STABILITY OF L^∞ SOLUTIONS FOR HYPERBOLIC SYSTEMS WITH COINCIDING SHOCKS AND RAREFACTIONS

STEFANO BIANCHINI

Abstract. We consider a hyperbolic system of conservation laws

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0,\cdot) = u_0 \end{cases}$$

where each characteristic field is either linearly degenerate or genuinely nonlinear. Under the assumption of coinciding shock and rarefaction curves and the existence of a set of Riemann coordinates w, we prove that there exists a semigroup of solutions $u(t) = \mathcal{S}_t u_0$, defined on initial data $u_0 \in L^\infty$. The semigroup $\mathcal{S}$ is continuous w.r.t. time and the initial data u_0 in the L^1_{loc} topology. Moreover $\mathcal{S}$ is unique and its trajectories are obtained as limits of wave front tracking approximations.

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1. Introduction

Consider the Cauchy problem for a strictly hyperbolic system of conservation laws

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0,\cdot) = u_0 \end{cases} \tag{1.1}$$

where $u \in \mathbb{R}^n$ and $f : \Omega \mapsto \mathbb{R}^n$ is sufficiently smooth, Ω open. If the initial data u_0 is of small total variation, the global existence was proved first in [17]. Moreover a series of paper [5, 6, 8, 9, 14] establishes the uniqueness and well posedness of the Cauchy problem (1.1). However, when u_0 has large total variation or even more generally u_0 belongs to L^∞, the solution u may not exist globally in L^∞ [19]: only for special system it is possible to consider initial data with large total variation. We recall some of the results available in this direction.

1) For scalar conservation laws, the entropic solution to (1.1) generates a contractive semigroup w.r.t the L^1 distance, on a domain of L^∞ data [20].
2) For general Temple class system, in [2, 4, 23] it is proved the existence and stability of the entropic solution for initial data with arbitrarily large but bounded total variation.
3) If all characteristic families are genuinely nonlinear and the system is Temple class, the existence and stability for initial data in L^∞ is proved in [11].
4) For special 2×2 systems, in which one of the equation is autonomous, various results have been proved in [3, 15], with initial data with unbounded total variation.

An open question is if the semigroup of solutions to the systems of case 2), defined on all the initial data u_0 with total variation arbitrary large but bounded, can be extended to data in L^∞. In many model systems, in fact, some of the characteristic fields are linearly degenerate, so that the result in [11] does not apply: for example, the traffic model considered in [1] or a 2×2 model for chromatography. Aim of this paper is to prove that, at least in the case where the flux function f is convex and shocks and rarefactions coincide, the solution to (1.1) can be defined for $u_0 \in L^\infty$.

To illustrate the heart of the matter, we assume that the system (1.1) admits a system of Riemann coordinates $w \in \mathbb{R}^n$, and that shock and rarefaction curves coincide in Ω. Moreover we assume that each characteristic field is linearly degenerate or genuinely nonlinear. We do not assume that rarefaction

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curves are straight lines. We consider a set E of the form

$$E \doteq \Big\{ u \in \Omega : \ w(u) \in [a_i, b_i], \ i = 1, \dots, n \Big\}.$$

With $L^\infty(\mathbb{R}; E)$ we denote the space of L^∞ functions with values in E. The main result of this paper is the following:

Theorem 1.1. *There exists a unique semigroup $\mathcal{S} : [0, +\infty) \times L^\infty(\mathbb{R}; E) \longmapsto L^\infty(\mathbb{R}; E)$ such that the following properties are satisfied:*

i) *for all $u_n, u \in L^\infty(\mathbb{R}; E)$, $t_n, t \in [0, +\infty)$, with $u_n \to u$ in L^1_{loc}, $|t - t_n| \to 0$ as $n \to +\infty$,*
$$\lim_{n \to +\infty} \mathcal{S}_{t_n} u_n = \mathcal{S}_t u \quad \text{in } L^1_{loc};$$

ii) *each trajectory $\mathcal{S}_t u_0$ is a weak entropic solution to the Cauchy problem (1.1) with $u_0 \in L^\infty(\mathbb{R}; E)$;*

iii) *if u_0 is piecewise constant, then, for t sufficiently small, $\mathcal{S}_t u_0$ coincides with the function obtained by piecing together the solutions of the corresponding Riemann problems.*

The first two properties imply the well posedness of the Cauchy problem (1.1) with initial data in L^∞. The uniqueness follows because $\mathcal{S}$ satisfies iii) and it is limit of wave front approximations.

As it is shown in the last example of [11], the semigroup $\mathcal{S}$ cannot be uniformly continuous: thus we cannot apply any compactness argument to construct the solution $u(t) \doteq \mathcal{S}_t u_0$. The fundamental problem is that, differently from [11], the total variation of the Riemann invariants corresponding to linearly degenerate families does not decrease in time.

The main idea of this paper is to study how the solution to the characteristic equation

$$\dot{x}(t) = \lambda_i(u(t, x(t))), \quad x(0) = y, \tag{1.2}$$

depends on the solution u of (1.1). It will be shown that, for a fixed time τ, the map $y \mapsto x(\tau)$ depends Lipschitz continuously on the initial data u_0, and moreover the Lipschitz constant is independent of the total variation of u_0. Since the Riemann invariant w_i is the broad solution to

$$(w_i)_t + \lambda_i(u(t, x))(w_i)_x = 0,$$

a simple argument gives the convergence of the wave front tracking approximations. Note that this also implies the well posedness of the ODE (1.2) when $u(t, x)$ is an L^∞ solution of the system (1.1). As noted in [15], for general system the solution to (1.2) does not exist.

The paper is organized as follows. Section 2 contains the basic assumptions on the system (1.1). Moreover we construct the wave front approximation of the solution $u(t)$. In Section 3 we analyze carefully the shift differential map, i.e. the evolution of a perturbation in u_0 in which only the position of the initial jumps has changed. The method we use is essentially the one in [11], with slight modifications due to the fact that in our system the rarefaction curves do not need to be straight lines. The main result is here the explicit computation of the shift differential map.

Section 4 is concerned with the equation for characteristics (1.2). We prove the Lipschitz dependence of the map $y \mapsto x(t)$ w.r.t. both the initial data u_0 and y. Moreover we will show that the Lipschitz constant is independent from the total variation of u_0. Finally, in Section 5, we prove Theorem 1.1.

2. Basic assumptions and wave front approximations

We consider a strictly hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \tag{2.1}$$

where $f : \Omega \to \mathbb{R}^n$ is a smooth vector field defined on some open set $\Omega \subseteq \mathbb{R}^n$. Let $A(u) \doteq DF(u)$ be the Jacobian matrix of f and denote with $\lambda_i(u)$ its eigenvalues and with $r_i(u)$, $l^i(u)$ its right and left eigenvectors, respectively. We assume that the eigenvalues λ_i can be either genuinely nonlinear or linearly degenerate. In the following the i-th rarefaction curve through $u \in \Omega$ will be written as $R_i(s)u$, with $R_i(0)u = u$, while the i-th shock curve will be denoted by $S_i(s)u$, and its speed by $\sigma_i(s, u)$. The directional derivative of a function $\phi(u)$ in the direction of $r_i(u)$ will be denoted as

$$r_i \bullet \phi(u) \doteq \lim_{h \to 0} \frac{\phi(u + h r_i(u)) - \phi(u)}{h},$$

while the left and right limit of a BV function f in a point x will be written as

$$f(x-) = \lim_{y \to x-} f(y), \qquad f(x+) = \lim_{y \to x+} f(y).$$

We assume that the rarefaction curves R_i generate a system of Riemann coordinates $w(u)$. We recall that a necessary and sufficient condition for the local existence of Riemann coordinates is the Frobenius involutive condition: if $[X,Y]$ denotes the Lie bracket of the vector fields X,Y, the condition is

$$[r_i, r_j] \in \operatorname{span}\{r_i, r_j\} \quad \text{for all } i, j = 1, \dots, n.$$

In the following we will use indifferently the conserved coordinates u or the Riemann coordinates w.

Fix a domain

$$E \doteq \Big\{ u \in \Omega : \ w(u) \in [a_i, b_i], \ i = 1, \dots, n \Big\}. \tag{2.2}$$

Since E is compact, there is a constant $c > 0$ such that

$$r_i \bullet \lambda_i(u) > c \quad \forall u \in E, \quad \text{if } \lambda_i \text{ is genuinly nonlinear.} \tag{2.3}$$

We suppose that the system (2.1) is uniformly strictly hyperbolic in Ω: this means that there exists a constant d such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geq d, \quad \forall u, v \in E, \ i = 1, \dots, n-1. \tag{2.4}$$

We also assume that in the system (2.1) shock and rarefaction curves coincide: this implies [26] that either the rarefaction curve $R_i(s)u$ is a straight line or the eigenvalue is linearly degenerate. In fact, one can prove that

$$\frac{d^2}{ds^2}\sigma_i(s,u)\bigg|_{s=0} = \frac{1}{6}(r_i \bullet \lambda_i(u)) \langle l^i(u), r_i \bullet r_i(u) \rangle + \frac{1}{3} r_i \bullet (r_i \bullet \lambda_i(u)), \tag{2.5}$$

and for the shock curve $S_i(s)u$ we have

$$\langle l^j(u), S'''(0)u - R'''(0)u \rangle = \frac{1}{2(\lambda_j(u) - \lambda_i(u))} (r_i \bullet \lambda_i(u)) \langle l^j(u), r_i \bullet r_i(u) \rangle. \tag{2.6}$$

If λ_i is genuinely nonlinear, the left hand side of (2.6) is zero if and only if the rarefaction curve is a straight line, because $r_i \bullet r_i(u)$ is orthogonal to $r_i(u)$.

The flux function f thus satisfies the following assumptions:

H1) the eigenvalues λ_i of Df are linearly degenerate or genuinely nonlinear;
H2) the rarefaction curves form a system of coordinates;
H3) shock and rarefaction curves coincide.

The system (2.1) has thus n_{ld} linearly degenerate fields λ_i, corresponding to the Riemann invariants w_i, and $n_{gnl} = n - n_{ld}$ genuinely nonlinear fields λ_k, corresponding to the Riemann invariants w_k. In the latter case we have $r_k \bullet r_k(u) = 0$ for all $u \in E$.

Remark 2.1. If $\Omega \subseteq \mathbb{R}^2$, then the rarefaction curves $R_i(s)u$ always generate a system of Riemann coordinates. Thus our assumptions are satisfied by the following classes of systems:

i) both eigenvalues are linearly degenerate;
ii) one eigenvalue is linearly degenerate, the other genuinely nonlinear and the rarefaction curves of the latter are straight lines;
iii) both eigenvalues are genuinely nonlinear and the system is of Temple class.

The various situations are shown in fig. 1. Case ii) corresponds to the traffic model considered in [1], while case i) corresponds to 2×2 chromatography.

Given the two points $u^-, u^+ \in E$, with coordinates $u^- = u(w_1^-, \dots, w_n^-)$ and $u^+ = u(w_1^+, \dots, w_n^+)$, with $w_i^+ \neq w_i^-$, consider the intermediate states $u(\omega_i)$, where

$$\omega_0 = w(u^-), \quad \omega_i = (w_1^+, \dots, w_i^+, w_{i+1}^-, \dots, w_n^-), \quad i = 1, \dots, n. \tag{2.7}$$

For all $i = 1, \dots, n$, we denote with $v_i(u^-, u^+)$ the vectors defined as

$$v_i(u^-, u^+) = u(\omega_i) - u(\omega_{i-1}), \tag{2.8}$$

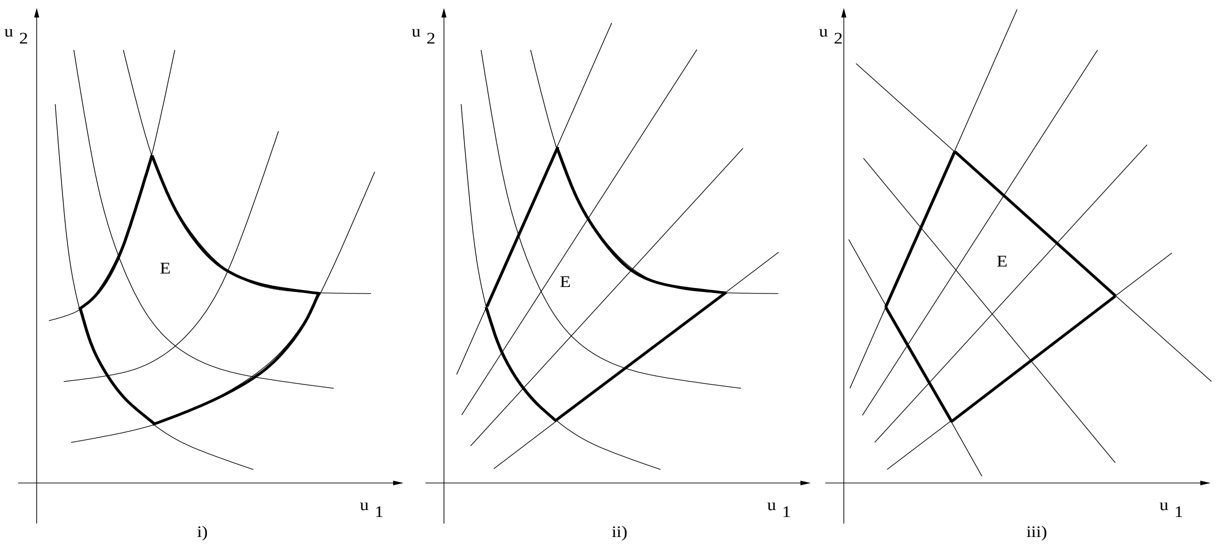

FIGURE 1. The various situations for a 2×2 system considered in Remark 2.1.

and we define $r_i(u^-, u^+)$ as

$$r_i(u^-, u^+) = \begin{cases} \dfrac{v_i(u^-, u^+)}{|v_i(u^-, u^+)|} = \dfrac{u(\omega_i) - u(\omega_{i-1})}{|u(\omega_i) - u(\omega_{i-1})|} & \text{if } w_i^- \neq w_i^+ \\ r_i(\omega_{i-1}) = r_i(\omega_i) & \text{if } w_i^- = w_i^+ \end{cases} \tag{2.9}$$

where $r_i(u)$ is the i-th eigenvector of $DF(u)$. We assume that the vectors $r_i(u^-, u^+)$ are linearly independent for all $u^-, u^+ \in E$. This condition is satisfied for data in a sufficiently small neighborhood of a given point $\bar{u} \in \Omega$. We denote also with $\{l^i(u^-, u^+), i = 1, \dots, n\}$ the dual base.

We now define an approximated semigroup of solutions $\mathcal{S}^\nu$ on a set $E^\nu \subseteq E$. The construction is similar to the one in [2]. For any integer $\nu \in \mathbb{N}$, set

$$E^\nu \doteq \Big\{ u \in E : w_i(u) \in 2^{-\nu}\mathbb{Z}, i = 1, \dots, n \Big\}, \tag{2.10}$$

and let $D^{\nu,M}$ be the domain defined as

$$D^{\nu,M} \doteq \Big\{ u : \mathbb{R} \longmapsto E^\nu : u \text{ piecewise constant and Tot.Var.}(u) \leq M \Big\}. \tag{2.11}$$

Given $\bar{u} \in E^\nu$, we construct a solution $u(t)$ by wave front tracking. We first define how to solve the Riemann problem $[u^-, u^+]$, with $u^-, u^+ \in E^\nu$.

The solution to the Riemann problem u^-, u^+ is constructed by piecing together the solutions to the simple Riemann problems $[\omega_{i-1}, \omega_i]$, where ω_i is defined in (2.7). If the i-th field is linearly degenerate, then $[\omega_{i-1}, \omega_i]$ is solved by a contact discontinuity travelling with speed $\lambda_i(\omega_i)$. If the i-th field is genuinely nonlinear and $w_i^+ < w_i^-$, then $[\omega_{i-1}, \omega_i]$ is solved by a shock travelling with the Rankine-Hugoniot speed $\sigma_i(\omega_{i-1}, \omega_i)$. Finally, if the i-th field is genuinely nonlinear and $w_i^+ > w_i^-$, then $[\omega_{i-1}, \omega_i]$ is solved by a rarefaction fan: if $w_i^+ = w_i^- + p_i 2^{-\nu}$, $p_i \in \mathbb{N}$, consider the states

$$\omega_{i,0} = \omega_{i-1}, \quad \omega_{i,l} = (w_1^+, \dots, w_{i-1}^+, w_i^- + \ell 2^{-\nu}, w_{i+1}^-, \dots, w_n^-), \quad \ell = 1, \dots, p_i.$$

The solution will consist of p_i shock waves $[\omega_{i,l-1}, \omega_{i,l}]$, travelling with the corresponding shock speed $\sigma_i(\omega_{i,l-1}, \omega_{i,l})$.

At time $t = 0$ we solve the initial Riemann problems of $\bar{u}$. Note that the number of wave fronts is bounded by $2^\nu \cdot \text{Tot.Var.}(\bar{u})$. When two or more fronts interact, we solve again the Riemann problem they generate, and so on. It is easy to show that at each interaction at least one of the following alternatives holds:

i) the number of waves decreases at least by 1;
ii) the total variation of the solution $u(t)$ decreases by $2^{1-\nu}$,

iii) the interaction potential $Q(t)$, defined as

$$Q(t) \doteq \sum_{\alpha,\beta \text{ approaching}} |\sigma_\alpha||\sigma_\beta| \leq M^2, \tag{2.12}$$

decreases by $2^{-\nu}$. We recall that two waves σ_α, σ_β of the families k_α, k_β, located at points x_α, X_β, are considered as *approaching* if $x_\alpha < x_\beta$ and $k_\alpha > k_\beta$.

This implies that there are at most a finite number of interactions, so that we can construct our approximate solution for all $t \geq 0$. Note that $\mathcal{S}_t^\nu u = u(t)$ is a semigroup of solutions, but not entropic due to the presence of rarefaction fronts.

If the i-th family is linearly degenerate, the i-th Riemann coordinate $w_i(t,\cdot)$ of the solution can be constructed by solving the semilinear system

$$\begin{cases} (w_i)_t + \lambda_i(u(t,x))(w_i)_x = 0 \\ w_i(0,x) = w_i(x) \end{cases} \tag{2.13}$$

Since u is a piecewise constant solution, with a finite number of jumps, the broad solution to (2.13) is well defined [7]: if we denote with $x(t,y)$ the solution to the ODE

$$\dot{x} = \lambda_i(u(t,x)), \qquad x(0) = y, \tag{2.14}$$

then the solution to (2.13) is given by

$$w_i(t, x(t,y)) = w_i(y). \tag{2.15}$$

In the following sections we will consider the dependence on the initial data u_0 of the genuinely nonlinear Riemann coordinates $w_k(t,\cdot)$ and the map $h_i^t(y)$ defined as

$$h_i^t(y) \doteq x_i(t,y), \tag{2.16}$$

where $x_i(t,y)$ is the solution to (2.13).

3. Estimates on the shift differential map

In this section we prove some properties of the shift differential map. These properties are closely related to the structure of (2.1), i.e. the conservation form, the coinciding shock and rarefaction assumption, which prevents the creation of shock when two jumps of the same family collide, and the existence of Riemann invariants, which prevents the creation of shock when two jumps of different families interact.

Consider a wave front solution $u(t,\cdot)$ of (2.1), and assume that the initial datum $u(0,\cdot)$ has a finite number N of jumps σ_α, located in y_α:

$$u(0,x) = \sum_{\alpha=1}^N \sigma_\alpha \chi_{[y_\alpha,+\infty)}(x).$$

If ξ_α is the shift rate of the jump σ_α, define $u^\theta(t,\cdot)$ as the front tracking solution with initial datum

$$u^\theta(0,x) = \sum_{\alpha=1}^N \sigma_\alpha \chi_{[y_\alpha+\theta\xi_\alpha,+\infty)}(x). \tag{3.1}$$

In the following, we will use the integral shift function, defined by

$$v(t,x) \doteq \lim_{\theta\to 0} \left\{ -\frac{1}{\theta} \int_{-\infty}^x u^\theta(t,y) - u(t,y) dy \right\}. \tag{3.2}$$

If $u(t,\cdot)$ has a shock σ_β, located in y_β, and if ξ_β is its shift rate, it is clear that the following relation holds:

$$\sigma_\beta \xi_\beta = v(t, y_\beta+) - v(t, y_\beta-). \tag{3.3}$$

We first recall the following result in [11], obtained using the conservation equation:

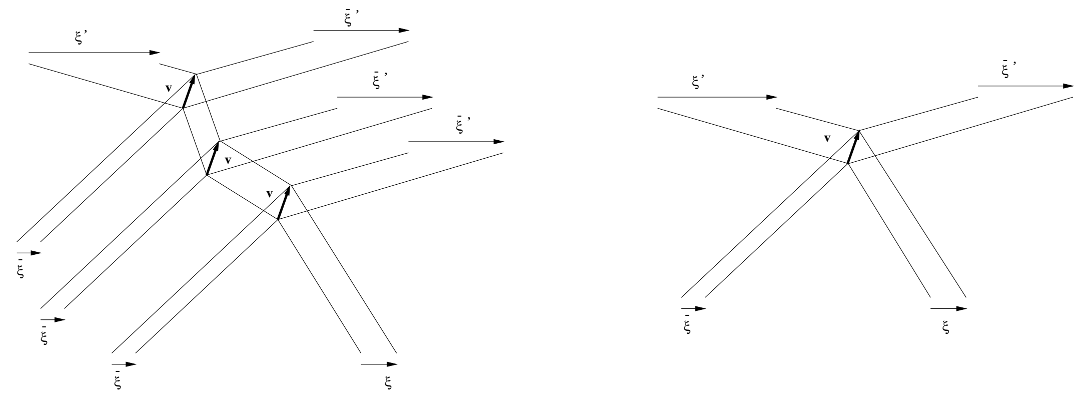

FIGURE 2. Interaction with a sheaf of shocks.

Lemma 3.1. *Consider a bounded, open region Γ in the t-x plane. Call σ_α, $\alpha = 1, \dots, N$, the fronts entering Γ and let ξ_α be their shifts. Assume that the fronts leaving Γ, say σ'_β, $\beta = 1, \dots, N'$, are linearly independent. Then their shifts ξ'_β are uniquely determined by the linear relation*

$$\sum_{\beta=1}^{N'} \xi'_\beta \sigma'_\beta = \sum_{\alpha=1}^{N} \xi_\alpha \sigma_\alpha. \tag{3.4}$$

Remark 3.2. As observed in [11], formula (3.4) implies that the shift rates of the outgoing fronts depend only on the shift rates of the incoming ones, and not on the order in which these wave-fronts interact inside Γ. In particular we can perform the following operations, without changing the shift rates of the outgoing fronts:

O1) switch the order of which three or more fronts interact;
O2) invert the order of two fronts at time 0, if they have zero shift rate.

The second lemma is concerned with a configuration where a sequence of contact discontinuities interacts with a wave of another family.

Lemma 3.3. *Consider a family of parallel contact discontinuities σ_α, $\alpha = 1, \dots, N$ of the i-th linearly degenerate family and a single wave-front σ of the k-th family, $k \neq i$. Let ξ_α and ξ be their initial shifts, respectively, and let ξ'_α, ξ' be their shifts after interaction. Assume that $\xi_\alpha = \bar{\xi}$ for all α. Then after the interactions all the shift rates ξ'_α of the i-th family have the same value $\bar{\xi}'$ and*

$$\xi'_\alpha = \bar{\xi}' = \frac{\bar{\xi}(\bar{\Lambda}' - \Lambda) - \xi(\bar{\Lambda}' - \bar{\Lambda})}{\Lambda - \bar{\Lambda}}, \qquad \xi' = \frac{\bar{\xi}(\Lambda' - \Lambda) - \xi(\bar{\Lambda}' - \bar{\Lambda})}{\Lambda - \bar{\Lambda}}, \tag{3.5}$$

where $\bar{\Lambda}$, Λ and $\bar{\Lambda}'$, Λ' are the speeds of the shocks σ_α and σ, before and after interaction, respectively.

Proof. Define the vector $\mathbf{v}$ in the t-x plane as the shift of the first collision point. By a direct computation one finds

$$\mathbf{v} = \left(\frac{\xi - \bar{\xi}}{\bar{\Lambda} - \Lambda}, \frac{\bar{\Lambda}\xi - \Lambda\bar{\xi}}{\bar{\Lambda} - \Lambda} \right). \tag{3.6}$$

Since all the incoming shock of the linearly degenerate family have the same speed $\bar{\Lambda}$, by simple geometrical considerations it follows that the vector $\mathbf{v}$ is constant during all interactions (fig. 2). Formula (3.5) follows easily. □

Remark 3.4. Note that this lemma allows us to perform the following new operation, without changing the shift rates:

O3) replace a family of contact discontinuities σ_α of a linearly degenerate, all with the same shift rate $\bar{\xi}$, by a single wave $\sigma = \sum \sigma_\alpha$ with shift rate $\bar{\xi}$.

In the next lemma we will show that the existence of Riemann coordinates w implies a strong relation among shocks of different families.

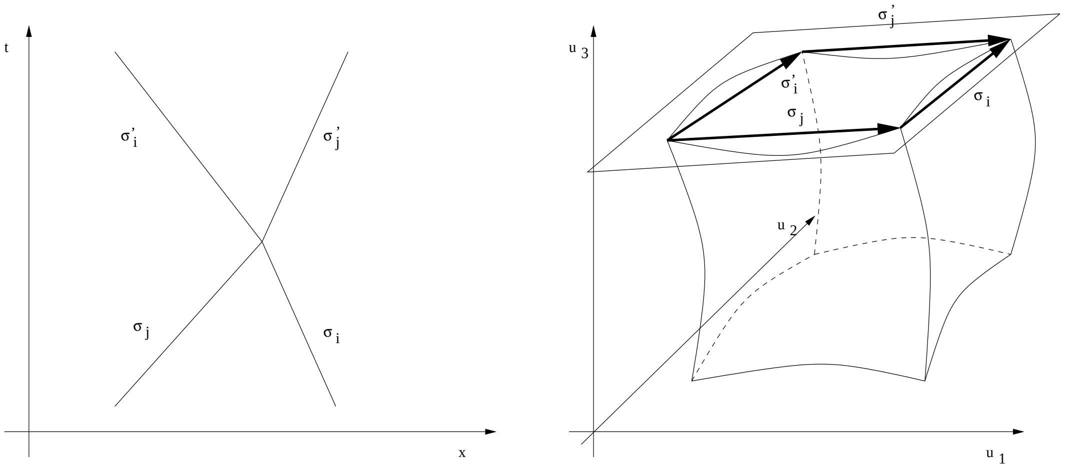

FIGURE 3. Vector relations among shocks.

Lemma 3.5. *Consider two adjacent jumps belonging to different families, σ_i and σ_j, $i < j$, located at $x_i > x_j$. Let σ'_i, σ'_j be their strength after interaction. Then the following holds:*

$$\operatorname{span}\{\sigma_i, \sigma_j\} = \operatorname{span}\{\sigma'_i, \sigma'_j\}. \tag{3.7}$$

Proof. If ξ_i, ξ_j are the shift rates before interaction, and ξ'_i, ξ'_j after interaction, then (3.7) follows easily from the conservation relation

$$\sigma_i \xi_i + \sigma_j \xi_j = \sigma'_i \xi'_i + \sigma'_j \xi'_j \qquad \forall \xi_i, \xi_j \in \mathbb{R}, \tag{3.8}$$

because by assumptions no waves of other families are generated. By condition (2.8) the conclusion follows. □

Remark 3.6. Note that the previous lemma implies that the conservation relation (3.8) is bidimensional, i.e. the shocks σ_i, σ_j and σ'_i, σ'_j lie on a two dimensional plane (fig. 3). We can obtain then an identity which relates the the strengths σ with the speeds Λ: substituting (3.5) in (3.8), since $\bar{\xi}$, ξ are arbitrary, we get

$$\begin{aligned} \sigma_i(\Lambda_j - \Lambda_i) &= \sigma'_i(\Lambda'_i - \Lambda_j) + \sigma'_j(\Lambda'_j - \Lambda_j), \\ \sigma_j(\Lambda_i - \Lambda_j) &= \sigma'_i(\Lambda'_i - \Lambda_i) + \sigma'_j(\Lambda'_j - \Lambda_i). \end{aligned} \tag{3.9}$$

One can show that if a Riemann solver verifies (3.9) for all couple of waves i, j, then there exists a flux function f such that the wave front approximation is a weak solution to (2.1).

An important property of the shift differential map for Temple class systems is the fact that a perturbation to the initial data, initially localized in $[a, b]$, remains in the neighborhood of the set $\cup_i [x_i(t, a), x_i(t, b)]$, where $x_i(t, y)$ is the solution of the i-th characteristic equation starting at y. We now extend this property to hyperbolic systems satisfying the hypotheses **H1)**, **H2)**, **H3)** of section 2.

Consider N jumps σ_α, $\alpha = 1, \dots, N$, of some linearly degenerate family i, located at x_α and corresponding to the jumps $c(\alpha)e_i$ in the Riemann coordinates w:

$$\sigma_\alpha = u(w(x_\alpha -) + c(\alpha)e_i) - u(w(x_\alpha -)), \tag{3.10}$$

for some constants $c(\alpha)$, $\alpha = 1, \dots, N$.

Definition 3.7. We say that the jumps σ_α defined in (3.10) are in involution if

$$\sum_{\alpha=1}^{N} c(\alpha) = 0, \tag{3.11}$$

i.e. the initial and final Riemann coordinate w_i is the same: $w_i(x_1-) = w_i(x_N+)$.

Note that, by the existence of Riemann coordinates, this relation does not depend on the positions and strength of the shocks of the other families. We can now extend Lemma 2 in [11] to our systems:

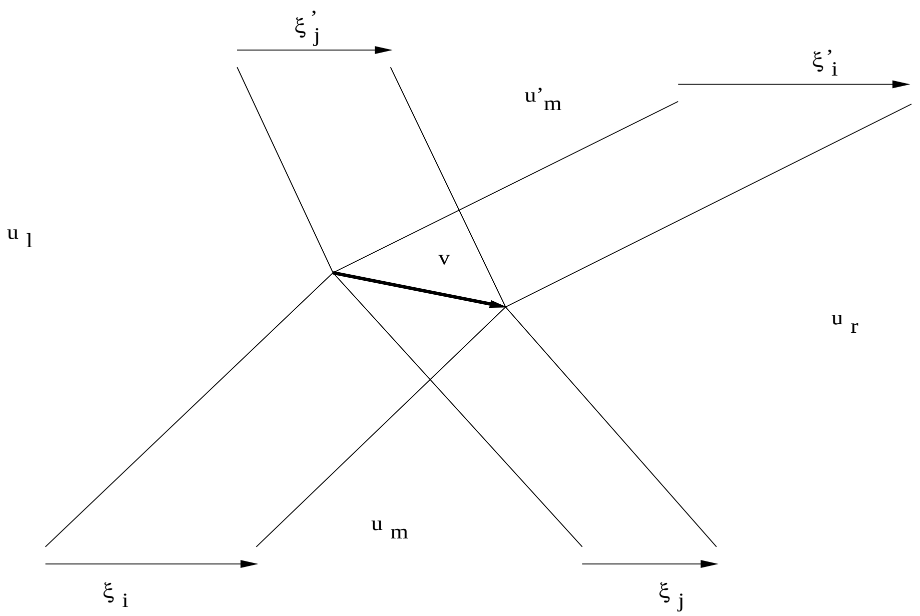

FIGURE 4. Computation of the shift rate.

Lemma 3.8. *Consider a wave front tracking solution u. Assume that there are N shocks σ_α*

i) *either of the i-th linearly degenerate family in involution,*
ii) *or of the k-th genuinely nonlinear family,*

and let $x_\alpha(t)$, $0 \leq t \leq T$, be the position of the shock σ_α, $\alpha = 1, \dots, N$. Then it is possible to assign shift rates to all shocks such that $\xi_1 = 1$ and the shift of all fronts outside the strip $\Gamma \doteq \{(t,x); t \in [0,T], x_1(t) \leq x \leq x_N(t)\}$ is zero.

Proof. We consider only the case of linearly degenerate family i, since in the other case the proof is exactly the one given in [11].

Let $x_\alpha(t)$, $\alpha = 1, \dots, N$, be the position of the shock σ_α of the i-th family in involution, and let $\bar{w}_i$ be the value of the Riemann coordinate at $x_1(t)- = x_1(0)-$. For $w \in E$, define $\tilde{w}$ as the projection of w on the hyperplane $\{w_i = \bar{w}_i\}$, and $\tilde{u} = u(\tilde{w})$.

We choose the shift rates such that

$$-\frac{d}{d\theta} \int_{-\infty}^{x} u^\theta dy = \sum_{x_i(t) \leq x} \xi(i)(t)\sigma_i(t) = c(t,x)(u(t,x) - \tilde{u}(t,x)), \tag{3.12}$$

where $c(t,x)$ is a scalar function different from 0 only in $[x_1(t), x_N(t)]$, and we recall that $\tilde{u}(t,x) = u(\tilde{w}(t,x))$.

By imposing the value $\xi_1 = 1$, i.e. $c(0, x_1(0)-) = 0$, $c(0, x_1(0)+) = 1$, we need to prove that (3.12) can be satisfied at time $t = 0$. We have two cases.

1) If the jump σ_i belongs to the i-th family and is inside $[x_0(0), x_N(0)]$, then set $\xi = c(t, x_i-)$.
2) If the jump σ_i belong to the k-th family with $k \neq i$, then by assumption (2.8) and by (3.7) there exists a unique shift ξ_i and a unique constant $c(0, x+)$ such that

$$\xi_i\sigma_i + c(0,x-)(u(0,x-) - \tilde{u}(0,x-)) = c(0,x+)(u(0,x+) - \tilde{u}(0,x)).$$

Since we assume that the shocks are in involution, setting $\xi_N = c(0, x_n-)$ we have that (3.12) holds at time $t = 0$: in fact the last jump has size $\tilde{u}(0, x_N(0)-) - u(0, x_N(0)-)$.

We now show that this property is conserved for all $t \geq 0$. This follows easily from conservation and Lemma 3.5. The proof is exactly the same as in [11]: we repeat it for completeness. Consider the interaction between two shocks σ_i and σ_j in the point (τ, y), see fig. 4. By inductive assumption, we have for the states u_l, u_m and u_l that

$$\sum_{x_\gamma(\tau)<y} \sigma_\gamma(\tau)\xi_\gamma(\tau) = c_l(u_l - \tilde{u}_l), \tag{3.13}$$
$$c_l(u_l - \tilde{u}_l) + \sigma_i\xi_i = c_m(u_m - \tilde{u}_m),$$
$$c_m(u_m - \tilde{u}_m) + \sigma_j\xi_j = c_r(u_r - \tilde{u}_r).$$

Using conservation we have

$$\xi_i\sigma_i + \xi_j\sigma_j = \xi'_j\sigma'_j + \xi'_i\sigma'_i, \tag{3.14}$$

so that for the new middle state u'_m we have

$$c_l(u_l - \tilde{u}_l) + \sigma'_j\xi'_j = c'_m(u'_m - \tilde{u}'_m) = c_r(u_r - \tilde{u}_r) - \sigma'_i\xi'_i, \tag{3.15}$$

and using Lemma 3.5 we conclude

$$\mathrm{span}\Big\{u_l - \tilde{u}_l, \sigma'_j\xi'_j\Big\} \bigcap \mathrm{span}\Big\{u_r - \tilde{u}_r, \sigma'_i\Big\} = \mathrm{span}\big\{u_m - \tilde{u}'_m\big\}.$$

The same relation proves that they vanish outside Γ: in fact, assume for example that $c_l = 0$ and $j < i$. Then from (3.15) we get

$$\sigma'_j\xi'_j = c'_m(u'_m - \tilde{u}'_m),$$

which implies that $c'_m = 0$. This concludes the proof. □

Remark 3.9. Note that for discontinuities of a linearly degenerate family all shift rates has the same sign. Note moreover that if no waves of other families are present, then we shift all jumps σ_α by unit rate 1. This corresponds to the case considered in Lemma 3.3, i.e. to the substitution of a family of contact discontinuities with a single jump, whose strength in this case is 0 by the involution assumption.

Using conservation and the previous lemmas, we obtain explicitly the shift differential map at a given time τ. We recall that, given the states $u^-, u^+ \in E$, we denote with $r_i(u^-, u^+)$ the vectors defined in (2.9), and with $l^i(u^-, u^+)$ its dual base. Let $P_j(u^-, u^+)$ be the projection operator on $\mathrm{span}\{r_i(u^-, u^+), i = 1, \dots, j\}$:

$$P_j(u^-, u^+)v \doteq \sum_{i=1}^{j} \langle l^i(u^-, u^+), v\rangle r_i(u^-, u^+), \tag{3.16}$$

where $\langle\cdot,\cdot\rangle$ denotes the the scalar product in $\mathbb{R}^n$.

Given a point (t, x), with $u(t, x)$ continuous in x, define x_i the intersection of the backward i-th characteristics starting at (t, x) with the real axis $\{(0, x)\}$, and for all $(0, y)$ let $j(y)$ the index such that $x_{j(y)} \le y < x_{j(y)-1}$, $j(y) = 1, \dots, n+1$, with $x_0 = +\infty$ and $x_{n+1} = -\infty$. Without any loss of generality, we can assume that in $(0, y)$ there is a jump σ of the k-th family.

Define the points $w_l, w_r \in E$ by

$$w_l(x,y) \doteq \begin{cases} w(t,x) & j(y) = 1 \\ (w_1(0,y-), \dots, w_{j(y)-1}(0,y-), w_{j(y)}(t,x), \dots, w_n(t,x)) & 2 \le j(y) \le n \\ w(0,y-) & j(y) = n+1 \end{cases} \tag{3.17}$$

$$w_r(x,y) \doteq \begin{cases} w(0,y+) & j(y) = 1 \\ (w_1(t,x), \dots, w_{j(y)-1}(t,x), w_{j(y)}(0,y+), \dots, w_n(0,y+)) & 2 \le j(y) \le n \\ w(t,x) & j(y) = n+1 \end{cases}$$

Moreover define the point $w_m \in E$ by

(3.18)
$$w_m(x,y) \doteq (w_1(t,x), \dots, w_k(0,y+), \dots, w_{j(y)-1}(t,x), w_{j(y)}(0,y+), \dots, w_n(0,y+)) \quad 2 \le j(y) \le n,$$

if $k < j(y)$, and in a similar way, if $k \ge j(y)$,

(3.19)
$$w_m(x,y) \doteq (w_1(t,x), \dots, w_{j(y)-1}(t,x), w_{j(y)}(0,y+), \dots, w_k(0,y-), \dots, w_n(0,y+)) \quad 2 \le j(y) \le n.$$

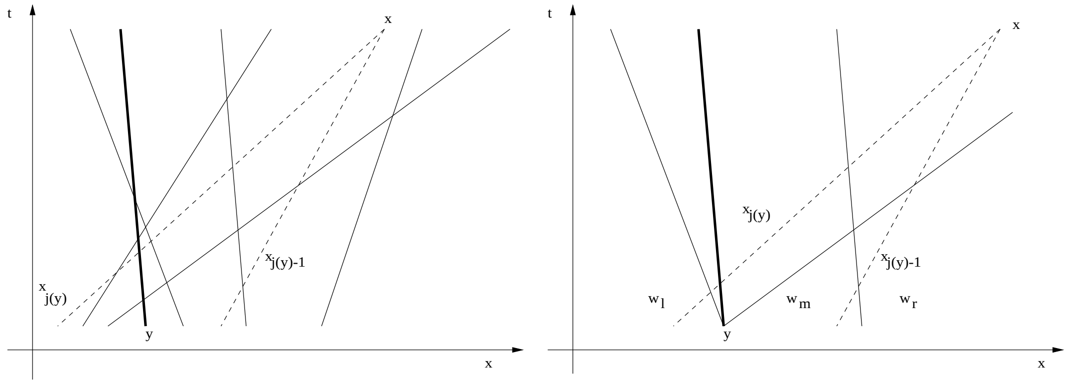

FIGURE 5. Wave pattern for the computation of formula 3.16.

Define $P(x,y)$ as the vector

(3.20)

$$P(x,y) \doteq \begin{cases} 0 & j(y)=1 \\ P_{j(y)-1}(w_l,w_m)\sigma + P_{j(y)-1}(w_m,w_r)\big(\sigma - P_{j(y)-1}(w_l,w_m)\sigma\big) & 2 \leq j(y) \leq n+1, k < j(y) \\ P_{j(y)-1}(w_l,w_m)\big(P_{j(y)-1}(w_m,w_r)\sigma\big) & 2 \leq j(y) \leq n, k \geq j(y) \end{cases}$$

where $w_l = w_l(x,y)$, $w_m = w_m(x,y)$, $w_r = w_r(x,y)$ and σ is the initial jump in $(0,y)$. Consider now a front tracking solution u^θ, obtained by shifting the initial jumps σ_α in y_α with rates ξ_α.

Theorem 3.10. *If $v(t,x)$ is the integral shift function of $u^\theta(t,\cdot)$, defined in* (3.2), *then*

$$v(t,x) = \lim_{\theta\to 0}\left\{-\frac{1}{\theta}\int_{-\infty}^{x} u^\theta(t,y) - u(t,y)dy\right\} = \sum_\alpha P(x,y_\alpha)\xi_\alpha. \tag{3.21}$$

Proof. The theorem will be proved outside the times of interaction, because the Lipschitz dependence in L^1 of the approximate semigroup implies the validity of (3.21) for all $t \geq 0$.

If is sufficient to show that $\sum_{y_\alpha} P(x,y_\alpha)\xi_\alpha$ is piecewise constant, with jumps only at the points x_β where $u(t,\cdot)$ has a shock σ_β, and the following relation holds:

$$\sum_{y_\alpha}\big(P(x_\beta+,y_\alpha) - P(x_\beta-,y_\alpha)\big)\xi_\alpha = \sigma_\beta\xi_\beta, \qquad \lim_{x\to-\infty}\sum_{y_\alpha} P(x,y_\alpha)\xi_\alpha = 0, \tag{3.22}$$

where ξ_β is the shift rate of σ_β, located in x. Note that by (3.20) the second equality of (3.22) is trivially satisfied.

By linearity in the shift rates ξ_α, we can consider the case in which a single shock is shifted, let us say σ at y: (3.21) becomes

$$v(t,x) = P(x,y)\xi. \tag{3.23}$$

Formula (3.16) follows from the following considerations: consider a wave front pattern, fig. 5, where for simplicity we assume that $k < j(y)$. The states w_l, w_m are computed considering the Riemann problem generated by adding to the k-jump σ in $(0,y)$ all the i-waves starting from the left of $(0,y)$ and ending in the right of (t,x) and all the i-waves, with $i \neq k$, starting from the right of $(0,y)$ and ending in the left of (t,x). The jump w_m,w_r is a single wave of the k-th family formed by adding all the k-waves between $(0,y)$ and (t,x). Using the definition of $v(t,x)$ given (3.2), one obtains easily the second case of (3.20): in fact the shift rates of the shocks in the left of (t,x) is given by the shift rates of the jumps of the Riemann problem w_l,w_m ending in the left of (t,x), $P_{j(y)-1}(w_l,w_m)\sigma$, plus the shift rate of the shock w_m,w_r, $P_k(w_m,w_r)(\sigma - P_{j(y)-1}(w_l,w_m)\sigma)$. Since only the i-waves with $i \geq j(y) > k$ are present in $\sigma - P_{j(y)-1}(w_l,w_m)\sigma$, then $P_k(w_m,w_r)(\sigma - P_{j(y)-1}(w_l,w_m)\sigma) = P_{j(y)-1}(w_m,w_r)(\sigma - P_{j(y)-1}(w_l,w_m)\sigma)$. The other cases can be computed in a similar way: in this case one solves the Riemann problem w_m,w_r in $(0,y)$, and consider the k-wave w_l,w_m starting in the left of $(0,y)$ and ending in the right of (t,x).

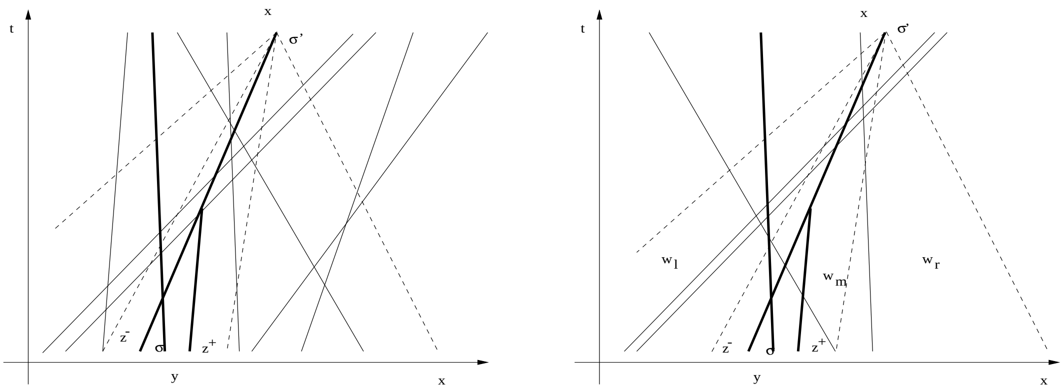

FIGURE 6. Computation of the shift rate in the case of Lemma 3.11.

From the above considerations it is clear that $P(x,y)$ is piecewise constant, with jumps only when in (t,x) there is a i-shock σ': in fact otherwise the wave front pattern used to compute $P(x,y)$ remains the same. Let $\{z_p : p = 1,\dots,M\}$ be the set of the starting points of all shocks arriving in (t,x), and define

$$z^- = \min_p z_p, \qquad z^+ = \max_p z_p. \tag{3.24}$$

We consider two cases:

1) the shocks arriving in (t,x) start on both sides of $(0,y)$: $z^- \le y \le z^+$. In this case, $(P(x+,y) - P(x-,y))\xi$ is the shift rate of the i-shock starting in the Riemann problem $w_l(x-,y), w_m(x+,y)$ if $i > k$ ($w_m(x-,y), w_r(x+,y)$ if $i < k$) which collides with a k-shock $w_m(x+,y), w_r(x+,y)$ (w_l, w_m if $i < k$): in fact the only difference is that in $w_m(x-,y), w_m(x+,y)$ there is a shock of the i-th family starting in $(0,y)$, and i is genuinely nonlinear. Finally, using $r_i \bullet r_i(u) = 0$ and Lemma 3.5, one can change position to the i-wave and the remaining k-wave w_m, w_r, whose strength does not change.

   If $i = k$, there are no k-shocks starting on the right (left) of $(0,y)$ and ending on the right (left) of (t,x), so that $(P(x+,y) - P(x-,y))\xi$ is the shift rate of the i-shock of the Riemann problem $w_l(x-,y), w_r(x+,y)$.
2) the shocks of the i-th family arriving in (t,x) start either in $(-\infty,y)$ or $(y,+\infty)$: assume for definiteness that $y < z^-$. In this case the difference $(P(x+,y) - P(x-,y))\xi$ is the shift rate of the shock σ' colliding with the shifted shocks of the Riemann problem $w_l(x-,y), w_m(x-,y)$ in $(0,y)$, crossing the jump $w_m(x-,y), w_r(x-,y)$, and finally overtaking σ'. In fact one can use Lemma 3.5 (and $r_i \bullet r_i(u) = 0$ if i is genuinely nonlinear) to obtain the wave pattern of fig. 7.

The various cases will be proved in the following lemmas.

Lemma 3.11. *Assume that $z^- \le y \le z^+$, i.e. case 1). If the shock σ' is of the i-th family, then its shift ξ' is*

$$\xi'\sigma' = \big(P(x+,y) - P(x-,y)\big)\xi. \tag{3.25}$$

Proof. We follow closely the method of [11]. Assume for definiteness $k < j(y)$, the other cases being similar. The basic idea is to reduce the computation to the single Riemann problem $w_l(x-,y), w_m(x+,y)$, with eventually a single k-wave w_m, w_r.

Consider fig. 6. By Lemma 3.1, we can simplify the wave configuration considering only the fronts crossing starting in the right of $(0,y)$ and ending in the left of (t,x): in fact we can move the other fronts to $\pm\infty$ without changing the shift rate of σ'.

We can now shift the initial position of the waves of the i-th family merging in x such that their initial position coincide with y, without changing the shift rate ξ'. This operation can be repeated for all shocks of genuinely nonlinear families.

Finally, we can move the shocks of the linearly degenerate families such that they have the same sequence of interaction with the other shocks. This means that, if x_i^j is the position of the j-th shock of the i-th linearly degenerate family, the only interactions among shocks occurring in the sector $[x_i^1(t), x_i^n(t)]$

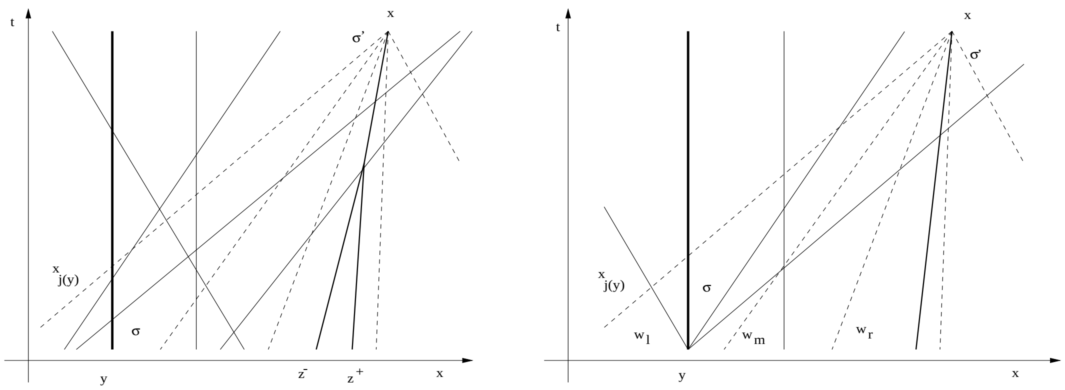

FIGURE 7. Computation of the shift rate in the case of Lemma 3.12.

are the one involving one i-th wave and one k-th wave, with $k \neq i$. Using Lemma 3.3, we can at this point substitute them with a single shock, whose strength is the sum of the strengths of the i-waves. Finally we move their position at $t = 0$ such that it coincides with y: we obtain the wave patterns of fig. 6. To conclude, we just need to prove that the Riemann problem obtained in this way is exactly $w_l(x-, y), w_m(x+, y)$ and that the remaining k-wave is $w_m(x+, y), w_r(x+, y)$.

By the previous argument, the strength of the shock of the j-th family $j < i$, $j \neq k$, is given by the j-waves starting in the right of y and ending in the left of x: since they are the only j-wave crossing the segment $[(0, y-), (t, x+)]$, it follows

$$w_{l,j}(x, y) = w_j(0, y-), \qquad w_{r,j}(x, y) = w_j(t, x+).$$

The other relations for $j = k$ and $j > i$ follows in the same way. Finally, for $j = i$ the jump is $w_i(t, x+) - w_i(t, x-)$. Note that the wave pattern is the same obtained in 1). □

We consider only the case $y < z^-$, since the other is entirely similar.

Lemma 3.12. *Assume that $y < z^-$. Then the shift ξ' of σ' is given by*

$$\xi'\sigma' = \big(P(x+, y) - P(x-, y)\big)\xi. \tag{3.26}$$

Proof. The hypothesis implies that the i-th shocks ending at x starts in the right of y. With the same simplification considered in Lemma 3.11, we reduce to the Riemann problem $w_l(x, y), w_r(x, y)$ in $\bar{y}$, such that the waves of the j-th families, $j > i$, generated at $\bar{y}$ collide with the i-wave in x_α (see fig. 7), after overtaking the k-wave w_m, w_r. The conclusion follows easily, since the wave pattern is the same considered in 2). □

This concludes the proof of Theorem 3.10. □

Finally we extend to our case the following result proved in [11]:

Proposition 3.13. *Let u be a wave-front tracking solution, and consider two wave-fronts, $x(t)$ and $y(t)$, $t \in [0, T]$. Then there exists a second front tracking solution $\tilde{u}$ such that the initial and final positions of the two shocks is the same, and* Tot.Var.$(\tilde{u})$ *is uniformly bounded.*

Proof. For genuinely nonlinear fields, the proof is the same as in [11]. We then restrict the proof to the case of a linearly degenerate fields i.

Assume that there exists two jumps σ_1, σ_2 of the i-th family, with positions $z_1(t) < z_2(t)$, such that

$$x(0) \notin [z_1(0), z_2(0)] \text{ and } y(0) \notin [z_1(0), z_2(0)], \quad x(T) \notin [z_1(T), z_2(T)] \text{ and } y(T) \notin [z_1(T), z_2(T)]. \tag{3.27}$$

For definiteness, assume $w_i(0, z_1-) < w_i(0, z_1+)$, and the following conditions is satisfied:

$$w_i(0, z_1-) \in [w_i(0, z_2-), w(0, z_2+)]. \tag{3.28}$$

Let σ_α, $\alpha = 1, \dots, N$ be the jumps of linearly degenerate family i in the strip $[z_1(0), z_2(0))$: if we define

$$\sigma_{N+1} = u(w_i(0, z_1-)) - u(w_i(0, z_2-)),$$

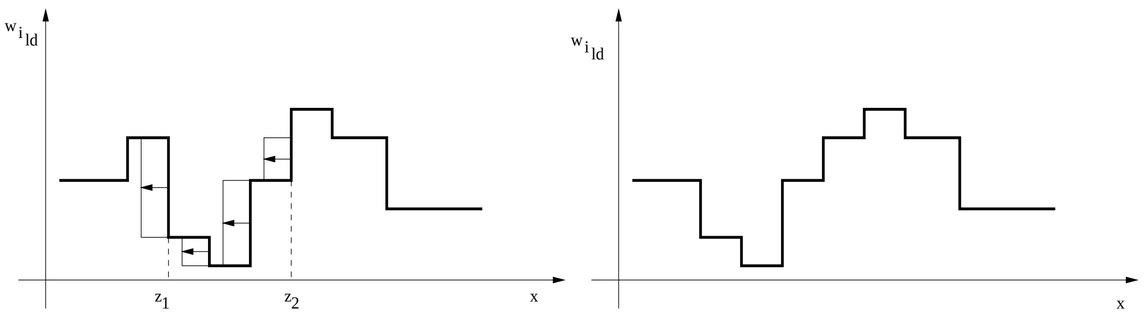

FIGURE 8. Cancellation among contact discontinuities.

it is easy to verify that the shocks σ_α, $\alpha = 1, \ldots, N+1$, are in involution. By Lemma 3.8, we can then moving the jumps to the left until either $z_1(t)$ meets the wave fronts $x(t)$, or $z_1(t)$ coincides with another shock of the i-th family (fig. 8). It is clear that we can repeat the same procedure also in the following cases:

i) $w_i(0, z_1-) > w_i(0, z_1+)$ and $w_i(0, z_1-) \in [w_i(0, z_2+), w(0, z_2-)]$;
ii) $w_i(0, z_2-) < w_i(0, z_2+)$ and $w_i(0, z_2+) \in [w_i(0, z_1+), w(0, z_1-)]$;
iii) $w_i(0, z_2-) > w_i(0, z_2+)$ and $w_i(0, z_2+) \in [w_i(0, z_1-), w(0, z_1+)]$.

It is now easy to prove that the total variation of the jumps of the i-th family satisfying (3.27) can at most be $3\|w\|_\infty$. Since $x(t)$, $y(t)$ divide the lines $t = 0$ and $t = \tau$ in three regions, the total variation of w_i is bounded by $27\|w\|_\infty$. □

4. Estimates on characteristics

In this section we prove some estimates on the solution $x_i(t, y)$ of the characteristic equation:

$$\begin{cases} \dot{x}_i = \lambda_i(u(t, x_i)) \\ x_i(0) = y \end{cases} \tag{4.1}$$

We assume for simplicity that the i-th family is linearly degenerate, however the same results are valid for characteristics of a genuinely nonlinear family if the following condition holds: for all τ there exists an ϵ such that in the strip $\{(t,x); \tau \le t \le T, x_i(t,y) - \epsilon \le x \le x_i(t,y) + \epsilon\}$ there are no shock waves of the i-th family. Given front tracking approximation u, $x_i(t,y)$ is unique, since it crosses only a finite number of transversal jumps, and it depends Lipschitz continuously on the initial data y (see [7]).

We want to give uniform estimates on this dependence. The idea is to suppose that in y there is a shock σ^ϵ of the i-family of size ϵ: $w_i(0, y+) - w_i(0, y-) = \epsilon$. Since by assumption no shocks of the i-family collide with σ^ϵ, it is easy to construct a wave front solution: for $x < x(t,y)$, the solution $u^\epsilon(t, \cdot)$ takes values in

$$E^{\nu,-} \doteq \Big\{ u : w_j(u) \in [a_j, b_j] \cap 2^{-\nu}\mathbb{Z}, j = 1, \ldots, n \Big\},$$

while for $x > x(t,y)$, enlarging E and assuming ϵ sufficiently small,

$$E^{\nu,+} \doteq \Big\{ u : w_j(u) \in [a_j, b_j] \cap 2^{-\nu}\mathbb{Z}, j \neq i, w_i(u) \in [a_i, b_i] \cap \big\{ 2^{-\nu}\mathbb{Z} + \epsilon \big\} \Big\}.$$

The following lemma proves the continuous dependence of the solution $u^\epsilon(t)$ and the position $x_i^\epsilon(t,y)$ of the shock σ^ϵ w.r.t. ϵ.

Lemma 4.1. *Consider a front tracking solution u, with initial data u_0 and the characteristic lines $x_i(t, y_1) < x_i(t, y_2)$, defined in (4.1) for a linearly degenerate family i. Let u^ϵ the wave front solution with initial data $u(w_0^\epsilon)$, where w_0^ϵ is defined as*

$$w_0^\epsilon(x) \doteq \begin{cases} w(u_0(x)) & x \le y_1 \\ w(u_0(x)) + \epsilon e_i & y_1 < x \le y_2 \\ w(u_0(x)) & x > y_2 \end{cases} \tag{4.2}$$

Then there exists constants L, L', depending only on the total variation of the initial data u_0, such that for all $t \geq 0$

$$\int_{\mathbb{R}} \big|u(t,x) - u^\epsilon(t,x)\big| dx \leq L\epsilon \qquad \textit{and} \qquad |x_i^\epsilon(t,y_j) - x_i(t,y_j)| \leq L'\epsilon t, \quad j = 1,2, \tag{4.3}$$

where $x_i^\epsilon(t,y_j)$ is the position of the shock σ_j^ϵ starting in $(0,y_j)$.

Proof. The first inequality is an easy consequence of the L^1 continuous dependence for front tracking solutions, see [2]. For the second one, note that all the shocks different from σ^ϵ have size uniformly bigger than 0, so that their position is shifted of the order ϵ. Thus the second inequality follows by standard ODE perturbation estimates, see [7]. □

An easy application of the previous lemma together with Proposition 3.13 implies that to compute $x_1(t,y_1)$ and $x_2(t,y_2)$, we can actually consider in (4.1) a solution $\tilde{u}$ with uniformly bounded total variation, so that the constant L' in (4.3) is independent on the total variation of u_0.

We now estimate the dependence of $x_i(t,y)$ w.r.t. u.

Proposition 4.2. *Let ξ_α be the shift rate of the jump σ_α in $u(0,\cdot)$, and denote with x_i^θ the solution to*

$$\begin{cases} \dot{x}_i^\theta = \lambda_i(u^\theta(t,x_i^\theta)) \\ x_i^\theta(0) = y \end{cases}$$

where $u^\theta(t)$ is the shifted front tracking solution. Then there exists a constant D independent of the total variation of u such that

$$\left|\lim_{\theta\to 0} \frac{x_i^\theta(t,y) - x_i(t,y)}{\theta}\right| \leq D \sum_\alpha \big|\sigma_\alpha \xi_\alpha\big|. \tag{4.4}$$

Proof. We give a sketch of the proof, for details one can see [11].

If ϵ is the size of the shock σ^ϵ located in $(0,y)$, then we can apply Theorem 3.10 to compute its shift ξ^ϵ: by formula (3.21) we obtain

$$\xi^\epsilon \sigma^\epsilon = \sum_\alpha \big(P(x+,y_\alpha) - P(x-,y_\alpha)\big)\xi_\alpha. \tag{4.5}$$

If θ is sufficiently small, then we have

$$\xi^\epsilon = \frac{x_i^{\theta,\epsilon}(t,y) - x_i^\epsilon(t,y)}{\theta},$$

where $x_i^{\theta,\epsilon}(t,y)$ is the position of the shifted shock and $x_i^\epsilon(t,y)$ is its original position. Note that $(P(x+,y_\alpha) - P(x-,y_\alpha))\xi_\alpha$ is the shift rate of the shock σ^ϵ, after colliding with the shocks of the Riemann problems w_l, w_m and w_m, w_r. Their total shift is proportional to $|\sigma_\alpha\xi_\alpha|$, and after the interaction with σ^ϵ, the shift of the latter is proportional to $|\sigma^\epsilon||\sigma_\alpha\xi_\alpha|$. Thus taking the limit as ϵ tends to 0 of (4.5), we obtain for ϵ sufficiently small

$$\left|\frac{x_i^\theta(t,y) - x_i(t,y)}{\theta}\right| \leq D \sum_\alpha \big|\sigma_\alpha\xi_\alpha\big|,$$

which implies (4.4). □

We prove now the uniform Lipschitz continuity of the map $y \longmapsto x_i(t,y)$ for all $t \geq 0$.

Proposition 4.3. *Consider two characteristic lines $x_i(t,y_1)$ $x_i(t,y_2)$, solution to (4.1). There exists $C > 0$, depending only on the system and the set E, such that*

$$\frac{1}{C} \leq \frac{x_i^2(t,y_2) - x_i^1(t,y_1)}{y_2 - y_1} \leq C. \tag{4.6}$$

Proof. As in the previous proposition, let ϵ be the size of the shock $\sigma^\epsilon(t)$ located in $(0,y)$ in Riemann coordinates. If $\xi(t)$ is its shift rate, then for θ sufficiently small by Theorem 3.10 we obtain

$$\frac{x_i^\epsilon(t,y+\theta\xi) - x_i^\epsilon(t,y)}{\theta}\sigma^\epsilon(t) = \xi^\epsilon(t)\sigma^\epsilon(t) = r_i(w_l,w_r)\big\langle l^i(w_l,w_r), \sigma(0)\big\rangle\xi(0). \tag{4.7}$$

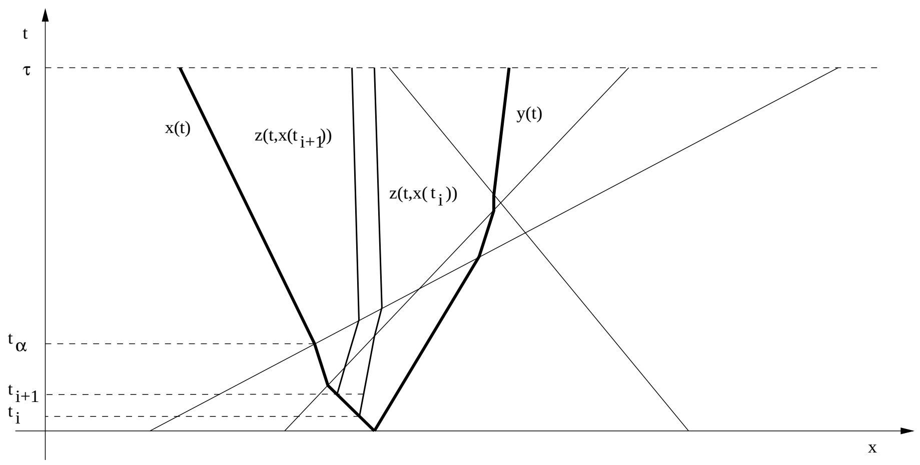

FIGURE 9. Decay of positive waves.

In fact, by assumption, in the simplified wave patterns to compute the shift rate of σ^ϵ, there are no waves of the i-th family different from σ^ϵ. Dividing by ϵ and taking the limit as ϵ tends to 0, we obtain

$$\frac{x_i(t,y+\theta\xi)-x_i(t,y)}{\theta}\frac{\partial}{\partial w_i}u(t,x)=\frac{\partial}{\partial w_i}u(0,y)\big\langle l^i(w_l,w_r),r_i(0,y)\big\rangle\xi(0),$$

which implies

$$\frac{d}{dy}x_i(t,y)=\frac{\partial u(0,y)/\partial w_i}{\partial u(t,x)/\partial w_i}\big\langle l^i(w_l,w_r),r_i(0,y)\big\rangle. \tag{4.8}$$

We use the fact that $\sigma^\epsilon(t)/\epsilon$ tends to $\partial u(0,y)/\partial w_i \cdot r_i(w_l,w_r)$ as $\epsilon\to 0$. Since E is compact, the conclusion (4.6) follows easily. □

Remark 4.4. The above proposition implies that the map h_i^t defined in (2.16) is uniformly Lipschitz, independent on the total variation of u_0, together with its inverse map $(h_i^t)^{-1}$.

To end this section, we give a different proof of the following result given in [11]:

Proposition 4.5. *If $x(t)$, $y(t)$ are the positions of two adjacent k-rarefaction waves, then for some constant $\kappa>0$ one has*

$$y(\tau)-x(\tau)\geq\kappa\tau 2^{-\nu}. \tag{4.9}$$

Thus for all $\tau>0$ the total variation of the Riemann invariant w_k of the k-th genuinely nonlinear family with N shocks at $t=0$ is bounded by

$$\text{Tot.Var.}\{w_k(\tau,\cdot);[a,b]\}\leq\frac{2(b-a)}{\kappa\tau}+\|w_k\|_{L^\infty}+(N+1)2^{1-\nu}. \tag{4.10}$$

Proof. Consider two adjacent k-rarefaction fronts $x(t)$ and $y(t)$, and let t_α, $\alpha=1,\dots,N$, be the interaction times of $x(t)$, $y(t)$ with other waves in the interval $[0,\tau]$. Fixed $t_i\in(t_{\bar\alpha},t_{\bar\alpha+1})$ for some $\bar\alpha$, let $z(t,x(t_i))$ be the characteristic line of the k-th genuinely nonlinear family starting in $(t_i,x(t_i))$ (see fig. 9). Assume $t_{i+1}>t_i$ sufficiently close to t_i such that $t_{i+1}\in(t_{\bar\alpha},t_{\bar\alpha+1})$ and $z(t,x(t_i))$ does not collide with shocks of other families for $t\in[t_i,t_{i+1}]$. Let $z(t,x(t_{i+1}))$ be the characteristic curve starting in $(t_{i+1},x(t_{i+1}))$. By the assumption of genuinely nonlinearity, at time t_{i+1} we have

$$z(t_{i+1},x(t_i))-z(t_{i+1},x(t_{i+1}))\geq c(t_{i+1}-t_i)2^{-\nu-1},$$

for some constant c, depending only on E. Using Proposition 4.3, at time τ we have

$$z(\tau,x(t_i))-z(\tau,x(t_{i+1}))\geq\frac{c}{C}(t_{i+1}-t_i)2^{-\nu-1}. \tag{4.11}$$

Repeating the process, it is possible to find a countable number of times t_i such that

$$\lim_{i\to-\infty}t_i=t_{\bar\alpha},\qquad\lim_{i\to+\infty}t_i=t_{\bar\alpha+1},$$

and using (4.11) we get

$$z(\tau,x(t_{\bar\alpha})-z(\tau,x(t_{\bar\alpha+1}))\geq\frac{c}{C}(t_{\bar\alpha+1}-t_{\bar\alpha})2^{-\nu-1}. \tag{4.12}$$

Repeating the process for $y(t)$ and for all intervals $(t_{\alpha+1},t_\alpha)$, we obtain (4.9) where $\kappa=c/C$.

The second equation follows noticing that the total amount of positive jumps in the interval $[a,b]$ is bounded by $(1+N)2^{-\nu}+(b-a)/\kappa\tau$. □

5. Proof of the main theorem

In this section we construct the semigroup $\mathcal{S}$ on $L^\infty(\mathbb{R};E)$. In [2] it is shown that for all M, there exists a semigroup $\mathcal{S}^M$ defined on the domain

$$D^M\doteq\Big\{u:\mathbb{R}\mapsto E:\text{Tot.Var.}(u)\leq M\Big\}, \tag{5.1}$$

which is the only limit of the wave front tracking approximations constructed in section 2. We study now the dependence of the solution on the initial data $u\in D^M$. We consider separately the case for genuinely nonlinear and linearly degenerate families.

Proposition 5.1. *Consider a front tracking solution u, such that $u(0,\cdot)$ has N jumps σ_α, $\alpha=1,\dots,N$, and let ξ_α be their shift rates. Given $\tau\geq0$, denote with σ_β the jumps in the Riemann invariant $w_k(\tau,\cdot)$ of the k-th genuinely nonlinear family. Then there exists a constant K, depending only on the system and the domain E such that*

$$\sum_\beta\big|\xi_\beta\sigma_\beta\big|\leq K(1+N2^{-\nu})\sum_{\alpha=1}^N\big|\xi_\alpha\sigma_\alpha\big|. \tag{5.2}$$

Proof. The proof follows by Theorem 3.10 and Proposition 4.5. In fact, fixed a shock $\sigma_{\bar\alpha}$, using Theorem 3.10, we have that at time τ for a shock σ_β of the i-th family there exist D'

$$\big|\xi_\beta\sigma_\beta\big|\leq D'\big|\xi_{\bar\alpha}\sigma_{\bar\alpha}\big|, \tag{5.3}$$

if the shock σ_β starts on both sides of $\sigma_{\bar\alpha}$, or, using the same estimate of Proposition 4.2,

$$\big|\xi_\beta\big|\leq D\big|\sigma_{\bar\alpha}\xi_{\bar\alpha}\big|, \tag{5.4}$$

if σ_β start on one side of $\sigma_{\bar\alpha}$. Since there is at most 1 shocks such that (5.3) holds, and the interval of influence is $[x_{\bar\alpha}-\hat\lambda\tau,x_{\bar\alpha}+\hat\lambda\tau]$, using Proposition 4.5 together with (5.3) and (5.4) we obtain

$$\sum_\beta\big|\xi_\beta\sigma_\beta\big|\leq D'\big|\xi_{\bar\alpha}\sigma_{\bar\alpha}\big|+D\big|\sigma_{\bar\alpha}\xi_{\bar\alpha}\big|\cdot\text{Tot.Var.}\Big\{w_k,[x_{\bar\alpha}-\hat\lambda\tau,x_{\bar\alpha}+\hat\lambda\tau]\Big\}\leq F(1+2^{-\nu})\big|\xi_{\bar\alpha}\sigma_{\bar\alpha}\big|.$$

The conclusion follows the linearity of the shift differential map. □

Using the results of the previous section, the following result is trivial:

Proposition 5.2. *Consider a wave front solution u, such that $u(0,\cdot)$ has N jumps σ_α, $\alpha=1,\dots,N$, and let ξ_α be their shifts. Consider the equation (4.1), whit the eigenvalue λ_i linearly degenerate. Fixed $\tau\geq0$, then the shift ξ_i of $x_i(\tau,y)$ is bounded by*

$$\big|\xi_i\big|\leq D\sum_{\alpha=1}^N\big|\xi_\alpha\sigma_\alpha\big|. \tag{5.5}$$

Proof. This is a corollary of Proposition 4.2. □

Using the above propositions, we can prove the following theorem:

Theorem 5.3. *Consider two initial data u_1 and u_2, and denote with $w_{j,k}(t,\cdot)$, the k-th Riemann coordinate of $\mathcal{S}^Mu_j$, $j=1,2$, corresponding to the k-th genuinely non linear family. Moreover, let $h^\tau_{j,i}$, $j=1,2$, the map defined in (4.1) for the i-th linearly degenerate family. Then there exists a constant K', independent of M, such that the following estimates hold:*

$$\int_{\mathbb{R}}\big|w_{1,k}(t,x)-w_{2,k}(t,x)\big|dx\leq K'\int_{\mathbb{R}}\big|u_1(x)-u_2(x)\big|dx, \tag{5.6}$$

$$\sup_{t\geq 0, x\in\mathbb{R}} \left|h^t_{1,i}(x) - h^t_{2,i}(x)\right| \leq K' \int_{\mathbb{R}} \left|u_1(x) - u_2(x)\right| dx. \tag{5.7}$$

Proof. Consider two piecewise constant initial data u_1^ν, u_2^ν in $D^{M,\nu}$, and construct a pseudo polygonal path $\gamma_0 : \theta \longmapsto u_\theta^\nu$, connecting u_1 and u_2, such that

$$\left\|\gamma_0\right\|_{L^1} \leq E\left\|u_1^\nu - u_2^\nu\right\|_{L^1}.$$

We can assume that u_θ^ν has a finite number N of jumps. If we denote with γ_τ^ν the path $\theta \longmapsto \mathcal{S}_\tau^\nu u_\theta^\nu$, we have by Proposition 5.1

$$\begin{aligned} \left\|w^\nu_{2,k}(\tau) - w^\nu_{1,k}(\tau)\right\|_{L^1} &\leq \left\|\left(\gamma_\tau^\nu\right)_k\right\|_{L^1} \leq K(1 + N2^{-\nu})\left\|\gamma_0\right\|_{L^1} \\ &\leq K'(1 + N2^{-\nu})\left\|u_2 - u_1\right\|_{L^1}. \end{aligned} \tag{5.8}$$

If now $\nu \to +\infty$, since $w^\nu_{j,k}(\tau)$ converges to $w_{j,k}(\tau)$, we obtain (5.6). Since this estimate does not depend on the number of initial jumps N, we can extend it uniformly on D^M.

Using the same pseudo polygonal path, in a similar way we can prove that

$$\left|x^\nu_{2,i}(\tau, y) - x^\nu_{1,i}(\tau, y)\right| \leq K'\left\|u_2 - u_1\right\|_{L^1}.$$

This shows that $x^\nu_i(\tau, \cdot)$ converges uniformly to the solution $x_i(\tau, \cdot)$ as $\nu \to +\infty$ and $u^\nu \to u$. It also implies that

$$\left|x_{2,i}(\tau, y) - x_{1,i}(\tau, y)\right| \leq K'\left\|u_2 - u_1\right\|_{L^1},$$

This concludes the proof. $\square$

We can now define $\mathcal{S}$ on the domain $L^\infty(\mathbb{R}; E)$:

Definition 5.4. For all $u \in L^\infty(\mathbb{R}, E)$, let $u^M \in D^M$ be such that

$$\lim_{M\to+\infty} u^M = u \quad \text{in } L^1_{\text{loc}}. \tag{5.9}$$

Define $\mathcal{S}_t u$ as

$$\mathcal{S}_t u = \lim_{M\to+\infty} \mathcal{S}^M_t u, \tag{5.10}$$

where the limit is in L^1_{loc}.

It is easy to prove that the right hand side of (5.10) is a Cauchy sequence in every compact set $[a, b]$: in fact, using the finite speed of propagation, we can consider u with compact support $[a - \hat\lambda t, b + \hat\lambda t]$. For the components w_k of the k-th genuinely nonlinear family, it follows directly from (5.6), while for a linearly degenerate component w_i, let $\tilde w$ be a Lipschitz continuous function such that

$$\int_{\mathbb{R}} \left|w_i(0, x) - \tilde w(x)\right| dx \leq \epsilon.$$

By Theorem 5.3 we have for $u_1, u_2 \in D^M$ such that $\|u - u_i\|_{L^1} < \delta$, $i = 1, 2$,

$$\sup_{t\geq 0, x\in\mathbb{R}} \left|h^t_{1,i}(x) - h^t_{2,i}(x)\right| < K'\delta,$$

and it follows by easy computations that

$$\begin{aligned} \left\|w_{1,i}(t) - w_{2,i}(t)\right\|_{L^1} \leq{}& \left\|w_{1,i}(t) - \tilde w \circ \left(h^t_{1,i}\right)^{-1}\right\|_{L^1} + \left\|w_{2,i}(t) - \tilde w \circ \left(h^t_{2,i}\right)^{-1}\right\|_{L^1} + \\ & \left\|\tilde w \circ \left(h^t_{1,i}\right)^{-1} - \tilde w \circ \left(h^t_{2,i}\right)^{-1}\right\|_{L^1} \\ \leq{}& C\left\|w_{1,i}(0) - \tilde w\right\|_{L^1} + C\left\|w_{2,i}(0) - \tilde w\right\|_{L^1} + L(b-a)G\left\|u_2 - u_1\right\|_{L^1} \\ \leq{}& 2C(\epsilon + \delta) + L(b-a)G\delta, \end{aligned} \tag{5.11}$$

where L is the Lipschitz constant of $\tilde w$. This shows that $w^M_i(t)$ is a Cauchy sequence for all $t \geq 0$, because the right hand side of (5.11) can be made arbitrarily small. We can now prove the main theorem:

Theorem 5.5. *The semigroup $\mathcal{S} : [0, +\infty) \otimes L^\infty(\mathbb{R}; E) \longmapsto L^\infty(\mathbb{R}; E)$ defined in (5.10) is the only continuous semigroup on $L^\infty(\mathbb{R}; E)$ such that the following properties are satisfied:*

i) *for all* $\bar u^n, \bar u \in L^\infty(\mathbb{R}; E)$, $t_n, t \in [0, +\infty)$, *with* $\bar u_n \to \bar u$ *in* L^1_{loc}, $|t - t_n| \to 0$ *as* $n \to +\infty$,

$$\lim_{n\to+\infty} \mathcal{S}_{t_n}\bar u_n = \mathcal{S}_t \bar u \quad \text{in } L^1_{loc}; \tag{5.12}$$

ii) *each trajectory* $t \mapsto \mathcal{S}_t u_0$ *is a weak entropic solution to the Cauchy problem*

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{5.13}$$

with $u_0 \in L^\infty(\mathbb{R}; E)$;

iii) *if* u_0 *is piecewise constant, then, for* t *sufficiently small,* $\mathcal{S}_t u_0$ *coincides with the function obtained by piecing together the solutions of the corresponding Riemann problems.*

Proof. The statement follows easily, since we proved that $\mathcal{S}_t u$ is the unique limit of wave front approximations, and for data with bounded total variation we can apply the results in [2]. □

Remark 5.6. Note that what we also proved that the characteristic equation (4.1) is well posed for L^∞ data: the solution $x_i(t, y)$ is Lipschitz continuous w.r.t. both variables. This is not trivial, since even for 2×2 systems not in conservation form the dependence is Hölder continuous, while for general $n \times n$ the solution does not exists.

Note moreover that semigroup $\mathcal{S}$ is continuous, but not uniformly continuous. However if the initial data takes values is a compact set of $L^1 \cap L^\infty$, then the semigroup becomes uniformly continuous. This extend the Lipschitz continuity when the initial data have bounded total variation.

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S.I.S.S.A. - I.S.A.S., via Beirut 2-4, 34014 TRIESTE (ITALY)
E-mail address: `bianchin@sissa.it`
URL: `http://www.sissa.it/~bianchin/`