned}
\mathscr{L}^1(\{t \in [0,T] : \gamma_2'(t) \neq 0\}) &= \int_0^T \mathbb{1}_{\{\gamma_2' \neq 0\}}(t)\, dt \\
&= \int_0^T \mathbb{1}_{\{\gamma_2' \neq 0\}}(t) \frac{|\gamma_2'(t)|}{|b(\gamma_2(t))|} dt \\
&= \int_{\gamma_2((0,T))} \frac{d\mathscr{H}^1(w)}{|b(w)|} \\
&\leq \int_{\gamma_1((0,\frac{T}{2}))} \frac{d\mathscr{H}^1(w)}{|b(w)|} = \frac{T}{2},
\end{aligned}
$$

where the last equality follows by Lemma 7.8. □

We can now prove

Lemma 8.3. *For ξ-a.e. $\mathfrak{a} \in \mathfrak{A}_{\mathrm{NP}}$ the interval $I_\mathfrak{a}$ coincides with the entire real line, i.e. $I_\mathfrak{a} = \mathbb{R}$.*

Proof. Consider the sets

$$\mathfrak{A}^- := \{\mathfrak{a} \in \mathfrak{A}_{\mathrm{NP}} : \alpha_\mathfrak{a} > -\infty\}$$

and

$$\mathfrak{A}^+ := \{\mathfrak{a} \in \mathfrak{A}_{\mathrm{NP}} : \beta_\mathfrak{a} < +\infty\}.$$

To get the desired conclusion, it is enough to prove that $\xi(\mathfrak{A}^-) = 0$ (for $\mathfrak{A}^+$ the proof is analogous).

Let us argue by contradiction: suppose that $\xi(\mathfrak{A}^-) > 0$ and consider the set of points

$$G := \bigcup_{\mathfrak{a} \in \mathfrak{A}^-} \gamma_\mathfrak{a}\left(\left(\alpha_\mathfrak{a}, \alpha_\mathfrak{a} + \frac{T}{2}\right]\right).$$

Being $\xi(\mathfrak{A}^-) > 0$ we have by disintegration

$$\mathscr{L}^2(G) = \int_{\mathfrak{A}^-} \left[\int_G c_\mathfrak{a} \mathscr{H}^1 \llcorner F_\mathfrak{a}\right] d\xi(\mathfrak{a}) > 0,$$

because $\mathscr{H}^1(G \cap F_\mathfrak{a}) > 0$ for every $\mathfrak{a}$ due to injectivity of $\gamma_\mathfrak{a}$. By Lemma 6.6 we have $\eta(\Gamma_G^T) > 0$.

Now we prove that $\widetilde{\Gamma}_G^T \subset \widetilde{\Gamma}_M$: indeed, for every $\gamma \in \widetilde{\Gamma}_G^T$ we have that

$$\mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in M\}) \geq \frac{T}{2}$$

by Lemma 8.2. Therefore,

$$\eta(\widetilde{\Gamma}_G^T) \leq \eta(\widetilde{\Gamma}_M) = 0$$

by Lemma 6.5. Furthermore, we have also $\eta(\dot{\Gamma}_G^T) = 0$ because on G we have $b \neq 0$ a.e. We have thus reached a contradiction and the lemma is proved. □

Remark 8.4. Let us consider a label $\mathfrak{a} \in \mathfrak{A}_{\mathrm{NP}}$ such that the orbit $\gamma_\mathfrak{a}(\mathbb{R})$ is bounded: Lemma 8.3 actually shows that the limit points of $\gamma_\mathfrak{a}$ as $s \to \pm\infty$ are not reached in finite time. The same conclusion holds in the case where only one between $\gamma_\mathfrak{a}(\{t > 0\})$ or $\gamma_\mathfrak{a}(\{t < 0\})$ is compact.

9. Renormalization and proof of the main theorem

Lemma 9.1. *Suppose that $b \in L^\infty(\mathbb{T}^2) \cap \mathrm{BV}(\mathbb{T}^2)$ is a nearly incompressible vector field. Let μ be a Radon measure on $\mathbb{T}^2$ and $u \in L^\infty(\mathbb{T}^2)$. Assume that u solves*

$$\partial_s(\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|) = \hat{\mu}_{\mathfrak{a}} \qquad \text{in } \mathscr{D}'(I_{\mathfrak{a}}) \text{ for } \xi\text{-a.e. } \mathfrak{a} \neq +\infty, \tag{9.1}$$

where $I_{\mathfrak{a}} = \mathbb{R}$ or $I_a = \mathbb{R}/\ell_{\mathfrak{a}}\mathbb{Z}$, with $\ell_{\mathfrak{a}} > 0$. Then

$$\operatorname{div}\left(uc_{\mathfrak{a}}b\mathscr{H}^1 \llcorner F_{\mathfrak{a}}\right) = \mu_{\mathfrak{a}} \qquad \text{in } \mathscr{D}'(\mathbb{T}^2) \text{ for } \xi\text{-a.e. } \mathfrak{a} \neq +\infty. \tag{9.2}$$

Recall that $\hat{u} = u \circ \gamma$, $\hat{c}_a = c_a \circ \gamma$, $\hat{b} = b \circ \gamma$ are defined as $L^1(\mathscr{H}^1 \llcorner F_{\mathfrak{a}})$ for ξ-a.e. $\mathfrak{a} \neq +\infty$, while $\hat{\mu}_{\mathfrak{a}}$ is pull back of the conditional probability $\mu_{\mathfrak{a}}$.

Proof. We note first that it is enough to prove that (9.2) holds in $\mathscr{D}'(B)$ for every ball $B \subset \mathbb{T}^2$.

Therefore, fix a ball B and, for ξ-a.e. $\mathfrak{a} \neq +\infty$, set $J_{\mathfrak{a}} := \gamma_{\mathfrak{a}}^{-1}(B)$ which is an open set; by assumption, $\partial_s(\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|) = \hat{\mu}_{\mathfrak{a}}$ holds in $\mathscr{D}'(J_{\mathfrak{a}})$. Moreover, we have that $\mu_{\mathfrak{a}} \llcorner (F_{\mathfrak{a}} \cap B)$ is finite and also $c_{\mathfrak{a}}\mathscr{H}^1 \llcorner (F_{\mathfrak{a}} \cap B)$ is finite (because they are disintegration of finite measures): this implies

$$c_{\mathfrak{a}} \in L^1(\mathscr{H}^1 \llcorner (F_{\mathfrak{a}} \cap B)).$$

Therefore, $\hat{c}_{\mathfrak{a}} \in L^1(J_{\mathfrak{a}})$ and $\hat{\mu}_{\mathfrak{a}}$ is a finite measure on $J_{\mathfrak{a}}$ (since it is the pull back of a finite measure by an injective function). In particular, we have that

$$(\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|) \in L^1(J_{\mathfrak{a}}),$$

and its distributional derivative is a finite measure: therefore

$$(\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|) \in \mathrm{BV} \cap L^1(J_{\mathfrak{a}}).$$

If we take $\phi \in C_c^\infty(B)$ we observe that, in general, $\hat{\phi} := \phi(\gamma_a)$ is a Lipschitz function on $J_{\mathfrak{a}}$ but it is not necessarily *compactly supported* in $J_{\mathfrak{a}}$. Therefore, we cannot conclude directly from (9.1). On the other hand, we can consider the following functional

$$\Lambda_{\mathfrak{a}}(\phi) := \int_{J_{\mathfrak{a}}} \hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|\partial_s\hat{\phi}\, ds + \int_{J_{\mathfrak{a}}} \hat{\phi} d\hat{\mu}_{\mathfrak{a}}, \qquad \forall \phi \in C_c^\infty(B). \tag{9.3}$$

We now define

$$\partial J_{\mathfrak{a}}^B := \{s \in \mathbb{R} : \gamma_{\mathfrak{a}}(s) \in \partial B\};$$

moreover, if $J_{\mathfrak{a}}$ is unbounded, we call $\partial J_{\mathfrak{a}}^\infty$ the set of its non-finite endpoints. We observe that $(\hat{u}\hat{c}_a|\hat{b}|\hat{\phi})_{\partial J_a^B} = 0$ because ϕ has compact support in the ball. Therefore, integrating by parts the first integral in (9.3), we obtain

$$\Lambda_{\mathfrak{a}}(\phi) = (\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|\hat{\phi})_{\partial J_{\mathfrak{a}}^\infty}.$$

On the other hand, using Remark 8.4, we see that the function $\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|$ is defined on an unbounded interval hence, being BV, it must be $(\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|)_{\partial J_{\mathfrak{a}}^\infty} = 0$. Therefore, we get that for every $\phi \in C_c^\infty(B)$, $\Lambda_{\mathfrak{a}}(\phi) = 0$, i.e.

$$\int_{J_{\mathfrak{a}}} \hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|\partial_s\hat{\phi}\, ds + \int_{J_{\mathfrak{a}}} \hat{\phi}\, d\hat{\mu}_{\mathfrak{a}} = 0$$

which gives, coming back from parametrizations,

$$\int_B uc_{\mathfrak{a}}b\cdot\nabla\phi d\mathscr{H}^1 \llcorner F_{\mathfrak{a}} + \int_B \phi\, d\mu_{\mathfrak{a}} = 0$$

i.e. $\operatorname{div}(uc_{\mathfrak{a}}b\mathscr{H}^1 \llcorner F_{\mathfrak{a}}) = \mu_{\mathfrak{a}}$ in $\mathscr{D}'(B)$. □

The proof of Lemma 9.1 gives us also the following

Corollary 9.2. *If $\mathfrak{a}\in\mathfrak{A}_{\mathrm{NP}}$ is such that $I_{\mathfrak{a}}=\mathbb{R}$, then*

$$\lim_{s\to\pm\infty}\hat{c}_{\mathfrak{a}}|\hat{b}| = 0.$$

We recall that by (8.2), the measure μ has the following disintegration with respect to f:

$$\mu = \int_{\mathfrak{a}\neq+\infty}\mu_{\mathfrak{a}}\, d\xi(\mathfrak{a}) + \int_{\mathfrak{a}\neq+\infty}\nu_{\mathfrak{a}}\, d\sigma(\mathfrak{a}) + \mu\llcorner\{f=+\infty\}$$

where σ is defined in (8.3) as $\sigma = \left[f_{\#}(|\mu|\llcorner\{f\neq+\infty\})\right]^{\mathrm{sing}}$ (singular w.r.t. ξ).

We now prove

Lemma 9.3. *Suppose that $b\in L^\infty(\mathbb{T}^2)\cap \mathrm{BV}(\mathbb{T}^2)$ is a nearly incompressible vector field. Let μ be a Radon measure on $\mathbb{T}^2$ and $u\in L^\infty(\mathbb{T}^2)$. Then u solves equation*

$$\operatorname{div}(ub)=\mu \quad \text{in } \mathscr{D}'(\mathbb{T}^2) \tag{9.4}$$

if and only if

$$\begin{cases} \operatorname{div}\left(uc_{\mathfrak{a}}b\mathscr{H}^1\llcorner F_{\mathfrak{a}}\right) = \mu_{\mathfrak{a}} & \text{for } \xi\text{-a.e. } \mathfrak{a}\neq+\infty,\\ \sigma = 0, \\ \mu\llcorner\{f=+\infty\} = 0. \end{cases} \tag{9.5}$$

Proof. $\boxed{\Rightarrow}$. We show that $\operatorname{div}(ub)=\mu$ implies $\partial_s(\hat{u}|\hat{b}|\hat{c}_{\mathfrak{a}}) = \mu_{\mathfrak{a}}$ in $\mathscr{D}'(I_{\mathfrak{a}})$; then it is enough to apply Lemma 9.1 to get the equation (9.5). By (3.13) we have that, for every $B\in\mathscr{B}$,

$$\partial_s(\hat{u}|\hat{b}|\hat{c}_h) = \hat{\mu}_h$$

in $\mathscr{D}'(I)$, which means

$$\partial_s(\hat{u}|\hat{b}|\hat{c}_{\mathfrak{a}}) = \hat{\mu}_{\mathfrak{a}}, \qquad \text{in } \mathscr{D}'(I_{\mathfrak{a}}\cap\gamma_{\mathfrak{a}}^{-1}(B)). \tag{9.6}$$

Now take a compactly supported test function $\phi\in C_c^\infty(I_{\mathfrak{a}})$: by compactness, there exist $B_1,\dots,B_n\in\mathscr{B}$ such that $\{\gamma_{\mathfrak{a}}^{-1}(B_i)\}_i$ is a finite covering of $\operatorname{supp}\phi$. We consider a partition of unity $\{\rho_i\}$ subordinated to this covering and we write $\phi=\sum_i\rho_i\phi$ with $\operatorname{supp}\rho_i\subset I_{\mathfrak{a}}\cap\gamma_{\mathfrak{a}}^{-1}(B_i)$: due to (9.6), we get

$$\int \partial_s\phi\,\hat{u}|\hat{b}|\hat{c}_{\mathfrak{a}}\,ds = \int\phi\,d\mu_{\mathfrak{a}}.$$

Since $\phi\in C_c^\infty(I_{\mathfrak{a}})$ is arbitrary, this proves

$$\partial_s(\hat{u}|\hat{b}|\hat{c}_{\mathfrak{a}}) = \hat{\mu}_{\mathfrak{a}} \quad \text{in } \mathscr{D}'(I_{\mathfrak{a}}).$$

Finally, integrating $\operatorname{div}(uc_{\mathfrak{a}}b\mathscr{H}^1\llcorner F_{\mathfrak{a}}) = \mu_{\mathfrak{a}}$ in $d\xi$ over $\mathfrak{a}\neq+\infty$ we get

$$\operatorname{div}(ub) = \mu_{\mathfrak{a}}\llcorner\{f\neq+\infty\}.$$

Subtracting this from (9.4), we get

$$\int_{\mathfrak{a}\neq+\infty} \mu_{\mathfrak{a}}\, d\sigma(\mathfrak{a}) + \mu \llcorner \{f=+\infty\} = 0,$$

which implies the desired thesis, since the two measures are mutually singular.

$\boxed{\Leftarrow}$. For any test function $\phi \in C_c^\infty(\mathbb{T}^2)$, integrating the equation in $d\xi$ over $\mathfrak{a} \neq +\infty$, we get

$$\int_{\mathfrak{a}\neq+\infty} \left[\int u(x) c_{\mathfrak{a}}(x) (b(x)\cdot\nabla\phi(x)) d(\mathscr{H}^1 \llcorner F_{\mathfrak{a}})(x) \right] d\xi(\mathfrak{a}) = \int_{\{f\neq+\infty\}} \phi\, d\mu.$$

Taking into account the formula of the disintegration of $\mathscr{L}^2$ and that $\sigma = 0$ and $\mu \llcorner \{f=+\infty\} = 0$ we obtain

$$\int u(x)(b(x)\cdot\nabla\phi(x)) dx = \int \phi\, d\mu$$

which is (9.4). □

Remark 9.4. The proof above shows actually that (9.4) ⇒ (9.1) ⇒ (9.5) and that (9.5) ⇒ (9.4). In particular, this means that (9.4), (9.1), (9.5) are indeed *equivalent*.

Remark 9.5. If $\mu \ll \mathscr{L}^2$ then $\mu\llcorner\{f=+\infty\}=0$ is equivalent to $\mu\llcorner M=0$, since $\{f=+\infty\} = M \mod \mathscr{L}^2$.

Proposition 9.6. *Suppose that b is bounded,* BV*, nearly incompressible and $u \in L^\infty([0,T]\times\mathbb{T}^2)$. Let ν be a Radon measure on $\mathbb{T}^2$. Then u solves*

$$\begin{cases} \partial_t u + \operatorname{div}(ub) = \nu \\ u(0,\cdot) = u_0(\cdot) \end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times\mathbb{T}^2)$$

if and only if

- $\partial_t u \llcorner \{f=+\infty\} = \nu \llcorner \{f=+\infty\}$;
- $\sigma = 0$;
- $\hat{u}$ *solves*

$$\begin{cases} \partial_t(\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|) + \partial_s(\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|) = \hat{\nu}_a \\ \hat{u}(0,\cdot) = \hat{u}_0(\cdot) \end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times I_{\mathfrak{a}}),$$

where $I_{\mathfrak{a}}$ is domain of parametrization of $F_{\mathfrak{a}}$ for ξ-a.e. $\mathfrak{a}\neq+\infty$.

By a direct argument one can prove that the only weak solution to the initial value problem $\partial_t v + \partial_s v = 0$, $v|_{t=0} = 0$ is $v \equiv 0$. Hence we immediately obtain the following uniqueness result:

Corollary 9.7. *If $b\colon \mathbb{T}^2 \to \mathbb{R}^2$ is bounded BV nearly incompressible vector field then only distributional solution of the continuity equation with zero initial data $\bar{u}\equiv 0$ is $u\equiv 0$.*

Proof of Proposition 9.6. Multiplying the continuity equation by a function $\psi \in C_c^\infty([0,T))$ and formally integrating by parts we get

$$u_t\psi + \operatorname{div}(u\psi b) = \psi\nu \;\Rightarrow\; \operatorname{div}\left(\int_0^T u\psi\, dt\, b\right) = \int_0^T u\psi_t\, dt - \psi(0)u_0 + \left(\int_0^T \psi\, dt\right)\nu,$$

i.e. $\mathrm{div}(wb) = \mu$ where $w := \int_0^T u\psi\, dt$ and

$$\mu := \left(\int_0^T u\psi_t\, dt - \psi(0)u_0 \right) \mathscr{L}^2 + \left(\int_0^T \psi\, dt \right) \nu.$$

Applying Proposition 9.3, we thus obtain that continuity equation is equivalent to

$$\begin{cases} \mathrm{div}\,(wc_{\mathfrak{a}} b \mathscr{H}^1 \llcorner F_a) = \mu_{\mathfrak{a}} & \text{in } \mathscr{D}'(\mathbb{T}^2) \text{ for } \xi\text{-a.e. } \mathfrak{a} \neq +\infty, \\ \sigma = 0, \\ \mu \llcorner \{f = +\infty\} = 0. \end{cases} \tag{9.7}$$

The measure $\mu_{\mathfrak{a}}$ can be computed explicitly:

$$\begin{aligned} \mu_{\mathfrak{a}} &= \left(\int_0^T u\psi_t\, dt - \psi(0)u_0 \right) \Lambda_{\mathfrak{a}} + \left(\int_0^T \psi\, dt \right) \nu_{\mathfrak{a}} \\ &= \left(\int_0^T u\psi_t\, dt - \psi(0)u_0 \right) c_{\mathfrak{a}} \mathscr{H}^1 \llcorner F_{\mathfrak{a}} + \left(\int_0^T \psi\, dt \right) \nu_{\mathfrak{a}}. \end{aligned}$$

Therefore, we get

$$\mathrm{div}\,(wc_{\mathfrak{a}} b \mathscr{H}^1 \llcorner F_{\mathfrak{a}}) = \left(\int_0^T u\psi_t\, dt - \psi(0)u_0 \right) c_{\mathfrak{a}} \mathscr{H}^1 \llcorner F_{\mathfrak{a}} + \left(\int_0^T \psi\, dt \right) \nu_{\mathfrak{a}}.$$

This means that for every $\phi \in C_c^\infty(\mathbb{T}^2)$, we have

$$\begin{aligned} \int_0^T \left[\int_{\mathbb{T}^2} c_{\mathfrak{a}} \psi (b \cdot \nabla\phi) d\mathscr{H}^1 \llcorner F_{\mathfrak{a}} \right] dt = &\int_0^T \left[\int_{\mathbb{T}^2} u\psi_t \phi c_{\mathfrak{a}} d\mathscr{H}^1 \llcorner F_{\mathfrak{a}} \right] dt \\ &- \int_{\mathbb{T}^2} \psi(0) u_0 \phi c_{\mathfrak{a}}\, d\mathscr{H}^1 \llcorner F_{\mathfrak{a}} \\ &+ \int_0^T \left[\int_{\mathbb{T}^2} \psi\phi\, d\nu_{\mathfrak{a}} \right] dt, \end{aligned}$$

hence

$$\begin{aligned} \int_0^T \left[\int_{\mathbb{T}^2} c_{\mathfrak{a}} (b \cdot \nabla(\psi\phi)) d\mathscr{H}^1 \llcorner F_{\mathfrak{a}} \right] dt = &\int_0^T \left[\int_{\mathbb{T}^2} u(\phi\psi)_t c_{\mathfrak{a}}\, d\mathscr{H}^1 \llcorner F_{\mathfrak{a}} \right] dt \\ &- \int_{\mathbb{T}^2} (\phi\psi)(0) u_0 c_{\mathfrak{a}}\, d\mathscr{H}^1 \llcorner F_{\mathfrak{a}} \\ &+ \int_0^T \left[\int_{\mathbb{T}^2} \psi\phi\, d\nu_{\mathfrak{a}} \right] dt. \end{aligned}$$

Since functions of the form $\psi(t)\phi(x)$ are dense in $C_c^\infty((0,T)\times\mathbb{T}^2)$, we deduce

$$\begin{cases} \partial_t (uc_{\mathfrak{a}} \mathscr{H}^1 \llcorner F_{\mathfrak{a}}) + \mathrm{div}\,(uc_{\mathfrak{a}} b \mathscr{H}^1 \llcorner F_{\mathfrak{a}}) = \nu_{\mathfrak{a}}, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

Now being $\gamma_{\mathfrak{a}}$ is Lipschitz and injective, we have

$$(\gamma_{\mathfrak{a}}^{-1})_\# (\mathscr{H}^1 \llcorner F_{\mathfrak{a}}) = |\gamma_{\mathfrak{a}}'| \mathscr{L}^1,$$

and this allows us to compute explicitly

$$\begin{aligned}\hat{\mu}_{\mathfrak{a}} &= (\gamma_{\mathfrak{a}}^{-1})_{\#}\mu_{\mathfrak{a}} \\ &= (\gamma_{\mathfrak{a}}^{-1})_{\#}\left(\int_0^T u\psi_t\,dt\,c_{\mathfrak{a}}\mathscr{H}^1 \llcorner F_{\mathfrak{a}} - \int_{\mathbb{T}^2}\psi(0)u_0 c_{\mathfrak{a}} d\mathscr{H}^1 \llcorner F_{\mathfrak{a}} + \nu_{\mathfrak{a}}\right) \\ &= \int_0^T u(\tau,\gamma(s))\psi_\tau(\tau)c_{\mathfrak{a}}(\gamma_{\mathfrak{a}}(s))|b(\gamma_{\mathfrak{a}}(s))|\,d\tau - \psi(0)u_0(\gamma_{\mathfrak{a}}(s))c_{\mathfrak{a}}(\gamma(s)) + \hat{\nu}_{\mathfrak{a}},\end{aligned} \tag{9.8}$$

where, by definition, we have set

$$\hat{\nu}_{\mathfrak{a}} := (\gamma_{\mathfrak{a}}^{-1})_{\#}\,(\nu_{\mathfrak{a}})\,.$$

From (9.8), we thus obtain that

$$\hat{\mu}_{\mathfrak{a}} = -\int_0^T \partial_t(\hat{u}|\hat{b}|\hat{c}_{\mathfrak{a}}) + \hat{\nu}_{\mathfrak{a}}.$$

Due to Remark 9.4, (9.7) is *equivalent* to

$$\begin{cases}\partial_t(\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|) + \partial_s(\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|) = \hat{\nu}_{\mathfrak{a}}, \\ \hat{u}(0,\cdot) = \hat{u}_0(\cdot),\end{cases}$$

in $\mathscr{D}'((0,T)\times I_{\mathfrak{a}})$ and this concludes the proof. □

9.1. **Final result.** We are now in position to prove the main result of this paper, which is the following

Theorem 9.8. *Every bounded, autonomous, nearly incompressible* BV *vector field on* $\mathbb{T}^2$ *has the renormalization property.*

Proof. Let $u \in L^\infty([0,T]\times\mathbb{T}^2)$ be a solution of the problem

$$\begin{cases}u_t + b\cdot\nabla u = 0, \\ u(0,\cdot) = u_0(\cdot),\end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times\mathbb{T}^2).$$

According to Definition 1.2, this means that

$$(\rho u)_t + \operatorname{div}(\rho u b) = 0 \qquad \text{in } \mathscr{D}'(I\times\mathbb{T}^2), \tag{9.9}$$

with initial condition $\rho(0,\cdot)u_0$. Now applying Proposition 9.6, equation (9.9) is equivalent to

$$\begin{cases}(\hat{\rho}\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|)_t + (\hat{\rho}\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|)_s = 0 & \xi\text{-a.e. } \mathfrak{a} \neq +\infty, \\ (\rho u)_t = 0 & \text{on } M,\end{cases}$$

with initial condition $(\hat{\rho}\hat{u})(0,\cdot) = \hat{\rho}(0,\cdot)\hat{u}_0(\cdot)$. On the other hand, by nearly incompressibility, we have

$$\rho_t + \operatorname{div}(\rho b) = 0$$

which is

$$\begin{cases}(\hat{\rho}\hat{c}_{\mathfrak{a}}|\hat{b}|)_t + (\hat{\rho}\hat{c}_{\mathfrak{a}}|\hat{b}|)_s = 0 & \xi\text{-a.e. } \mathfrak{a} \neq +\infty, \\ \rho_t = 0 & \text{on } M.\end{cases}$$

In particular, we have that ρu is constant on M and also ρ is constant on M (in particular, it is positive, since ρ is bounded). Therefore, we get that u is identically equal to u_0 on M and hence also $\rho\beta(u)$ is constant on M and it is equal to $\rho(0,\cdot)\beta(u_0)$.

On the other hand, comparing the two equations

$$\left(\hat{\rho}\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|\right)_t + \left(\hat{\rho}\hat{u}\hat{c}_{\mathfrak{a}}|\hat{b}|\right)_s = 0$$

and

$$\left(\hat{\rho}\hat{c}_{\mathfrak{a}}|\hat{b}|\right)_t + \left(\hat{\rho}\hat{c}_{\mathfrak{a}}|\hat{b}|\right)_s = 0$$

we get that for ξ-a.e. $\mathfrak{a}$,

$$\hat{u}(t,s) = \hat{u}(0, s-t),$$

which clearly gives, for every $\beta \in C^1(\mathbb{R})$,

$$\beta(\hat{u})(t,s) = \beta(\hat{u})(0, s-t).$$

This implies

$$\begin{cases} (\rho\beta(u))_t + \operatorname{div}(\rho\beta(u)b) = 0, \\ (\hat{\rho}\beta(\hat{u}))\,(0,\cdot) = \hat{\rho}(0,\cdot)\beta(\hat{u}_0(\cdot)), \end{cases}$$

which means

$$\begin{cases} (\beta(u))_t + b\cdot\nabla\beta(u) = 0, \\ \beta(u)(0,\cdot) = \beta(u_0)(\cdot), \end{cases}$$

and this concludes the proof. □

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