ON THE SHIFT DIFFERENTIABILITY OF THE FLOW GENERATED BY A HYPERBOLIC SYSTEM OF CONSERVATION LAWS

Stefano Bianchini

S.I.S.S.A. (I.S.A.S.), Via Beirut 2/4, 34013 Trieste, Italy

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Abstract. We consider the notion of shift tangent vector introduced in [7] for real valued BV functions and introduced in [9] for vector valued BV functions. These tangent vectors act on a function $u \in L^1$ shifting horizontally the points of its graph at different rates, generating in such a way a continuous path in L^1. The main result of [7] is that if the semigroup $\mathcal{S}$ generated by a scalar strictly convex conservation law is shift differentiable, i.e. paths generated by shift tangent vectors at u_0 are mapped in paths generated by shift tangent vectors at $\mathcal{S}_t u_0$ for almost every $t \geq 0$. This leads to the introduction of a sort of differential, the "shift differential", of the map $u_0 \longrightarrow \mathcal{S}_t u_0$.

In this paper, using a simple decomposition of $u \in \mathrm{BV}$ in terms of its derivative, we extend the results of [9] and we give a unified definition of shift tangent vector, valid both in the scalar and vector case. This extension allows us to study the shift differentiability of the flow generated by a hyperbolic system of conservation laws.

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1. Introduction

In this paper we address th question of differentiability of the flow generated by a strictly hyperbolic system of conservation laws. The primary motivation for the introduction of shift differentials comes from the theory of hyperbolic conservation laws [6,7,14], in particular the evolution of first order perturbation of initial data. Other potential applications are variational problems and optimal control of solutions.

It is well known that if $\mathcal{S}$ is L^1 contractive semigroup generated by a scalar conservation, then in general the map $u \longrightarrow \mathcal{S}_t u$, for fixed t, is not differentiable in the usual L^1 differential structure. However, for any Lipschitz continuous map, one can introduce a new differential structure on the space BV, defining a tangent space T_u at $u \in$ BV: given a set Γ of continuous paths $\gamma : [0, \theta^*] \ni \theta \to \gamma(\theta) \in$ BV, with $\gamma(0) = u$, consider the equivalence relation $\overset{\wedge}{\sim}$ defined as

$$\gamma \overset{\wedge}{\sim} \widetilde{\gamma} \quad \text{if} \quad \lim_{\theta \to 0} \frac{1}{\theta} \|\gamma^\theta(u) - \widetilde{\gamma}^\theta(u)\|_{L^1} = 0. \tag{1.1}$$

The tangent space T_u at u is by definition the elements of the set $\Gamma / \overset{\wedge}{\sim}$. Every equivalence class can be regarded as a "first-order tangent vector" at the point u. The standard choice is to consider a family T_u of tangent vectors which can be put in a one-to-one correspondence with $L^1(\mathbb{R})$. More precisely, T_u is defined as the family of all equivalence classes of the maps

$$\theta \longrightarrow \gamma_v(\theta) \doteq u + \theta v, \qquad v \in L^1(\mathbb{R}). \tag{1.2}$$

As it is shown in [7], this choice is not adequate for describing a first-order variation of the flow map S_t: in fact it happens that $u \longrightarrow \mathcal{S}_t u$ does not map $T_u = L^1$ into $T_{\mathcal{S}_t u} = L^1$.

For example consider Burgers' equation

$$u_t + \left[u^2/2\right]_x = 0, \tag{1.3}$$

with the family of initial conditions

$$u^\theta(0, x) = \theta x \cdot \chi_{[0,1]}(x). \tag{1.4}$$

By $\chi_I(x)$ we denote here and in the following the characteristic function of the interval I. Choose for example $\overline{u} \doteq x \cdot \chi_{[0,1]}(x)$. In this case u^θ corresponds to the path (1.1) with tangent vector $v = x \cdot \chi_{[0,1]}(x) \in L^1(\mathbb{R})$. Assuming $\theta > 0$, the corresponding solution of (1.3)-(1.4) is

$$u^\theta(t, x) = \frac{\theta x}{1 + \theta t} \cdot \chi_{[0, \sqrt{1+\theta t}]}(x). \tag{1.5}$$

Observe that, for $t > 0$, the map $\theta \longrightarrow u^\theta(t, \cdot)$ defined in (1.5) is Lipschitz continuous but nowhere differentiable because the location $x^\theta(t) = \sqrt{1 + \theta t}$ of the shock varies with θ. Therefore, the limit

$$\lim_{\Delta\theta \to 0} \frac{u^{\theta+\Delta\theta} - u^\theta}{\Delta\theta}, \qquad t > 0,$$

is not well defined as an element of the space $T_{\mathcal{S}_t \bar{u}} = L^1(\mathbb{R})$. In [7] it is studied a different space T_u of tangent vectors, $u \in \mathrm{BV}(\mathbb{R})$, which can be put into a one-to-one correspondence with $L^1(Du)$. Here Du denotes the (signed) Radon measure corresponding to the distributional derivative of u. The basic idea is the following. In the special case where v is continuously differentiable with compact support, to v it is associated the equivalence class of the map $\theta \mapsto u^\theta$, where u^θ is implicitly defined as

$$u^\theta(x + \theta v(x)) = u(x), \tag{1.6}$$

for all $\theta \geq 0$ sufficiently small. It is then shown that this correspondence can be uniquely extended to the whole space $L^1(Du)$. Observe that in (1.2) the graph of u^θ is obtained by lifting the graph of u vertically by θv. On the other hand, in (1.6), the graph of u is shifted horizontally by θv. This motivates the term "shift-differential" used in the sequel. A map which is differentiable w.r.t. the tangent vectors in T_u is said to be shift differentiable.

The main result in [7] shows that the flow generated by a single conservation law $u \longrightarrow \mathcal{S}_t u$ is shift differentiable "almost everywhere" w.r.t. t, i.e. outside a countable set $\{t_k\}_{k\in\mathbb{N}}$.

In [4] it is introduced a different approach to first order perturbations of a strictly convex scalar conservation law. The main result of [4] is that if the initial datum u_0 satisfies $(u_0)_x \leq C$, then the limit

$$w \doteq \lim_{\theta \to 0} \frac{\mathcal{S}_t(u_0 + \theta v) - \mathcal{S}_t u_0}{\theta} \tag{1.7}$$

exists in the weak* sense of measures, for all $v \in L^1 \cap L^{+\infty}$, and the map $v \longrightarrow w$ is a linear and bounded operator from $L^{+\infty}$ to the space of bounded Radon measures on $\mathbb{R}$. This means that the operator $u_0 \longrightarrow \mathcal{S}_t u_0$ is Gâteaux differentiable in u_0, in some weak sense. It can be shown that if $u + \theta v$ generates a shift tangent vector, then the measure w is equal to aDu, where $a \in L^1(Du)$ is the shift tangent vector generated by $\mathcal{S}_t(u_0 + \theta v)$. Of course the limit (1.1) is much stronger than (1.7).

In [9] the construction of shift differentials is extended to the case of vector valued function u. If $u : \mathbb{R} \longrightarrow \mathbb{R}^n$ is a BV function, a shift tangent vector is defined in terms of:

i) a decomposition of u into n scalar components;

ii) a n-tuple of functions $(v_1, \dots, v_n) \in L^1(Du)$, determining the rate at which each component of u is shifted.

This is accomplished by assigning a matrix valued function $A : \mathbb{R} \longrightarrow \mathbb{M}_d^{n\times n}$, where we denote with $\mathbb{M}_d^{n\times n}$ the set of diagonalizable $n \times n$ matrices with real eigenvalues. The eigenvectors of A correspond to the local scalar decomposition of u, while the eigenvalues of A determine the shift rate of the correspondent component of u. Denoting with $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$, the assumption on A are:

A1) if l^i, r_i denote the left and right eigenvectors of A, normalized such that $|r_i| = 1$ and $\langle l^j, r_i \rangle = \delta_i^j$, then they are Borel measurable and uniformly bounded;

A2) the eigenvalues λ_i belong to $L^1(Du)$.

Given $A \in \mathbb{M}_d^{n\times n}$ satisfying A1, A2, the main result of [9] is the construction of an equivalent class of paths $\gamma : \theta \longrightarrow u^\theta \in L^1_{loc}$ which determine the shift tangent vector A.

In this paper we consider a class of paths $\gamma : \theta \longrightarrow u^\theta \in L^1_{loc}$, and show how these paths generate a space T_u of tangent vectors at $u \in \mathrm{BV}$. Relying on the previous observations and the results of section 4, we will call the elements of T_u shift tangent vectors.

Before introducing the space T_u, in section 2 we study a simple case: we consider a constant matrix A and a piecewise constant function u with a single jump. In this case the path $\theta \to u^\theta \in L^1_{loc}(\mathbb{R}, \mathbb{R}^n)$ is defined considering the solution of the linear equation

$$\frac{\partial u^\theta}{\partial \theta} + A \frac{\partial u^\theta}{\partial x} = 0. \tag{1.8}$$

The study of this simple case leads to the introduction of a new distance $\widehat{d}$ on the space $\mathbb{M}_d^{n\times n}$ such that $(\mathbb{M}_d^{n\times n}, \widehat{d})$ is a complete metric space.

In section 3, relying on a simple decomposition of u in terms of its measure derivative, we define the matrix valued functions A which can generate shift tangent vectors at u. These vector valued functions are called admissible generators. Our definition is much less restrictive than the one in [9].

It is obvious that two matrix valued functions A and $\widetilde{A}$ define the same path if, roughly speaking, their difference acts on the vector space "orthogonal" to u: for example, if $\bar{l} \in \mathbb{R}^n$ is a left eigenvector for the constant matrix A and $Du(x)$ is orthogonal to $\bar{l}$ for all $x \in \mathbb{R}$, then the path generated by (1.8) is independent from the value of the eigenvalue $\bar{\lambda}$ corresponding to $\bar{l}$. It follows that, differently from the scalar case, a path $\theta \longrightarrow u^\theta \in L^1_{loc}$ does not determine uniquely the admissible generator, in general. However, we give two criteria to say whether two matrix valued functions A and $\widetilde{A}$ define the same shift tangent vector or not.

In section 4 we show that our definition coincides with the one given in [7] for the scalar case. In some sense our construction gives the most natural path, as the path $\gamma_v^\theta(u)$ considered in (1.2) is the easiest choice in that case. Moreover we show that the matrix valued functions considered in [9] are admissible generators and the definition of shift tangent vector given there coincides with ours.

Finally in section 5 we address the question of the application to hyperbolic systems of conservation laws. Extending [7], we introduce the shift differential of a map $\Phi : L^1_{loc} \longrightarrow L^1_{loc}$. Roughly speaking, Φ is shift differentiable at $u \in \mathrm{BV}$ if $\Phi(u) \in \mathrm{BV}$ and for all shift tangent vectors $\mathcal{A} \in T_u$, Φ maps a path generating $\mathcal{A}$ into a path generating a shift tangent vector $\mathcal{B} \in T_{\Phi(u)}$. In general the semigroup S_t generated by an hyperbolic system of conservation laws is not defined on the whole L^1_{loc}, but on an L^1–closed subset of BV. Since with our definition there are shift tangent vectors generated only by paths $\gamma(\theta)$ with

$$\lim_{\theta \to 0} \mathrm{Tot.Var.}(\gamma(\theta)) = +\infty,$$

we need to restrict the space T_u, i.e. we need to consider a subset $\mathrm{M}(u)$ of T_u. The shift differential of a map Φ is now defined using $\mathrm{M}(u)$ instead of the entire T_u. Finally we give three examples of the applications to the Lipschitz continuous semigroup $\mathcal{S}_t$ generated by a hyperbolic system of conservation laws.

The first example consider a simple 2×2 Temple class system, i.e. a system whose rarefaction curves are straight lines. In this case we show that the shift differentiability of the map $u \longrightarrow \mathcal{S}_t u$ occurs only if we restrict the space T_u to the shift tangent vectors that shift independently the two Riemann invariants. We show also that this subset $\mathrm{M}(u)$ of T_u is the biggest set $\mathrm{M}(u)$ such that the shift differentiability of the map $u \longrightarrow \mathcal{S}_t u$ occurs. The last two examples show that in general it is difficult to determine which subset $\mathrm{M}(u)$ of T_u should be considered to prove the shift differentiability of the map $u \longrightarrow \mathcal{S}_t u$, and this subset could be very small.

2. The Riemann problem for linear systems

We begin with the most elementary case: a Riemann problem for a linear hyperbolic system with constant coefficients. Without any loss of generality, we consider a function $u \in L^1_{loc}(\mathbb{R}; \mathbb{R}^n)$ with a single jump in 0, i.e. $u(x) \doteq v\chi_{[0,+\infty)}(x)$, $v \in \mathbb{R}^n$, and a constant matrix $A \in \mathbb{M}_d^{n\times n}$. We recall that with $\mathbb{M}_d^{n\times n} \subseteq \mathbb{M}^{n\times n}$ we denote the space of $n \times n$ diagonalizable matrices with real eigenvalues: if with $r_i, l^i \in \mathbb{R}^n$ the left and right eigenvectors corresponding to the eigenvalues

$\lambda_i \in \mathbb{R}$, then

$$r_1 \wedge \cdots \wedge r_n > 0, \qquad |r_i| = 1, \qquad \langle l^j, r_i \rangle = \delta_i^j, \qquad i,j = 1, \ldots, n, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$, δ_i^j is the kronecher's delta and $|\cdot|$ is a norm of $\mathbb{R}^n$. For sake of definiteness, we shall use $|(v_1, \ldots, v_n)| = |v_1| + \cdots + |v_n|$, but all the following results are valid for any equivalent norm in $\mathbb{R}^n$. To define in this simple case the shift tangent vector A at u, we consider the linear system

$$\begin{cases} u_t + Au_x &= \quad 0 \\ u(0,x) &= \quad v\chi_{[0,+\infty)}(x), \end{cases} \tag{2.2}$$

whose solution is

$$u(t,x) = \sum_{i=1}^{n} \langle l^i, u(0, x - \lambda_i t) \rangle r_i = \sum_{i=1}^{n} \chi_{[\lambda_i t, +\infty)}(x) \langle l^i, v \rangle r_i, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$. We now introduce a notation for the functions obtained by (2.3). This notation is the obvious generalization of definition 2 in [9].

Definition 2.1. *If $u \in L^1_{loc}$ and $A \in \mathbb{M}_d^{n \times n}$ are as above, we denote by $A^\theta \star u$ the solution of (2.3) evaluated at time θ, i.e.*

$$A^\theta \star u(x) \doteq u(\theta, x) = \sum_{i=1}^{n} \chi_{[\theta\lambda_i, +\infty)}(x) \langle l^i, v \rangle r_i. \tag{2.4}$$

Note that this definition coincides with definition 6 of [9], since the matrix A is constant; in particular each component $\langle l^i, u \rangle r_i$ of u is shifted of the amount $\theta\lambda_i$.

In the following sections we shall need to estimate $\|A^\theta \star u - \widetilde{A}^\theta \star u\|_{L^1}$, where $A, \widetilde{A} \in \mathbb{M}_d^{n \times n}$. If we denote by $\widetilde{u}(t,x)$ the solution of

$$\begin{cases} \widetilde{u}_t + \widetilde{A}\widetilde{u}_x &= \quad 0 \\ \widetilde{u}(0,x) &= \quad v\chi_{[0,+\infty)}(x), \end{cases} \tag{2.5}$$

an explicit computation gives

$$\begin{aligned} \|A^\theta \star u - \widetilde{A}^\theta \star u\|_{L^1} &= \int_{\mathbb{R}} |u(\theta, x) - \widetilde{u}(\theta, x)| dx \\ &= \int_{\mathbb{R}} |u(1, x/\theta) - \widetilde{u}(1, x/\theta)| dx = \theta \|u(1) - \widetilde{u}(1)\|_{L^1}, \end{aligned} \tag{2.6}$$

where we use the fact that u and $\widetilde{u}$ are self similar. The last formula, divided by θ, can be used to define a distance on the space of diagonalizable matrices:

Definition 2.2. *Given two matrices* $A, \widetilde{A} \in \mathbb{M}_d^{n\times n}$ *and a vector* $v \in \mathbb{R}^n$*, consider the two Riemann problems defined in (2.2) and (2.5), and denote by* $u(t,x)$ *and* $\widetilde{u}(t,x)$ *their respective solutions. We define the function* $d(A,\widetilde{A};v)$ *as*

$$d(A,\widetilde{A};v) \doteq \|u(1)-\widetilde{u}(1)\|_{L^1} = \int_{\mathbb{R}} |u(1,x)-\widetilde{u}(1,x)|dx. \tag{2.7}$$

Moreover we define the distance $\widehat{d}(A,\widetilde{A})$ *by*

$$\widehat{d}(A,\widetilde{A}) \doteq \sup_{v:|v|=1} d(A,\widetilde{A};v). \tag{2.8}$$

In the following theorem we prove that $\widehat{d}$ is actually a distance in $\mathbb{M}_d^{n\times n}$.

Theorem 2.3. *The function* $\widehat{d} : \mathbb{M}_d^{n\times n} \times \mathbb{M}_d^{n\times n} \longrightarrow \mathbb{R}$ *defined in (2.8) is a distance on the space of diagonalizable matrices* $\mathbb{M}_d^{n\times n}$*. The space* $(\mathbb{M}_d^{n\times n}, \widehat{d})$ *is a complete metric space.*

Proof. It is obvious form (2.7) and (2.8) that $\widehat{d}(A,\widetilde{A}) \geq 0$ and $\widehat{d}(A,\widetilde{A}) = \widehat{d}(\widetilde{A},A)$.
Suppose now that $\widehat{d}(A,\widetilde{A}) = 0$. By definition, the distance $\widehat{d}(A,\widetilde{A})$ can be defined as the L^1 distance between the two solution of the Riemann problem (2.2) and (2.5), respectively: thus to prove $A = \widetilde{A}$ is equivalent to say that the matrix A is uniquely determined by the solution of the Riemann problems (2.2), when v varies in $\mathbb{R}^n$. Assume that v is equal to a right eigenvector of A, namely $v = r_\ell$, with $1 \leq \ell \leq n$. Then the solution u of (2.2) is a single shock traveling with speed λ_ℓ,

$$u(t,x) = r_\ell \chi_{[\lambda_\ell t, +\infty)}(x),$$

and since by hypotheses $d(A,\widetilde{A};r_l) = 0$, u must be a solution also of the equation (2.5): in fact u and $\widetilde{u}$ are self similar, and thus if they coincide at $t = 1$, they are equal for all $t \geq 0$. Consequently (2.3) implies that

$$\widetilde{A} r_\ell = \lambda_\ell r_\ell. \tag{2.9}$$

Since (2.9) holds for all $\ell \in \{1,\dots,n\}$ and $\widetilde{A} \in \mathbb{M}_d^{n\times n}$, it follows $A = \widetilde{A}$.
To prove $\widehat{d}(A,\widetilde{A}) \leq \widehat{d}(A,\widehat{A}) + \widehat{d}(\widehat{A},\widetilde{A})$, for all $A, \widetilde{A}, \widehat{A} \in \mathbb{M}_d^{n\times n}$, note that for all $v \in \mathbb{R}^n$ we have

$$\begin{aligned} d(A,\widetilde{A};v) &= \|u(t)-\widetilde{u}(t)\|_{L^1} \leq \|u(t)-\widehat{u}(t)\|_{L^1} + \|\widehat{u}(t)-\widetilde{u}(t)\|_{L^1} \\ &= d(A,\widehat{A};v) + d(\widehat{A},\widetilde{A};v) \leq \widehat{d}(A,\widehat{A}) + \widehat{d}(\widehat{A},\widetilde{A}), \end{aligned} \tag{2.10}$$

where $\widehat{u}$ is the solution of (2.2) with the matrix $\widehat{A}$. Taking the supremum of the left–hand side of (2.10), we conclude

$$\widehat{d}(A,\widetilde{A}) \leq \widehat{d}(A,\widehat{A}) + \widehat{d}(\widehat{A},\widetilde{A}).$$

This concludes the proof that $\widehat{d}$ is a metric. We now prove that this distance makes $\mathbb{M}_d^{n\times n}$ a complete metric space.

For any $v \in \mathbb{R}^n$ and $A, \widetilde{A} \in \mathbb{M}_d^{n\times n}$ we can write

$$|Av - \widetilde{A}v| \leq \int_{\mathbb{R}} |u(1,x) - \widetilde{u}(1,x)|dx = d(A, \widetilde{A}; v). \tag{2.11}$$

In fact, from (2.2) and (2.5) it follows

$$\int_{\mathbb{R}} u(t,x) - \widetilde{u}(t,x)dx = -\int_0^t \int_{\mathbb{R}} Au(t,x)_x - \widetilde{A}\widetilde{u}(t,x)_x dxdt = t(Av - \widetilde{A}v). \tag{2.12}$$

Now let $A_k \in \mathbb{M}_d^{n\times n}$ be a Cauchy sequence in $(\mathbb{M}_d^{n\times n}, \widehat{d})$, and define $u_k(t)$ as the self–similar solution of the Riemann problem (2.2) with the matrix A_k and initial datum $v\chi_{[0,+\infty)}(x)$. Since $\|u_k(t) - u_\ell(t)\|_{L^1} \leq t\widehat{d}(A_k, A_\ell)$, we have that, for any fixed t, $u_k(t) - u_1(t)$ is a Cauchy sequence in $L^1(\mathbb{R})$, converging to a unique limit $u(t)$:

$$\lim_{k\to+\infty} \int_{\mathbb{R}} |u_k(t,x) - u(t,x)|dx = 0. \tag{2.13}$$

Note that (2.11) implies that A_k is a Cauchy sequence in $\mathbb{M}^{n\times n}$, and then there exists a matrix A such that

$$\lim_{k\to+\infty} A^k = A. \tag{2.14}$$

If we write equation (2.2) with the matrix A^k in weak form and we let $k \to +\infty$, (2.13) and (2.14) imply for all $\phi \in C^1(\mathbb{R}, \mathbb{R})$ that

$$\iint_{\mathbb{R}^+\times\mathbb{R}} u(t,x)\phi_t(t,x) + Au(t,x)\phi_x(t,x)dxdt + \int_{\mathbb{R}^+} v\phi(0,x)dx = 0, \tag{2.15}$$

so that u is a weak solution of the system

$$\begin{cases} u_t + Au_x & = & 0 \\ u(0,x) & = & v\chi_{[0,+\infty)}(x), \end{cases} \tag{2.16}$$

This concludes the proof, because it is well known that the system (2.16) has a solution for all $v \in \mathbb{R}^n$ if and only if A is diagonalizable (see [13]). $\square$

Remark 2.4. It is easy to see that the space $\mathbb{M}_d^{n\times n}$ is not a vector space: for example

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

and while the two matrices are in $\mathbb{M}_d^{n\times n}$, their sum is not. The same result can be proved for the product of two matrices. Note that $\mathbb{M}_d^{n\times n}$ is star shaped, and in fact $\widehat{d}$ satisfies also

$$\widehat{d}(\alpha A, \alpha\widetilde{A}) = \alpha\widehat{d}(A, \widetilde{A}), \tag{2.17}$$

for any $\alpha > 0$, $A, \widetilde{A} \in \mathbb{M}_d^{n\times n}$. While $\mathbb{M}_d^{n\times n}$ is a manifold, the distance $\widehat{d}$ does not define a Riemannian structure: namely it can be shown that there are no metric tensors generating $\widehat{d}$.

A corollary of theorem 2.3 is

Corollary 2.5. *The distance $\widehat{d}$ is stronger that the usual operator norm $\|\cdot\|$: for any two matrices $A, \widetilde{A} \in \mathbb{M}_d^{n\times n}$*

$$\|A - \widetilde{A}\| = \sup_{v\in\mathbb{R}^n} |Av - \widetilde{A}v| \leq \widehat{d}(A, \widetilde{A}). \tag{2.18}$$

Proof. Formula (2.18) follows immediately from (2.11) and the definition (2.8). □

The following remark will be important in the following section.

Remark 2.6. Consider two matrices $A, \widetilde{A} \in \mathbb{M}_d^{n\times n}$. It is clear that $\|A-\widetilde{A}\| = 0$ implies $\widehat{d}(A, \widetilde{A}) = 0$, but in general, fixed a vector $v \in \mathbb{R}^n$, we could have $Av = \widetilde{A}v$ and $d(A, \widetilde{A}; v) > 0$. Consider for example the vector $v = (1, -1)$ and the following matrices

$$A_1 \doteq \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}, \quad A_2 \doteq \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}.$$

With easy computation one can verify that $d(A_1, A_2; v) = 44/5$ while $A_1 v = A_2 v$.

3. Shift tangent vectors: the vector case

In this section we introduce a space of shift tangent vectors for a function $u \in$BV$(\mathbb{R}; \mathbb{R}^n)$. These are functions in $L^1_{loc}(\mathbb{R}; \mathbb{R}^n)$ whose distributional derivative is a bounded vector measure on $\mathbb{R}$. We define $f \doteq Du/|Du|$, the Lebesgue decomposition of Du w.r.t. its total variation measure $|Du|$. Without any loss of generality, in the following we assume u right continuous.

We now give the following definition:

Definition 3.1. *Consider a matrix valued function $A : \mathbb{R} \longrightarrow \mathbb{M}_d^{n\times n}$. Given $u \in BV(\mathbb{R}; \mathbb{R}^n)$, we say that A is a admissible generator of a shift tangent vector at u if*

$$\int_{\mathbb{R}} d(A(y), 0; f(y))|Du|(y) < +\infty, \tag{3.1}$$

where $d(A(y), 0; f(y))$ is defined in (2.7). We denote the class of admissible generators for u as Adm(u).

For all $u \in$ BV, it is easy to prove the existence of the following limit

$$u(-\infty) \doteq \lim_{x\to-\infty} u(x).$$

Since our definition of shift tangent vector at u depends only on the derivative Du of u, we assume $u(-\infty) = 0$. Given $u \in$ BV, we can obviously write

$$u(x) = u(-\infty) + \int_{\mathbb{R}} f(y)\chi_{(-\infty,x]}(y)|Du|(y) = u(-\infty) + \int_{\mathbb{R}} f(y)\chi_{[y,+\infty)}(x)|Du|(y). \tag{3.2}$$

If $A(y)$ is an admissible generator for u, then we consider for any y the solution $w(t, x; y)$ of the Riemann problem

$$\begin{cases} w_t + A(y)w_x & = & 0 \\ w(0, x; y) & = & f(y)\chi_{[y,+\infty)}(x), \end{cases} \tag{3.3}$$

and we define the path $\theta \to A^\theta \star u \in L^1_{loc}$ as

$$\begin{aligned} A^\theta \star u(x) \doteq \int_{\mathbb{R}} w(\theta, x; y)|Du|(y) &= \int_{\mathbb{R}} \sum_{i=1}^{n} \langle l^i(y), f(y)\rangle \chi_{[y,+\infty)}(x - \lambda_i(y)\theta) r_i(y)|Du|(y) \\ &= \int_{\mathbb{R}} \sum_{i=1}^{n} \langle l^i(y), f(y)\rangle \chi_{[y+\lambda_i(y)\theta,+\infty)}(x) r_i(y)|Du|(y). \end{aligned} \tag{3.4}$$

If A is constant, (3.4) coincides with (2.4). With these notations we can rewrite condition (3.1) as

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |w(1, x; y) - w(0, x; y)| dx dy < +\infty. \tag{3.1$'$}$$

We first show that the function $A^\theta \star u$ is well defined as an element of L^1_{loc} for all $\theta \geq 0$.

Lemma 3.2. *Given a function $u \in L^1_{loc} \cap BV(\mathbb{R}; \mathbb{R}^n)$, the function $A^\theta \star u$ defined in (3.4) is a path in $L^1_{loc}(\mathbb{R}; \mathbb{R}^n)$ such that*

$$\|A^\theta \star u - u\|_{L^1} \leq \theta \int_{\mathbb{R}} d(A, 0; f(y))|Du|(y). \tag{3.5}$$

Proof. Since we have

$$\iint_{\mathbb{R}^2} \Big| w(\theta, x; y) - w(0, x; y) \Big| dx |Du|(y) = \theta \int_{\mathbb{R}} d(A, 0; f(y))|Du|(y) \leq +\infty,$$

the function $w(\theta, x; y) - w(0, x; y)$ is in $L^1(|Du| \times dx)$, and then (3.4) is in L^1_{loc}, thus defined a.e.. Finally, if we change the order of integration, we can write

$$\begin{aligned} \|A^\theta \star u - u\|_{L^1} = \int_{\mathbb{R}} |A^\theta \star u(x) - u(x)| dx &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \big(w(\theta, x; y) - w(0, x; y)\big) |Du|(y) \right| dx \\ \leq \iint_{\mathbb{R}^2} \Big| w(\theta, x; y) - w(0, x; y) \Big| dx |Du|(y) &= \theta \int_{\mathbb{R}} d(A(y), 0; f(y))|Du|(y). \end{aligned} \tag{3.6}$$

The conclusion follows. $\square$

Remark 3.3. In general the function $A^\theta \star u$ needs not to be in $\mathrm{BV}(\mathbb{R}; \mathbb{R}^n)$ if $n \geq 2$. In fact, consider the following example

$$u = (u_1, u_2) = \left(\sum_{i=1}^{+\infty} \frac{1}{2^i} \chi_{[i,+\infty)}(x), 0 \right) \in \mathrm{BV}(\mathbb{R}; \mathbb{R}^2),$$

and the matrix valued function defined as

$$A(i) = \begin{bmatrix} -2^{-i} & 0 \\ -2 & 2^{-i} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2^i & 1 \end{bmatrix} \begin{bmatrix} -2^{-i} & 0 \\ 0 & 2^{-i} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2^i & 1 \end{bmatrix}.$$

The solution of each Riemann problem (3.3) is

$$w(\theta, x; i) = \begin{cases} (0,0) & x < i - 2^{-i}\theta \\ (1, 2^i) & i - 2^{-i}\theta \le x < i + 2^{-i}\theta \\ (1,0) & x \ge i + 2^{-i}\theta \end{cases} \tag{3.7}$$

If we set $|(v_1, v_2)| = |v_1| + |v_2|$, an easy computation shows that $d(A(i), 0; f(i)) = 2 - i + 2 < 3$, and then A is an admissible generator at u. The weak derivative of (3.7) has measure norm $1 + 2^{i+1}$, and thus, for all $0 < \theta \le 1$, $A^\theta \star u$ is not in $\mathrm{BV}(\mathbb{R}; \mathbb{R}^2)$. The scalar case is a particular situation, since for all $\phi \in C^1_c(\mathbb{R})$ we have

$$\begin{aligned} \int_{\mathbb{R}} A^\theta \star u(x) \phi'(x) dx &= \iint_{\mathbb{R}^2} w(\theta, x; y) \phi'(x) dx |Du|(y) \\ &= \iint_{\mathbb{R}^2} f(y) \chi_{[y + \lambda_i(y)\theta, +\infty)}(x) \phi'(x) = \int_{\mathbb{R}} f(y) \int_{y + \lambda_i(y)\theta}^{+\infty} \phi'(x) dx |Du|(y) \\ &= -\int_{\mathbb{R}} f(y) \phi(y + \lambda_i(y)\theta) |Du|(y) \le \|\phi\|_{C^0} \mathrm{Tot.Var.}(u). \end{aligned}$$

This implies that $A^\theta \star u \in \mathrm{BV}(\mathbb{R})$.

In general we can prove the following theorem:

Theorem 3.4. *Assume that the left eigenvectors $l^i(y)$ of $A(y) \in \mathrm{Adm}(u)$, satisfying (2.1), are uniformly bounded by a constant M. Then the function $A^\theta \star u$ is in BV for all $\theta \ge 0$.*

Proof. We recall that by (2.1) the right eigenvectors are normalized. If ϕ is a $C^1_c(\mathbb{R})$ function and we denote with ϕ' its derivative, we have

$$\begin{aligned} \int_{\mathbb{R}} A^\theta \star u(x) \phi'(x) dx &= \iint_{\mathbb{R}^2} w(\theta, x; y) \phi'(x) dx |Du|(y) \\ &= \int \sum_{i=1}^n \langle l^i(y), f(y) \rangle r_i(y) \left(\int_{y + \theta\lambda_i(y)}^{+\infty} \phi(x) dx \right) |Du|(y) \\ &= \int \sum_{i=1}^n \langle l^i(y), f(y) \rangle r_i(y) \phi(y + \theta\lambda_i) |Du|(y) \le 2nM \|\phi\|_{C^0} \mathrm{Tot.Var.}(u). \end{aligned}$$

This concludes the proof, since the above formula is a definition of the space BV. $\square$

Remark 3.5. Definition 3.1 is the largest class of admissible generators at $u \in \mathrm{BV}$. One can restrict the class of admissible generators, for example assuming the matrix valued function $A \in \mathrm{Adm}(u)$ to be uniformly diagonalizable in the sense of theorem 3.4, or using the distance $\widehat{d}$ in (3.1) instead of $d(A, 0; f(y))$. However all the following results are independent of the set of admissible generators, and then we will use definition (3.1).

Before giving the definition of shift tangent vector, we prove the following theorem.

Theorem 3.6. *If A and $\widetilde{A}$ are two admissible generators at u, then*

$$\begin{aligned}\liminf_{\theta\to 0}\frac{1}{\theta}\|A^\theta\star u-\widetilde{A}^\theta\star u\|_{L^1}&\geq\int_{\mathbb{R}}\Big|A(y)f(y)-\widetilde{A}(y)f(y)\Big|\,|Du|(y),\\ \limsup_{\theta\to 0}\frac{1}{\theta}\|A^\theta\star u-\widetilde{A}^\theta\star u\|_{L^1}&\leq\int_{\mathbb{R}}d(A(y),\widetilde{A}(y);f(y))|Du|(y).\end{aligned}\tag{3.8}$$

Remark 3.7. Note that by formula (2.11) we have

$$\int_{\mathbb{R}}\Big|A(y)f(y)-\widetilde{A}(y)f(y)\Big|\,|Du|(y)\leq\int_{\mathbb{R}}d(A(y),\widetilde{A}(y);f(y))|Du|(y).$$

Note also that

$$\int_{\mathbb{R}}d(A(y),\widetilde{A}(y);f(y))|Du|(y)\leq\int_{\mathbb{R}}d(A(y),0;f(y))|Du|(y)+\int_{\mathbb{R}}d(\widetilde{A}(y),0;f(y))|Du|(y)<+\infty,$$

if A, $\widetilde{A}$ are admissible.

To prove the theorem we need a preliminary lemma.

Lemma 3.8. *The function $(A^\theta\star u-u)/\theta$ converges to $A(y)f(y)|Du|(y)$ in the weak sense of measures.*

Proof. If ϕ is a $C^0_c(\mathbb{R})$ function, we have

$$\begin{aligned}\int_{\mathbb{R}}\frac{A^\theta\star u(x)-u(x)}{\theta}\,\phi(x)dx&=\iint_{\mathbb{R}^2}\left(\frac{w(\theta,x;y)-w(0,x;y)}{\theta}\right)\phi(x)dx|Du|(y)\\ &=\int\sum_{i=1}^n\langle l^i(y),f(y)\rangle r_i(y)\left(\frac{1}{\theta}\int_{y+\theta\lambda_i(y)}^y\phi(x)dx\right)|Du|(y).\end{aligned}\tag{3.9}$$

Since ϕ is uniformly continuous in $\mathbb{R}$, we have that

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_{y+\theta\lambda_i(y)}^y\phi(x)dx=\lambda_i(y)\phi(y),$$

for every $y\in\mathbb{R}$. Note that (2.7) yields

$$\left|\sum_{i=1}^n\langle l^i(y),f(y)\rangle r_i(y)\left(\frac{1}{\theta}\int_{y+\theta\lambda_i(y)}^y\phi(x)dx\right)\right|\leq\|\phi\|_{C^0}d(A(y),0;f(y)),$$

and then we can use Lebesgue's dominated convergence theorem in (3.9) when $\theta\to 0$:

$$\begin{aligned}\lim_{\theta\to 0}\int_{\mathbb{R}}\frac{A^\theta\star u(x)-u(x)}{\theta}\,\phi(x)dx&=-\int_{\mathbb{R}}\sum_{i=1}^n\langle l^i(y),f(y)\rangle r_i(y)\phi(y)|Du|(y)\\ &=-\int_{\mathbb{R}}A(y)f(y)\phi(y)|Du|(y).\quad\square\end{aligned}\tag{3.10}$$

Proof of Theorem 3.6. Since $A(y)f(y)|Du|(y)$ is the weak limit of the sequence $(A^\theta \star u - u)/\theta$, the linearity of the integrals gives

$$\lim_{\theta \to 0} \int_{\mathbb{R}} \frac{A^\theta \star u(x) - \widetilde{A}^\theta \star u(x)}{\theta} \phi(x)dx = -\int_{\mathbb{R}} \big(A(y)f(y) - \widetilde{A}(y)f(y)\big)\phi(y)|Du|(y),$$

for all $\phi \in C^1_c(\mathbb{R})$, and a standard argument yields

$$\int_{\mathbb{R}} \left| A(y)f(y) - \widetilde{A}(y)f(y) \right| |Du|(y) \leq \liminf_{\theta \to 0} \int_{\mathbb{R}} \left| \frac{u^\theta(x) - \widetilde{u}^\theta(x)}{\theta} \right| dx.$$

The other inequality follows from a argument similar to (3.6): since

$$\begin{aligned} \int_{\mathbb{R}} \left| A^\theta \star u(x) - \widetilde{A}^\theta \star u(x) \right| dx &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \big(w(\theta, x; y) - \widetilde{w}(\theta, x; y)\big)|Du|(y) \right| dx \\ &\leq \iint_{\mathbb{R}^2} \left| w(\theta, x; y) - \widetilde{w}(\theta, x; y) \right| dx |Du|(y) = \theta \int_{\mathbb{R}} d(A(y), \widetilde{A}(y); f(y))|Du|(y), \end{aligned}$$

it follows

$$\limsup_{\theta \to 0} \frac{1}{\theta} \|A^\theta \star u - \widetilde{A}^\theta \star u\|_{L^1} \leq \int_{\mathbb{R}} d(A(y), \widetilde{A}(y); f(y))|Du|(y). \quad \square$$

In the following example, we prove that the limits in (3.8) may in general be different.

Example 3.9. Consider the vector function $u = (u_1, u_2)$ defined as

$$u_1(x) = \begin{cases} 0 & x < 0 \\ \displaystyle\sum_{i=1}^{n} \frac{1}{2^i} \min\{c_i, 1\} & \displaystyle\sum_{i=1}^{n-1} \frac{c_i}{3^i} + \frac{1}{3^n} < x < \sum_{i=1}^{n-1} \frac{c_i}{3^i} + \frac{2}{3^n}, \quad c_i = 0, 1, 2 \\ 1 & x > 1 \end{cases}$$

$$u_2(x) = 0,$$

and the fixed matrix A defined as

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The function u is essentially Vitali's function multiplied by the unitary vector $(1, 0)$. We now study the integral

$$I(\theta) = \frac{1}{\theta} \int_{\mathbb{R}} |u^\theta(x) - u(x)| dx,$$

where u^θ is the solution the system (2.2) evaluated at time θ, namely

$$u^\theta(x) = \frac{1}{2} u_1(x - \theta) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} u_1(x + \theta) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Note that by the definition of u, we have $I(3^{-n}) = I(3^{-1})$. An explicit computation of $I(3^{-1})$ gives

$$I(3^{-1}) = 3\int_{-3^{-1}}^{1+3^{-1}} |u_1^{3^{-1}}(x) - u(x)|dx + 3\int_{-3^{-1}}^{1+3^{-1}} |u_2^{3^{-1}}(x)|dx = 3\left(\frac{1}{6} + \frac{1}{3}\right) = \frac{3}{2}.$$

With the same construction we can compute

$$I(2/3^n) = I(2/9) = \frac{9}{2}\left(\frac{11}{108} + \frac{2}{9}\right) = \frac{35}{24}.$$

Note that

$$\int_{\mathbb{R}} |A(y)f(y)||Du|(y) = \int_{\mathbb{R}} Du(y) = 1 \leq \frac{35}{24},$$
$$\int_{\mathbb{R}} d(A(y), 0; f(y))|Du|(y) = \int_{\mathbb{R}} 2Du(y) = 2 \geq \frac{3}{2},$$

as theorem 3.6 requires.

We now introduce the main definitions of this paper.

Definition 3.10. *Fix $u \in BV(\mathbb{R};\mathbb{R}^n)$ and consider the partition of* $\mathrm{Adm}(u)$ *defined by the following equivalence relation* $\overset{\wedge}{\sim}$ *: if $A, \widetilde{A} \in \mathrm{Adm}(u)$, then $A \overset{\wedge}{\sim} \widetilde{A}$ if*

$$\limsup_{\theta\to 0} \frac{1}{\theta}\int_{\mathbb{R}} \left| A^\theta \star u(x) - \widetilde{A}^\theta \star u(x)\right| dx = 0. \tag{3.11}$$

We define the shift tangent vectors $\mathcal{A}$ at u as the elements $\mathcal{A}$ of the set $\mathrm{Adm}(u)/\overset{\wedge}{\sim}$.

Remark 3.11. In general, given $u \in \mathrm{BV}$, it is difficult to give a precise characterization of the equivalence class $\mathcal{A}$. For example, if $u \in C_c^1(\mathbb{R};\mathbb{R}^n)$ and $A, \widetilde{A} \in \mathbb{M}_d^{n\times n}$ are two constant matrices, the application of Lebesgue dominated convergence theorem shows that

$$\lim_{\theta\to 0} \frac{1}{\theta}\int_{\mathbb{R}} |A^\theta \star u(x) - \widetilde{A}^\theta \star u| = \int_{\mathbb{R}} |Au_x(y) - \widetilde{A}u_x(y)|dy,$$

and thus in this case $A \overset{\wedge}{\sim} \widetilde{A}$ if $Au_x(y) = \widetilde{A}u_x(y)$ for all $y \in \mathbb{R}$. Remark 2.6 shows instead that if u has a jump, in general A and $\widetilde{A}$ belong to different equivalence classes. However, using (3.8), we can say that if

$$\int_{\mathbb{R}} d(A(y), \widetilde{A}(y); f(y))|Du|(y) = 0,$$

then $A \overset{\wedge}{\sim} \widetilde{A}$. Conversely if

$$\int_{\mathbb{R}} \left|A(y)f(y) - \widetilde{A}(y)f(y)\right| |Du|(y) > 0,$$

then $A^\theta \star u$ and $\widetilde{A}^\theta \star u$ generate two different shift tangent vectors.

Definition 3.12. *Consider a path $\theta \to u^\theta \in L^1(\mathbb{R};\mathbb{R}^n)$, defined in some interval $\theta \in [0,\theta^\star]$. We say that the path u^θ generates the shift tangent vector $\mathcal{A} \in \mathrm{Adm}(u)/\overset{\wedge}{\sim}$ if for some admissible generator $A \in \mathcal{A}$ we have*

$$\lim_{\theta\to 0} \frac{1}{\theta}\|u^\theta - A^\theta \star u\|_{L^1} = 0. \tag{3.12}$$

Remark 3.13. It is easy to prove using (3.11) that the above definition does not depend on the choice of the representative $A \in \mathcal{A}$. Moreover, if it exists, the shift tangent vector $\mathcal{A}$, not the admissible generator, is uniquely determined by the curve $\theta \longrightarrow u^\theta$.

4. Equivalence of the other definitions

In this section we show that definition 3.12 coincides with the definition of shift tangent vector given in [7] for the scalar case and in [9] in the vector case.

If u is a function in $\mathrm{BV}(\mathbb{R};\mathbb{R})$, then instead of a matrix valued function $A \in \mathrm{Adm}(u)$ we have a function $a \in L^1(Du)$: in fact in this case condition (3.1) reduces to

$$\int_{\mathbb{R}} |a(y)||Du|(y) < +\infty, \tag{3.1$'$}$$

since $d(a,0;v) = |a||v|$. Moreover it is easy to verify that the set $\mathrm{Adm}(u)/\overset{\wedge}{\sim}$ coincides with $L^1(Du)$, because in the scalar case the limits (3.8) coincide and then $a \overset{\wedge}{\sim} \tilde{a}$ if and only if $a = \tilde{a}$ almost everywhere w.r.t. the measure $|Du|$. We recall that in [7] the shift tangent vector $a \in L^1(Du)$ is defined by the equivalence class w.r.t. the L^1 norm of the path $a^\theta \diamond u$, where

$$(a^\theta \diamond u)(x + \theta a^\theta(x)) = u(x), \tag{4.1}$$

and a^θ is a Lipschitz continuous function such that

$$\lim_{\theta\to 0} \int_{\mathbb{R}} \left|a^\theta(y) - a(y)\right| |Du|(y) = 0, \quad \limsup_{\theta\to 0} \mathrm{Lip}(\theta a^\theta) < 1, \quad \lim_{\theta\to 0} \|\theta a^\theta\|_\infty = 0. \tag{4.2}$$

In [7] it is shown that this class is uniquely determined by the function $a \in L^1(Du)$, so that the definition is consistent.

The following proposition shows that our definition of shift tangent vectors coincides with the one above.

Proposition 4.1. *Given $a \in L^1(Du)$, let a^θ be a Lipschitz continuous function such that (4.2) hold, and let $a^\theta \diamond u$ be the L^1_{loc} function defined in (4.1). Then*

$$\lim_{\theta\to 0} \frac{1}{\theta}\|a^\theta \diamond u - a^\theta \star u\|_{L^1} = 0, \tag{4.3}$$

where $a^\theta \star u$ is defined the path in L^1_{loc} defined in (3.4).

Proof. It is simple to verify that the function $a^\theta \diamond u$ can be written as

$$a^\theta \diamond u = \int_{-\infty}^{+\infty} f(y)\chi_{[y,+\infty)}(x - \theta a^\theta(y))|Du|(y),$$

and, using (4.2), with easy computation we obtain

$$\begin{aligned}\frac{1}{\theta}\|a^\theta \diamond u - a^\theta \star u\|dx &\leq \frac{1}{\theta}\iint_{\mathbb{R}^2} \left|f(y)\chi_{[y,+\infty)}(x - \theta a^\theta(y)) - f(y)\chi_{[y,+\infty)}(x - \theta a(y)))\right| |Du|(y)dx \\ &= \int_{\mathbb{R}} |a^\theta(y) - a(y)||Du|(y) = \|a^\theta - a\|_{L^1(Du)},\end{aligned}$$

and then (4.3) is verified. $\square$

We now consider the vector case. Given $u \in \mathrm{BV}(\mathbb{R};\mathbb{R}^n)$, we recall that in [9] a shift tangent vector is determined by a matrix valued function $A : \mathbb{R} \longrightarrow \mathbb{M}^{n\times n}_d$ having the r_i, l^i as right and left eigenvectors, and the λ_i as eigenvalues such that the functions r_i, $l^i : \mathbb{R} \longrightarrow \mathbb{R}^n$ are Borel measurable and uniformly bounded and $\lambda_i \in \text{Ł}^1(|Du|)$. In this case the path $A^\theta \diamond u$ is defined by considering a matrix valued function $A^\theta : \mathbb{R} \longrightarrow \mathbb{M}^{n\times n}_d$ such that

i) for each given $\theta \in (0, \theta^*]$, its right and left eigenfunctions r^θ_i, $l^{i,\theta} : \mathbb{R} \longrightarrow \mathbb{R}^n$ and its eigenvalues $\lambda^\theta_i : \mathbb{R} \longrightarrow \mathbb{R}$ are Lipschitz continuous and bounded. Moreover, they remain constant outside a compact interval $K^\theta \subseteq \mathbb{R}$;

ii) for each $i = 1, \ldots, n$ one has

$$\lim_{\theta\to 0} \int_{\mathbb{R}} \left\{\left|r^\theta_i(x) - r_i(x)\right| + \left|l^{i,\theta(x)} - l^i(x)\right| \ + \left|\lambda^\theta_i(x) - \lambda_i(x)\right|\right\} d\mu_u(x) = 0, \tag{4.4}$$

$$\lim_{\theta\to 0} \theta^{1/4}\left(\mathrm{Lip}(r^\theta_i) + \mathrm{Lip}(l^{i,\theta}) + \mathrm{Lip}(\lambda^\theta_i) + \|\lambda^\theta_i\|_\infty\right) = 0, \tag{4.5}$$

$$\lim_{\theta\to 0} \max_j\{\|\lambda^\theta_j\|_\infty\} \int_{\mathbb{R}} \left(\left|r^\theta_i(x) - r_i(x)\right| + \left|l^{i,\theta}(x) - l^i(x)\right|\right)|Du|(x) = 0, \tag{4.6}$$

$$\|r^\theta_i\|_\infty \equiv 1, \qquad\qquad \sup_{i,\theta} \|l^{i,\theta}\|_\infty < \infty, \tag{4.7}$$

Next, for any integer $k \in \mathbb{Z}$, let $P^\theta_k \in \mathbb{R}$ be the points $P^\theta_k \doteq \frac{k}{2}\theta^{3/4}$, and let I^θ_k and J^θ_k be the open intervals centered at P^θ_k and with lengths $\frac{1}{2}\theta^{3/4}$ and $\theta^{3/4}$, respectively. Finally set

$$A^\theta \diamond u(x) \doteq \sum_{i=1}^n \left\langle l^{i,\theta}(P^\theta_k), u(x - \theta\lambda^\theta_i(P^\theta_k))\right\rangle r^\theta_i(P^\theta_k), \qquad \text{for } x \in I^\theta_k. \tag{4.8}$$

The last result of this section shows that, following definition 3.14, $A^\theta \diamond u$ generates the shift tangent vector $\mathcal{A}$ determined by $A \in \mathcal{A}$.

Proposition 4.2. *If A is defined as above, then A is admissible. Moreover, if we consider the path $A^\theta \star u$ defined in (3.4), we have*

$$\lim_{\theta\to 0} \frac{1}{\theta} \|A^\theta \diamond u - A^\theta \star u\|_{L^1} = 0. \tag{4.9}$$

It follows that $A^\theta \diamond u$ generates $\mathcal{A}$, where $\mathcal{A}$ is the shift tangent vector determined by $A \in \mathcal{A}$.

Proof. By the assumptions on A, there exists a constant M such that

$$\sup_i \|l^i\|_\infty \leq M. \tag{4.10}$$

The matrix valued function $A : \mathbb{R} \longrightarrow \mathbb{M}_d^{n\times n}$ is then uniformly strictly hyperbolic on $\mathbb{R}$, and for every $y \in \mathbb{R}$ the distance $\widehat{d}(A, 0)$ is bounded by

$$\begin{aligned} \widehat{d}(A(y), 0) &= \sup_{\substack{v\in\mathbb{R}^n,\\ |v|=1}} d(A(y), 0; v) = \sup_{\substack{v\in\mathbb{R}^n,\\ |v|=1}} \left| \sum_{i=1}^n \lambda_i(y) \langle l^i(y), v\rangle r_i(y) \right| \\ &\leq \sum_{i=1}^n |\lambda_i(y)| \sup_{\substack{v\in\mathbb{R}^n,\\ |v|=1}} \langle l^i(y), v\rangle \leq M \sum_{i=1}^n |\lambda_i(y)|. \end{aligned} \tag{4.11}$$

Since λ_i belongs to $L^1(|Du|)$, (4.13) implies that A is admissible. Now we prove (4.9).

For any $y \in \mathbb{R}$ we evaluate the L^1 distance between the solution of the Riemann problem (2.2) with matrix $A(y)$ and the one with matrix $A^\theta(y)$: if $v \in \mathbb{R}^n$ has norm $|v| = 1$, we have

$$\begin{aligned} d(A(y), A^\theta(y); v) = &\Bigg\| \sum_{i=1}^n \chi_{[\lambda_i(y)t, +\infty)}(x) \langle l^i(y), v\rangle r_i(y) \\ &\qquad - \sum_{i=1}^n \chi_{[\lambda_i^\theta(y)t, +\infty)}(x) \langle l^{i,\theta}(y), v\rangle r_i^\theta(y) \Bigg\|_{L^1} \\ \leq &\sum_{i=1}^n |\lambda_i(y) - \lambda_i^\theta(y)| |\langle l^i(y), v\rangle| \\ &\quad + \left\| \sum_{i=1}^n \chi_{[\lambda_i^\theta(y)t, +\infty)}(x) \Big(\langle l^i(y), v\rangle r_i(y) - \langle l^{i,\theta}(y), v\rangle r_i^\theta(y) \Big) \right\|_{L^1} \\ \leq &\, M \sum_{i=1}^n |\lambda_i(y) - \lambda_i^\theta(y)| + \max_j \{\|\lambda_j^\theta\|_\infty\} \sum_{i=1}^n \Big(M \big| r_i^\theta(y) - r_i(y) \big| + \big| l^{i,\theta}(y) - l^i(y) \big| \Big). \end{aligned} \tag{4.12}$$

We note in fact that the integrand function is different from 0 only for

$$x \in [-\max_j \{\|\lambda_j^\theta\|_\infty, \max_j \{\|\lambda_j^\theta\|_\infty].$$

It follows from (4.12) and (4.4)–(4.7) that

$$\lim_{\theta\to 0} \int_{\mathbb{R}} \widehat{d}(A(y), A^\theta(y)) |Du|(y) = 0. \tag{4.13}$$

If M is a bound also for $l^{i,\theta}$ in (4.7), a similar argument gives

$$\begin{aligned} d(A^\theta(P_k^\theta), A^\theta(y); v) \leq M \left|y - P_k^\theta\right| \sum_{i=1}^{n} \mathrm{Lip}(\lambda_i^\theta) \\ + \left|y - P_k^\theta\right| \max_j \{\|\lambda_j^\theta\|_\infty\} \sum_{i=1}^{n} \Big(M\mathrm{Lip}(r_i^\theta) + \mathrm{Lip}(l^{i,\theta})\Big), \end{aligned} \tag{4.14}$$

If y is in J_k^θ, (4.14) and (4.4)–(4.7) yield

$$\lim_{\theta \to 0} \sum_k \int_{J_k^\theta} \widehat{d}(A^\theta(P_k^\theta), A^\theta(y))|Du|(y) = 0. \tag{4.15}$$

The conclusion follows easily from (4.13) and (4.15), noting that

$$\begin{aligned} \|A^\theta \diamond u - A^\theta \star u\|_{L^1} &= \sum_k \int_{I_k^\theta} \left|A^\theta \diamond u(x) - A^\theta \star u(x)\right| dx \\ &\leq \int_{\mathbb{R}} \widehat{d}(A(y), A^\theta(y))|Du|(y) + \sum_k \int_{J_k^\theta} \widehat{d}(A^\theta(y), A(y))|Du|(y). \quad \square \end{aligned}$$

5. Application to systems of conservation laws

In this section we study the shift–differentiability of the flow generated by a hyperbolic system of conservation laws. Following [7], we consider an operator $\Phi : L^1_{loc} \longrightarrow L^1_{loc}$.

Definition 5.1. *We say that Φ is shift–differentiable at $u \in BV(\mathbb{R};\mathbb{R}^n)$ along the shift tangent vector $\mathcal{A} \in \mathrm{Adm}(u)/\overset{\wedge}{\sim}$ if $\Phi(u) \in BV(\mathbb{R};\mathbb{R}^n)$ and there exists $\mathcal{B} \in \mathrm{Adm}(\Phi(u))/\overset{\wedge}{\sim}$ such that*

$$\lim_{\theta \to 0} \frac{1}{\theta} \left\|B^\theta \star \Phi(u) - \Phi(A^\theta \star u)\right\|_{L^1} = 0, \tag{5.1}$$

for some $A \in \mathcal{A}$, $B \in \mathcal{B}$. Moreover if there exists a map $\Lambda : \mathrm{Adm}(u)/\overset{\wedge}{\sim} \longrightarrow \mathrm{Adm}(\Phi(u))/\overset{\wedge}{\sim}$ such that for all $\mathcal{A} \in \mathrm{Adm}(u)/\overset{\wedge}{\sim}$ the limit (5.2) holds with $\mathcal{B} = \Lambda\mathcal{A}$, then Φ is shift differentiable at u.

In other words Φ is shift differentiable at u if it is shift differentiable along each direction $\mathcal{A} \in \mathrm{Adm}(u)/\overset{\wedge}{\sim}$. Note that $\mathrm{Adm}(u)/\overset{\wedge}{\sim}$ is not a vector space, except in the scalar case.

Remark 5.2. This definition points out a major difficulty when one is dealing with the semigroup $\mathcal{S}$ generated by a system of conservation laws: in fact the domain $\mathcal{D}$ of definition of $\mathcal{S}$ is an L^1 closed subset of BV, while in general the definition of shift tangent vector uses a path not in BV. Moreover using the same example given in remark 3.3, one can show that there exists shift tangent vectors such that if $\theta \longrightarrow u^\theta$ is any generating path, then

$$\lim_{\theta \to 0} \mathrm{Tot.Var}(u^\theta) = +\infty.$$

The above remark motivates the following definition:

Definition 5.3. *Let* $\Phi : L^1_{loc} \supseteq \mathrm{Dom}(\Phi) \longrightarrow BV$ *such that (5.1) holds for* $u, v \in \mathrm{Dom}(\Phi)$*. Given a subset* $\mathrm{M}(u) \subseteq \mathrm{Adm}(u)/\overset{\wedge}{\sim}$*, we say that* Φ *is* $\mathrm{M}(u)$*–shift differentiable at* u *if there exists a map* $\Lambda : \mathrm{Adm}(u)/\overset{\wedge}{\sim} \supseteq \mathrm{M}(u) \longrightarrow \mathrm{Adm}(\Phi(u))/\overset{\wedge}{\sim}$ *such that for all* $\mathcal{A} \in \mathrm{M}(u)$ *the limit (5.1) holds with* $\mathcal{B} = \Lambda\mathcal{A}$*.*

In the rest of this section we consider the application of these definitions to the flow generated by a hyperbolic system of conservation laws

$$\begin{cases} v_t + f(v)_x & = & 0 \\ v(0,x) & = & u(x) \end{cases} \tag{5.2}$$

where $v \in \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ smooth. We recall that, under various assumption of f (see [1,2,3,5,11]), (5.2) generates a unique Lipschitz continuous semigroup $\mathcal{S}_t : [0, +\infty) \otimes \mathcal{D} \longrightarrow \mathcal{D}$ such that

– $\mathcal{D}$ contains all the functions u with sufficiently small total variation with values in some compact set K;
– there exists a constant $\mathcal{L}$ such that

$$\|S_t u - S_s w\|_{L_1} \leq \mathcal{L}(|t-s| + \|u - w\|_{L_1});$$

– each trajectory $S_t u$ is a weak entropic solution of the Cauchy problem (5.2) with initial datum u;
– if u is a piecewise constant function, then, for small t, $S_t u$ coincides with the function obtained piecing together the solutions of the corresponding Riemann problems.

We are interested in the shift differentiability of the map $u \longrightarrow S_t u$, for fixed t. The first example shows the application of the above definitions to the semigroup $\mathcal{S}_t$ generated by a simple 2×2 Temple class system. We recall that in the case of Temple class systems the domain $\mathcal{D}$ of the semigroup can be extended to vector valued functions with arbitrary large total variation (see [2,3]).

Example 5.4. Consider the following Temple class system:

$$\begin{cases} u_t + \left(\dfrac{u}{u+v}\right)_x & = & 0 \\ v_t + \left(\dfrac{v}{u+v}\right)_x & = & 0 \end{cases} \tag{5.3}$$

If we choose the two Riemann coordinates $w_1 = u + v$ and $w_2 = v/u$, the system becomes

$$\begin{cases} (w_1)_t & = & 0 \\ (w_2)_t + \dfrac{1}{w_1}(w_2)_x & = & 0 \end{cases} \tag{5.4}$$

We assume that the initial data $w_{1,0}, w_{2,0} \in \mathrm{BV}(\mathbb{R};\mathbb{R})$ assume values in some square $[a,b]\times[a,b] \subseteq \mathbb{R}^2$, with $a,b>0$. It is well known that the domain $[a,b]\times[a,b]$ is invariant for the semigroup $\mathcal{S}_t$ generated by (5.3) ([2,3]). If we consider the characteristics lines of the second equation, i.e. the integral lines of the ODE

$$\begin{cases} \dot{x} &= \dfrac{1}{w_{1,0}(x)} \\ x(0) &= y \end{cases} \qquad \Longrightarrow \qquad t = \int_{y(t,x)}^{x} w_{1,0}(z)dz, \tag{5.5}$$

then the solution of (5.4) can be obtained by the method of characteristics:

$$w_1(t,x) = w_{1,0}(x), \qquad w_2(t,x) = w_{2,0}(y(t,x)). \tag{5.6}$$

The semigroup $\mathcal{S}_t$ is then

$$\begin{array}{rcccc} \mathcal{S}_t & : & \mathrm{BV}(\mathbb{R};[a,b]\times[a,b])\times[0,+\infty) & \longrightarrow & \mathrm{BV}(\mathbb{R};[a,b]\times[a,b]) \\ & & (w_{1,0},w_{2,0}) & \longrightarrow & (w_1(t,x),w_2(t,x)) \doteq (w_{1,0}(x),w_{2,0}(y(t,x))) \end{array} \tag{5.7}$$

The class of shift tangent vectors $\mathrm{M}(w_{1,0},w_{2,0}) \subseteq \mathrm{Adm}(w_{1,0},w_{2,0})/\stackrel{\wedge}{\sim}$ that we consider are those generated by shifting independently the two components w_1 and w_2: more precisely, $\mathrm{M}(w_{1,0},w_{2,0})$ is the class of shift tangent vectors generated by the set of admissible generators

$$\left\{ A \in \mathbb{M}_d^{2\times 2} : A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \lambda_1 \in L^1(Dw_1), \lambda_2 \in L^1(Dw_2) \right\}. \tag{5.8}$$

It is clear that in this case $\mathrm{M}(w_{1,0},w_{2,0})$ is homeomorphic to the space $L^1(Dw_{1,0}) \times L^1(Dw_{2,0})$, and then it is a vector space. The following result is essentially theorem 3 of [7].

Theorem 5.5. *Suppose that* $u \longrightarrow \mathcal{S}_t u$ *is* Y*–shift differentiable at* u*, where*

$$\mathrm{Y} \doteq \left\{ A \in \mathbb{M}_d^{2\times 2} : A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \lambda_i \in C_c^1(\mathbb{R}), i = 1,2 \right\}. \tag{5.9}$$

Then $u \longrightarrow \mathcal{S}_t u$ *is* $\mathcal{M}$*–shift differentiable.*

Proof. The proof is an easy extension of the proof of theorem 3 in [7]. □

If θ is less than $\min\{1/\mathrm{Lip}(\lambda_1), 1/\mathrm{Lip}(\lambda_2)\}$, then $\lambda_i^\theta \star w_i$ coincides with the function implicitly defined by

$$(\lambda_i^\theta \star w_i)(x + \theta\lambda_i(x)) = w_i(x).$$

If we write (5.5) for the shifted case, we have

$$\begin{cases} \dot{x} &= \dfrac{1}{\lambda_1 \star w_{1,0}(x)} \\ x(0) &= y \end{cases} \qquad \Longrightarrow \qquad t = \int_{y}^{x^\theta(t,y)} \lambda_1^\theta \star w_{1,0}(z)dz, \tag{5.5'}$$

and then the solution is

$$\mathcal{S}_t(\lambda_1^\theta \star w_{1,0}, \lambda_2^\theta \star w_{2,0})(x^\theta(t,y)) = (\lambda_1^\theta \star w_{1,0}(x^\theta(t,y)), \lambda_2^\theta \star w_{2,0}(y)). \tag{5.10}$$

We can write

$$\begin{aligned}
\mathcal{S}_t w_2(x) &= w_{2,0}(y(t,x)) = \lambda_2 \star w_{2,0}(y(t,x) + \theta\lambda_2(y(t,x))) \\
&= \mathcal{S}_t(\lambda_2^\theta \star w_{2,0})\Big(x^\theta\big(t, y(t,x) + \theta\lambda_2(y(t,x))\big)\Big) \\
&= \mathcal{S}_t(\lambda_2^\theta \star w_{2,0})\left(x + \theta\frac{x^\theta\big(t, y(t,x) + \theta\lambda_2(y(t,x))\big) - x}{\theta}\right).
\end{aligned} \tag{5.11}$$

Now we show that

$$\frac{x^\theta(t, y(x) + \theta\lambda_2(y(t,x))) - x}{\theta} \quad\to\quad \frac{1}{w_{1,0}(x)}\int_{y(t,x)}^x \lambda_1(z)Dw_{1,0}(z) + \frac{w_{1,0}(y(t,x))}{w_{1,0}(x)}\lambda_2(y(t,x)), \tag{5.12}$$

if $Dw_{1,0}$ is continuous in x, $y(t,x)$. Indeed (5.5') is differentiable at $\theta = 0$ if and only if $w_{1,0}$ is continuous in x, $y(t,x)$, and its derivative coincides with (5.12). It is clear that the convergence in (5.12) is in $L^1(Dw_2(t))$ if and only if the atomic part of Dw_1 and Dw_2 are disjoint at $t = 0$ and t. In fact the left hand side of (5.12) is uniformly bounded for all $\theta > 0$, and (5.12) holds outside the countable set of jumps of $w_{1,0}$. Using Lebesgue's dominated convergence theorem the conclusion follows.

The map $\Lambda : \mathrm{M}(w_{1,0}, w_{2,0}) \longrightarrow \mathrm{Adm}(\mathcal{S}_t u)$ is then

$$\begin{cases} v_1(x) \\ v_2(x) \end{cases} \longrightarrow \begin{cases} v_1(x) \\ \dfrac{1}{w_1(x)}\displaystyle\int_{y(t,x)}^x \lambda_1(z)Dw_1(z) + \dfrac{w_1(y(t,x))}{w_1(x)}\lambda_2(y(t,x)) \end{cases} \tag{5.13}$$

Using theorem 5.5 we can say that, given $w = (w_1, w_2)$ such that the atomic parts of Dw_1 and Dw_2 are disjoint, the map $w \longrightarrow \mathcal{S}_t w$ defined in (5.8) is $\mathrm{M}(w_1, w_2)$–shift differentiable for all t such that the atomic parts of $D(\mathcal{S}_t w)_1$ and $D(\mathcal{S}_t w)_2$ are disjoint. In this simple case it is easy to prove that Λ is a bounded linear operator from $\mathrm{M}(w_{1,0}, w_{2,0})$ to $\mathrm{M}(w_1(t), w_2(t))$.

Note that one cannot expect the map $w \to \mathcal{S}_t w$ to be $\mathrm{M}(w)$–shift differentiable for a bigger set $\mathrm{M}(w)$. Consider in fact figure 5.1a. In this case the initial condition u is one single shock along the first Riemann invariant. The solution $\mathcal{S}_t u$ is then a single travelling shock. In this case $\mathrm{Adm}(u)$ is all $\mathbb{M}_d^{n\times n}$. Consider a matrix $A \in \mathbb{M}_d^{n\times n}$, and assume that the function $A^\theta \star u$ has two jumps for $\theta > 0$: this implies that the A is not in the class $\mathrm{M}(w)$ considered in (5.8). The solution $\mathcal{S}_t(A^\theta \star u)$ is in figure 5.1b: in fact it is easy to prove that the new jumps are splitted in waves of the two families. It is then clear that for all $t > 0$ the map $u \to \mathcal{S}_t u$ is not shift differentiable along A: in fact no matrix $A \in \mathbb{M}_d^{n\times n}$ can generate the wave patterns of figure 5.1b if $t > 0$.

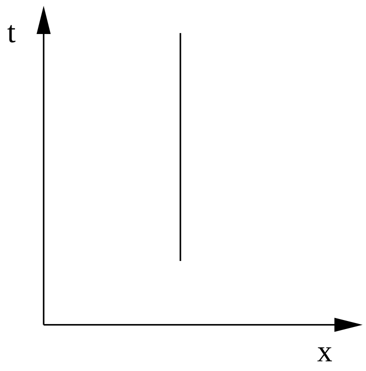

Figure 5.1a

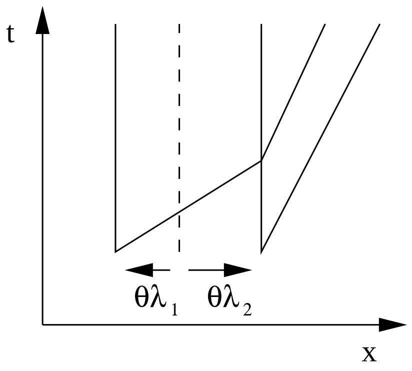

Figure 5.2b

The following examples show that in general the map $u \longrightarrow \mathcal{S}_t u$ is not M(u)–shift differentiable even if M(u) contains very simple tangent vectors. The conclusion is that, given a hyperbolic system (5.2), it is very difficult to determine the class M(u) in which the shift differentiability of the map $u \longrightarrow \mathcal{S}_t u$ occurs. Moreover the set M(u) could be extremely small: in the following example, M(u) is only bidimensional.

Example 5.6. Consider the wave–fronts configuration of figure 5.2a: three shocks of different families interact at the same point, but only two survive to the interaction. It is clear that, by standard Glimm's interaction estimates, one can construct this configuration if the vanishing shock has size of the order of the product of the sizes of the other two. For simplicity we assume that the two surviving shocks do not change in position or size: this can be achieved considering a systems in which two equations are independent of the others, for example the 3×3 system

$$\begin{cases} v_t - u_x &= 0 \\ u_t + \left(\dfrac{1}{v}\right)_x &= 0 \\ z_t + (-3+z)z_x + v_x &= 0 \end{cases}$$

Consider the configuration of figure 2.b, in which the vanishing shock has been shifted of an amount θ: the admissible generator A is then

$$A(y) = \begin{cases} 0 & x \leq \overline{x} \\ \\ I & x > \overline{x} \end{cases}$$

where I is the identity matrix, and $\bar{x}$ is a point between the vanishing shock and the other two. The shadowed region represents the centered rarefaction wave generated by the interaction of the two surviving shocks.

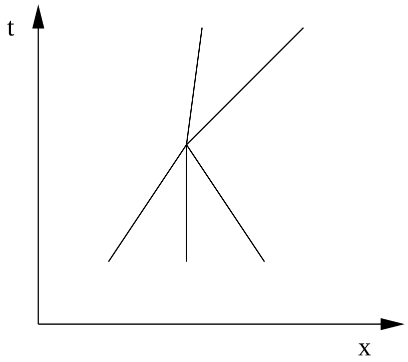

Figure 2.a

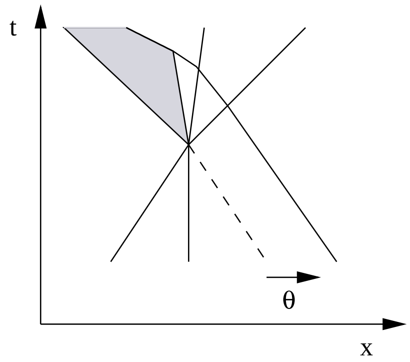

Figure 2.b

Assume without any loss of generality that the interaction occurs at $t = 1$. Since we are dealing with systems in conservation form, we have for all $t \geq 0$ that

$$\int_{\mathbb{R}} \mathcal{S}_t(A^\theta \star u)(x) - \mathcal{S}_t u(x) dx = \int_{\mathbb{R}} A^\theta \star u(x) - u(x) dx. \tag{5.14}$$

From the picture it is clear that the path $\theta \longrightarrow \mathcal{S}_t(A^\theta \star u)$ does not generate any shift tangent vector, if $t > 1$: in fact, since the two surviving shocks do not move, the shift tangent vector must be 0, but using the conservation property (5.14) one has

$$\frac{1}{\theta}\int_{\mathbb{R}} \left|\mathcal{S}_t(A^\theta \star u)(x) - \mathcal{S}_t u(x)\right| dx \geq \frac{1}{\theta}\left|\int_{\mathbb{R}} \mathcal{S}_t(A^\theta \star u)(x) - \mathcal{S}_t u(x) dx\right| = |\sigma|, \tag{5.15}$$

where σ is the size of the shifted shock. With the same analysis, it is clear that the $u \to \mathcal{S}_t u$ is shift differentiable along $\mathcal{A} \in \mathrm{Adm}(u)$ if and only if the three shocks are shifted in such a way that they will meet at the same point. If we denote with s_i the speed of the i–th shock, the shift rates of the three shocks are

$$\lambda_i = \xi_1 - s_i \xi_2, \qquad (\xi_1, \xi_2) \in \mathbb{R}^2,\ i = 1, 2, 3.$$

The vector (ξ_1, ξ_2) gives the direction in which the interaction point is shifted.

In example 5.6 the instability w.r.t. the shift differentiability can be related to the structural instability of the point of interaction of the three shocks (see [8]). We recall that the solution $S_t u$ of (5.4) is said to be structurally stable at the point $(\tau, \xi) \in \mathbb{R}^+ \times \mathbb{R}$ if, on the half plane $t < 0$, the function

$$\widetilde{u}(t, x) \doteq \lim_{\eta \to 0^+} u(\tau + \eta t, \xi + \eta x) \tag{5.16}$$

satisfies one of the following conditions:

– $\widetilde{u}$ is a constant function;

– $\widetilde{u}$ contains one incoming shock and no other wave;

– $\widetilde{u}$ contains two incoming shocks and no other wave.

In [8] it is shown that limit (5.16) exists and it is a self similar weak solution of

$$u_t + f(u)_x = 0.$$

In the case considered in example 5.6, the self similar solution contains 3 incoming shocks, and thus it is structurally unstable. In the next example we show that the same results can be proved for solution structurally stable in each point of the half plane $\mathbb{R}^+ \times \mathbb{R}$.

Example 5.7. Consider the (triangular) system

$$\begin{cases} u_t + (-3+u)u_x & = & -v_x \\ v_t + vv_x & = & 0 \end{cases}$$

In this case the second equation is decoupled. This implies that the 2–waves move with a speed independent on the value of u: then it is possible to construct the wave–pattern of figure 5.3a, for initial data chosen carefully and small enough: a Lipschitz continuous solution of the first equation, a shock and a centered rarefaction wave of the second equation interact in such a way that only the shock will survive.

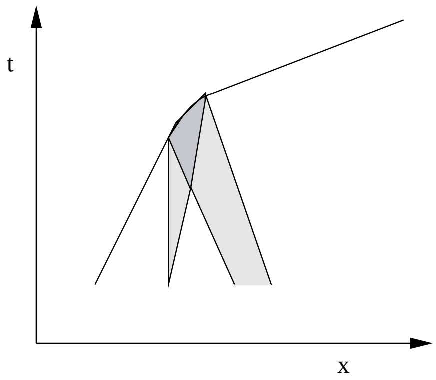

Figure 5.3a

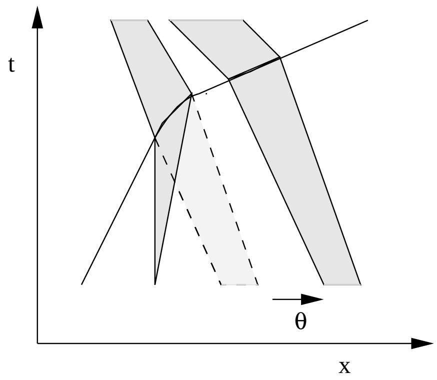

Figure 5.3b

In figure 5.3b it is represented the solution corresponding to $A^\theta \star u$, where A is the same as in example 5.6. Using the same analysis of example 5.6, one can show that the path $\theta \longrightarrow \mathcal{S}_t(A^\theta \star u)$ does not generate any shift tangent vector: in fact also in this case the surviving shock does not move, and then the shift tangent vector should be zero. A computation similar to (5.15) gives

$$\frac{1}{\theta}\int_{\mathbb{R}} \left|\mathcal{S}_t(A^\theta \star u)(x) - \mathcal{S}_t u(x)\right| dx \geq \int_{\mathbb{R}} |Du|(x).$$

Note that in this case the solution is structurally stable in the whole plane $\mathbb{R}^+ \times \mathbb{R}$.

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