A NOTE ON SINGULAR LIMITS TO HYPERBOLIC SYSTEMS

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Abstract. In this note we consider two particular approximations to hyperbolic system of conservation laws, namely the standard backward schemes for non linear semigroups and the semidiscrete scheme.

Under the assumption that the rarefaction curves of the corresponding hyperbolic system are straight lines, we prove the stability of the approximate solution and its convergence to the unique solution of the hyperbolic system for initial data with small total variation.

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1. Introduction

Consider a hyperbolic system of conservation laws in one space variable

$$\left\{ \begin{array}{rcl} u_t + f(u)_x & = & 0 \\ u(0,x) & = & u_0(x) \end{array} \right. \tag{1.1}$$

where $u \in \mathbb{R}^n$ and f is a smooth function from an open set $\Omega \subseteq \mathbb{R}^n$ with values in $\mathbb{R}^n$. Let K_0 be a compact set contained in Ω, and let δ_1 sufficiently small such that the compact set

$$K_1 \doteq \Big\{ u \in \mathbb{R}^n : \mathrm{dist}\big(u, K_0\big) \leq \delta_1 \Big\} \tag{1.2}$$

is entirely contained in Ω.

We assume that the Jacobian matrix $A = Df$ is uniformly strictly hyperbolic in K_1, i.e.

$$\min_{i<j} \Big\{ \lambda_j(u) - \lambda_i(v) \Big\} \geq c > 0, \qquad \forall u, v \in K_1, \tag{1.3}$$

where we denote by $\lambda_i(u)$ the eigenvalues of $A(u)$, $\lambda_i < \lambda_j$. Let $r_i(u)$, $l^i(u)$ be the its right, left eigenvectors, normalized such that

$$\big|r_i(u)\big| = 1, \qquad \big\langle l^j(u), r_i(u) \big\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In this setting it is well known that if $u_0(-\infty) \in K_0$ and Tot.Var.(u_0) is sufficiently small, there exists a unique "entropic" solution $u : [0, +\infty) \mapsto \mathrm{BV}(\mathbb{R}, \mathbb{R}^n)$ in the sense of [4]. Moreover these solutions can be constructed as limits of wave front tracking approximations and they depend Lipschitz continuously on the initial data in the L^1 topology.

For a special class of systems, called in [6] *Straight Line Systems*, i.e. systems such that the integral curves of the right eigenvectors $r_i(u)$ are straight lines, or equivalently

$$(Dr_i) r_i = 0, \tag{1.4}$$

very recently it has been proved that solutions to (1.1) can be constructed as L^1 limits of solutions to different singular approximations of the hyperbolic system:

- Vanishing viscosity approximation [3]. This is the limit as $\epsilon \to 0$ of the solution $u^\epsilon(t)$ of the system

$$u_t + f(u)_x - \epsilon u_{xx} = 0.$$

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- Relaxation approximation [2, 7]. While in case 1) the perturbation is parabolic, in this case we consider a hyperbolic perturbation, namely
$$u_t + f(u)_x = \epsilon\big(\Lambda^2 u_{xx} - u_{tt}\big),$$
where, for linear stability, Λ is strictly bigger than all the eigenvalues of $Df(u)$.
- Godunov scheme [6]. This is a discrete scheme obtained from (1.1) by considering differential ratio instead of derivatives:
$$u(n+1, j+1) = u(n, j+1) + \frac{\Delta t}{\Delta x}\Big[f\big(u(n,j)\big) - f\big(u(n, j+1)\big)\Big],$$
where, for stability of the scheme, it is assumed that $0 < \lambda_1 < \cdots < \lambda_n < \Delta x/\Delta t$.

The main task in showing the convergence of these approximations is to obtain uniform BV estimates for $t \geq 0$ (or for any $n \in \mathbb{N}$ in the last case), if the initial data u_0 are of sufficiently small total variation.

This task is achieved by decomposing the equations satisfied by u_x, or u_x and u_t in [2], or $u(n,j) - u(n, j-1)$ in [6], as n scalar equations, coupled by terms of higher order. These terms are then considered as the source of total variation. For the special case of straight line systems and for the vanishing viscosity approximation, a decomposition of u_x which makes the source terms integrable is the projection along the eigenvectors r_i of the Jacobian $Df(u)$:

$$u_x = \sum_i v^i r_i. \tag{1.5}$$

Once it is proved that the L^1 norm of the component v^i is bounded, by Helly's theorem there exists a subsequence u^{ϵ_k} converging to a weak solution $\bar{u}(t)$ of (1.1) as $k \to \infty$.

This result can be explained looking at the source of total variation for solutions to (1.1). In fact, due to the assumption (1.4), the shock and the rarefactions curves coincide: this implies that the total variation of the solution to (1.1) increases only when waves of different families interact. In the approximations considered here, the same condition (1.4) implies that the travelling profiles lie on the rarefaction curves, so that a decomposition of the form (1.5) generates coupling terms of the form $v^i v^j$, $i \neq j$: in fact, the source terms can be different from 0 only when two waves of different families are present at the same point (t, x). Finally, since the speeds of the components are different due to (1.3), one can show that the L^1 norm in the half plane $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ of these coupling terms is bounded and of squared order w.r.t. the L^1 norm of the components v^i. A simple argument implies the boundness of $\|v^i\|_{L^1}$.

A general method to show that the terms $v^i v^j$, $i \neq j$, which are called in [3] *transversal terms*, are integrable in the plane, is to consider the case of two linear equations satisfied by z^1, z^2, with drift speeds separated by c (1.3). In this case, by means of Fourier transform, the computation of the L^1 norm in the half plane of the product $z^1 \cdot z^2$ becomes a path integral in the complex plane, which can be evaluated by means of the calculus of residues. This approach introduces a Glimm type functional $Q(t)$, of second order w.r.t. the L^1 norm of the components z^1, z^2, which bounds the integral of the transversal term in the half plane $\tau \geq t$. Finally, for general speeds $\lambda_1(t, x)$, $\lambda_2(t, x)$, one can show that
$$\frac{d}{dt} Q(t) \leq -\int_{\mathbb{R}} \big|z^1(t,x)\big|\big|z^2(t,x)\big| dx,$$
so that
$$Q(0) \geq \iint_{t \geq 0} \big|z^1(t,x)\big|\big|z^2(t,x)\big| dx dt.$$

To prove the uniqueness of the limit $\bar{u}(t)$, one consider the equation for a perturbation h of the singular approximations. We observe that $h = u_x$ is a particular solution of such system. A generalization of the arguments used to prove an a priori bound on the total variation of u shows the boundness of the L^1 norm of the components h^i, where

$$h = \sum_i h^i r_i(u). \tag{1.6}$$

By a standard homotopy argument [3], this yields the stability of all solutions of the approximating system. Since the Lipschitz continuous dependence on the initial data is uniform w.r.t. both ϵ and t, we obtain as $\epsilon \to 0$ a uniform Lipschitz semigroup $\mathcal{S}$.

Finally, it is well known that a uniform Lipschitz semigroup of solutions to (1.1) is uniquely defined if we know the jumps conditions of the entropic shocks, see [5]. In our case, one can analyze the Green kernel of the linearized equation (1.6) when ϵ tends to 0 to prove that in the hyperbolic limit there exists a constant $\hat{\lambda} > 0$ such that

$$\int_a^b \big|\mathcal{S}_t u(x) - \mathcal{S}_t v(x)\big| dx \leq \int_{a-\hat{\lambda}}^{b+\hat{\lambda}} \big|u(x) - v(x)\big| dx.$$

The above equation implies a local dependence on the initial data for the limiting semigroup. This result and the fact that in the scalar case the solution u^ϵ converges to the entropic solution prove that the jump conditions coincide with the scalar jumps along the eigenvectors r_i, see [3].

Thus, under the assumption (1.4), the limit semigroup is independent on the approximation and coincides with the solution constructed by wave front tracking using the classical Lax Riemann solver.

In this note we want to extend the previous approach to the following cases:

1. Semigroup approximation [8, 9]. This is obtained as limit of the system
$$\frac{u(t,x) - u(t-\epsilon,x)}{\epsilon} + A\big(u(t,x)\big) u(t,x)_x = 0.$$
This is the standard backward scheme for non-linear semigroups.
2. Semi-discrete schemes [1], i.e. infinite dimensional ODE defined by
$$\frac{\partial}{\partial t} u(t,x) + \frac{1}{\epsilon}\Big(f\big(u(t,x)\big) - f\big(u(t,x-\epsilon)\big)\Big) = 0.$$

We shall prove that as $\epsilon \to 0$ the limits of the respective solutions converge to a unique solution to (1.1), and that this limits defines a Lipschitz continuous semigroup $\mathcal{S}$ on the space of function with small TV. Moreover, using the same arguments of [3], this semigroup is perfectly defined by a Riemann solver which, as explained above, coincides with the classical one.

The same can be proved for quasilinear systems as in [6], but for simplicity we consider only systems on conservation forms.

Without any loss of generality we assume that

$$\min_{u \in K_1} \big\{\lambda_1(u)\big\} = \alpha > 0, \tag{1.7}$$

for all u in the compact set K_1. Moreover we define the constant

$$\max_{u \in K_1} \big\{\lambda_n(u)\big\} = \beta > 0. \tag{1.8}$$

In Section 2 we consider the backward scheme for nonlinear semigroups, and in Section 3 we analyze the semidiscrete numerical scheme. The main focus is to show that these equations define a time-discrete Lipschitz semigroup in case 1) and a Lipschitz continuous semigroup on the piecewise constant functions in case 2), and the trajectories of these semigroups have uniformly bounded total variation. In fact, once these results are achieved, it is easy to prove the convergence of these solutions, by adapting the above explanations (see also [2, 3, 6]).

2. Semigroup approximation

We consider the following singular approximation to system of conservation laws:

$$\frac{u(t,x) - u(t-\epsilon,x)}{\epsilon} + A\big(u(t,x)\big) u(t,x)_x = 0, \tag{2.1}$$

where we recall that $u \in \mathbb{R}^n$ and $A(u) = Df(u)$. By the rescaling $t \to t/\epsilon$, $x \to x/\epsilon$, and setting for simplicity $u_n(x) = u(n,x)$, we obtain the evolutionary equations

$$u_n - u_{n-1} + A\big(u_n\big) u_{n,x} = 0. \tag{2.2}$$

It is easy to prove that if the BV norm of u_{n-1} is sufficiently small, then u_n exists: in fact, setting $w = u_n - u_{n-1}$, the equations satisfied by w are

$$w_x + A^{-1}\big(u_n\big) w = -u_{n-1,x},$$

Using the hyperbolicity of Df and (1.7), we can write

$$\big|w(x)\big| \le \frac{C}{c}\text{Tot.Var.}\big(u_{n-1,x}\big),$$

where the constant C depends on the norm of the left eigenvectors l^i. In particular, if we can show that the total variation of u_n is uniformly bounded, then the solution can be extended for all $n \ge 0$.

2.1. **Projection on rarefaction curves.** We now start the procedure explained in Section 1. By projecting the derivative along the eigenvectors $r_i(u_n) = r_{i,n}$ of $A(u_n)$

$$u_{n,x} = \sum_i v_n^i r_i(u_n) = \sum_i v_n^i r_{i,n}, \tag{2.3}$$

we obtain the equations satisfied by the components v_n^i:

$$\sum_i v_n^i r_{i,n} - \sum_i v_{n-1}^i r_{i,n-1} + \sum_i \big(\lambda_{i,n} v_n^i r_{i,n}\big)_x = 0.$$

This can be rewritten as

$$\sum_i \Big(v_n^i - v_{n-1}^i + \big(\lambda_{i,n} v_n^i\big)_x\Big) r_{i,n} = \sum_i v_{n-1}^i \big(r_{i,n-1} - r_{i,n}\big) - \sum_{i,j} \lambda_{i,n} v_n^i v_n^j \big(D r_{i,n}\big) r_{j,n} \tag{2.4}$$

The left-hand side is in conservation form, and we consider the right-hand side as the source of total variation. If we assume, as in the introduction, that $(Dr_i)r_i(u) = 0$, the function $r_i(u) - r_i(v)$ is zero when $u - v$ is parallel to $r_i(u) = r_i(v)$. Thus we have

$$r_i(u) - r_i(v) = \sum_{j \ne i} \alpha_j(u,v) \big\langle l^j(u), u - v\big\rangle, \tag{2.5}$$

where α_j is a smooth function and moreover $\alpha_j(u,u) = r_j(u)$. Using (2.2), the expansion (2.4) thus becomes

$$\begin{aligned} v_n^i - v_{n-1}^i + \big(\lambda_{i,n} v_n^i\big)_x &= \sum_{j \ne k} \Big(\lambda_{k,n} v_{n-1}^j v_n^k \big\langle l_n^i, \alpha_j(u_n, u_{n-1})\big\rangle - v_n^i v_n^j \big\langle l_n^i, \big(D r_{i,n}\big) r_{j,n}\big\rangle\Big) \\ &= \sum_{j \ne k} H_{jk}^i(n) v_{n-1}^j v_n^k + \sum_{j \ne k} K_{jk}^i(n) v_n^j v_n^k, \end{aligned} \tag{2.6}$$

with H, K uniformly bounded smooth functions.

To estimate the source terms in (2.6), we first consider the case of two linear equations.

2.2. **Analysis of the linear case.** Consider a single linear equation

$$z_n - z_{n-1} + \lambda z_{n,x} = 0, \qquad \lambda > 0. \tag{2.7}$$

We can find the fundamental solution to the previous equation by means of Fourier transform: we set

$$z_n(x) = \int_{\mathbb{R}} c(n,\xi) e^{-i\xi x} d\xi,$$

and by substituting into (2.7) we obtain the equations for the Fourier coefficients $c(n,\xi)$:

$$c(n,\xi) - c(n-1,\xi) - i\lambda\xi c(n,\xi) = 0 \quad \Longrightarrow \quad c(n,\xi) = \frac{c_0(\xi)}{\big(1 - i\lambda\xi\big)^n}.$$

In particular the fundamental solution has Fourier coefficients independents of ξ, $c_0(\xi) \equiv 1/2\pi$, so that

$$z_n(x) = \frac{1}{\lambda}\left(\frac{x}{\lambda}\right)^{n-1} \frac{e^{-x/\lambda}}{(n-1)!} \chi_{[0,+\infty)}(x). \tag{2.8}$$

Consider now two equations of the form (2.7),

$$\begin{aligned} z_n^1 - z_{n-1}^1 + \lambda_1 z_{n,x}^1 &= 0, \\ z_n^2 - z_{n-1}^2 + \lambda_2 z_{n,x}^2 &= 0, \end{aligned} \tag{2.9}$$

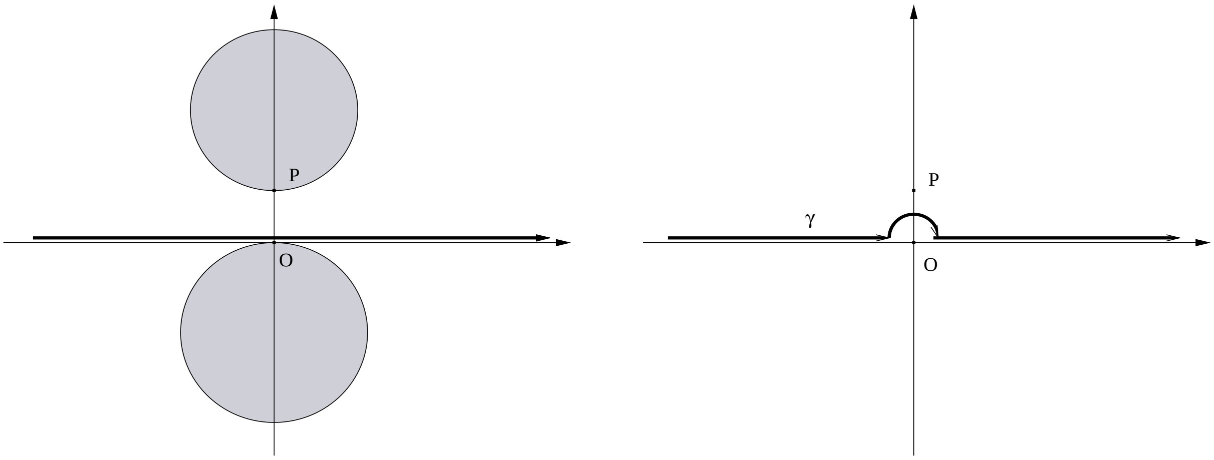

FIGURE 1. Integration path on the complex plane, where $P = i(\lambda_1 - \lambda_2)/\lambda_1\lambda_2$.

with initial data $z_0^1(x) = \delta(x)$ and $z_0^2(x) = \delta(x - x_0)$, and assume without any loss of generality that $\lambda_1 > \lambda_2 > 0$. We can compute the *transversal term* when $x_0 \neq 0$:

$$\sum_{n=0}^{+\infty} \int_{\mathbb{R}} z_n^1(x) z_n^2(x) dx.$$

Denoting with $c^i(n, \xi)$ the Fourier coefficients of $z_n^i(x)$ we have

$$\begin{aligned} \sum_{n=0}^{N} \int_{\mathbb{R}} z_n^1(x) z_n^2(x) dx &= \sum_{n=0}^{N} 2\pi \int_{\mathbb{R}} c^1(n,\xi) c^2(n,-\xi) e^{-i\xi x_0} d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{n=0}^{N} \frac{1}{\big(1 - i\lambda_1\xi\big)^n \big(1 + i\lambda_2\xi\big)^n} e^{-i\xi x_0} d\xi. \end{aligned} \tag{2.10}$$

If ξ is considered as a complex variable, we can let $N \to +\infty$ only in the region where

$$Z \doteq \Big\{ \xi \in \mathbb{Z} : \big|(1 - i\lambda_1\xi)(1 + i\lambda_2\xi)\big| < 1 \Big\}$$

i.e. outside the regions depicted in Figure 1. Deforming the path to avoid the region Z, we can pass to the limit:

$$\begin{aligned} \sum_{n=0}^{+\infty} \int_{\mathbb{R}} z_n^1(x) z_n^2(x) dx &= \frac{1}{2\pi} \int_{\gamma} \frac{e^{-i\xi x_0}}{1 - \frac{1}{(1-i\lambda_1\xi)(1+i\lambda_2\xi)}} d\xi \\ &= \frac{1}{2\pi} \int_{\gamma} \frac{(1 - i\lambda_1\xi)(1 + i\lambda_2\xi)}{(1 - i\lambda_1\xi)(1 + i\lambda_2\xi) - 1} e^{-i\xi x_0} d\xi. \end{aligned}$$

By means of calculus of residues we have finally that

$$P(x_0) \doteq \sum_{n=0}^{+\infty} \int_{\mathbb{R}} z_n^1(x) z_n^2(x) dx = \begin{cases} 1/(\lambda_1 - \lambda_2) \cdot \exp\Big((\lambda_1 - \lambda_2)/(\lambda_1\lambda_2) \cdot x_0\Big) & x_0 < 0 \\ 1/(\lambda_1 - \lambda_2) & x_0 \geq 0 \end{cases} \tag{2.11}$$

In fact, depending on the sign of x_0, the integration along the line γ is equivalent to the integration around the pole 0 or the pole $P = i(\lambda_1 - \lambda_2)/\lambda_1\lambda_2$.

2.3. **BV estimates.** Now to prove that (2.2) has a solution with uniformly bounded total variation. Define the functional

(2.12)

$$Q(n) = Q(u_n, u_{n-1}) \doteq \sum_{i<j} \iint_{\mathbb{R}^2} P_0(x-y) \Big\{ \big|v_n^i(x) v_n^j(y)\big| + \big|v_{n-1}^i(x) v_n^j(y)\big| + \big|v_n^i(x) v_{n-1}^j(y)\big| \Big\} dx dy,$$

where P is computed substituting to $\lambda-\mu$ the constant of separation of speeds c, and taking the minimal value of the exponent $(\lambda-\mu)/\lambda\mu$:

$$P_0(x) \doteq \begin{cases} 1/c \cdot \exp\Big(c/\big(\beta(\beta-c)\big)x_0\Big) & x_0 < 0 \\ 1/c & x_0 \geq 0 \end{cases}$$

We recall that c and β are defined in the introduction.

Using the same analysis of [2], we see immediately that

$$\begin{aligned} Q(n) - Q(n-1) = {} & \sum_{i<j} \iint_{\mathbb{R}^2} P_0(x-y)\Big\{ \big|v_n^i(x)v_n^j(y)\big| - \big|v_{n-1}^i(x)v_{n-1}^j(y)\big| \Big\} dxdy \\ & + \sum_{i<j} \iint_{\mathbb{R}^2} P_0(x-y)\Big\{ \big|v_{n-1}^i(x)v_n^j(y)\big| - \big|v_{n-2}^i(x)v_{n-1}^j(y)\big| \Big\} dxdy \\ & + \sum_{i<j} \iint_{\mathbb{R}^2} P_0(x-y)\Big\{ \big|v_n^i(x)v_{n-1}^j(y)\big| - \big|v_{n-1}^i(x)v_{n-2}^j(y)\big| \Big\} dxdy \\ \leq {} & -\Big(1 - C \max_{m=1,\dots,n} \text{Tot.Var.}(u_m)\Big) \left[\sum_{j\neq k} \big|v_{n-1}^j v_n^k\big| + \sum_{j \neq k} \big|v_n^j v_n^k\big| \right], \end{aligned} \tag{2.13}$$

where C is c constant depending only on H_{jk}^i, K_{jk}^i, α, K and c. For the sake of completeness, we show the computation for the terms of the first summation:

$$\begin{aligned} & \iint_{\mathbb{R}^2} P_0(x-y)\Big\{ \big|v_n^i(x)v_n^j(y)\big| - \big|v_{n-1}^i(x)v_{n-1}^j(y)\big| \Big\} dxdy \\ & = \iint_{\mathbb{R}^2} P_0(x-y)\Big\{ \Big(\big|v_n^i(x)\big| - \big|v_{n-1}^i(x)\big|\Big)\big|v_n^j(y)\big| + \big|v_{n-1}^i(x)\big|\Big(\big|v_n^j(y)\big| - \big|v_{n-1}^j(y)\big|\Big) \Big\} dxdy \\ & \leq \iint_{\mathbb{R}^2} P_0(x-y)\Big\{ -\Big(\lambda_i\big|v_n^i\big|\Big)_x \big|v_n^j(y)\big| - \big|v_{n-1}^i(x)\big|\Big(\lambda_j\big|v_n^j\big|\Big)_y dxdy + \text{Tot.Var.}(u_n)\cdot\big\{\text{source terms}\big\} \\ & = \iint_{\mathbb{R}^2} P_0'(x-y)\Big\{ (\lambda_i-\lambda_j)\big|v_n^i(x)v_n^j(y)\big| - \lambda_j\Big(\lambda_i\big|v_n^i\big|\Big)_x v_n^j(y) \Big\} dxdy + \text{Tot.Var.}(u_n)\cdot\big\{\text{source terms}\big\} \\ & \leq \iint_{\mathbb{R}^2} \Big\{ P_0'\big(\lambda_i-\lambda_j\big) + P_0''\lambda_i\lambda_j \Big\} \big|v_n^i(x)v_n^j(y)\big| dxdy + \text{Tot.Var.}(u_n)\cdot\big\{\text{source terms}\big\} \\ & \leq -\int_{\mathbb{R}} \big|v_n^i(x)\big|\big|v_n^j(x)\big| dx + \text{Tot.Var.}(u_n)\cdot\big\{\text{source terms}\big\}, \end{aligned}$$

because one has

$$P_0'(x-y)\big(\lambda_i-\lambda_j\big) + P_0''(x-y)\lambda_i\lambda_j \leq -\delta(x-y).$$

Thus if δ_0 is sufficiently small, using (2.13) we have

$$\text{Tot.Var.}(u_1) + C_0 Q(u_1,u_0) \leq \delta_1 \quad \text{and} \quad \frac{d}{dt}\Big\{\text{Tot.Var.}(u) + C_0 Q(u)\Big\} \leq 0,$$

where the constant C_0 is big enough, independent on δ_0. This proves that the solution u_n has uniformly bounded total variation for all $n \in \mathbb{N}$.

2.4. **Stability estimates.** We now consider the stability estimates of (2.2). Due to the observation in the Introduction, once we get these estimate we can prove that the Riemann solver of the limiting hyperbolic system is the Lax Riemann solver, so that the solutions of these approximations converge in L^1 to the unique entropic solution of (1.1).

The equations for a perturbation $u + \delta h$ as $\delta \to 0$ are

$$h_n - h_{n-1} + \big(A(u_n)h_n\big)_x = \big(DA(u_n)u_{n,x}\big)h - \big(DA(u_n)h_n\big)u_{n,x} = 0, \tag{2.14}$$

since the equations (2.2) are in conservation form. Using the same projection of (2.3), i.e.

$$h_n = \sum_i h_n^i r_{i,n},$$

we have that the equations for the components h^i_n are

$$h^i_n - h^i_{n-1} + \big(\lambda_{i,n} h^i_n\big)_x = \sum_{j\neq k}\Big(\lambda_{k,n} h^j_{n-1} v^k_n \langle l^i_n, \alpha_j(u_n, u_{n-1})\rangle - h^i_n v^j_n \langle l^i_n, (Dr_{i,n}) r_{j,n}\rangle\Big)$$
$$= \sum_{j\neq k} H(n) h^j_{n-1} v^k_n + \sum_{j\neq k} K'(n) h^j_n v^k_n. \tag{2.15}$$

Using the same analysis of [2], it is easy to prove that a functional as in Section 2.3 gives the stability of the solution. This concludes the analysis of the semigroup approximation.

3. Approximation by semi-discrete scheme

We now consider the case 2) of Section 1, i.e. the following singular approximation to system of conservation laws:

$$\frac{\partial}{\partial t} u(t,x) + \frac{1}{\epsilon}\Big(f\big(u(t,x)\big) - f\big(u(t,x-\epsilon)\big)\Big) = 0, \tag{3.1}$$

where $u \in \mathbb{R}^n$. By the rescaling $t \to t/\epsilon$, $x \to x/\epsilon$, we obtain the evolutionary equations

$$\dot u_n(t) + f\big(u_n(t)\big) - f\big(u_{n-1}(t)\big) = 0. \tag{3.2}$$

The equation for the function $v_n \doteq u_n - u_{n-1}$ are

$$\dot v_n(t) + f\big(u_n(t)\big) - 2f\big(u_{n-1}(t)\big) + f\big(u_{n-2}(t)\big) = 0. \tag{3.3}$$

3.1. **Projection on rarefaction curves.** The vector v_n is now decomposed along the eigenvectors $r_{i,n}$ of the Riemann problem u_{n-1}, u_n, i.e. the right eigenvectors of the averaged matrix

$$A\big(u_n, u_{n-1}\big) \doteq \int_0^1 Df\big(su_{n-1} + (1-s)u_n\big) ds. \tag{3.4}$$

If we denote with $\lambda_{i,n}$ the eigenvalue corresponding to $r_{i,n}$, it is well known that

$$f\big(u_n\big) - f\big(u_{n-1}\big) = \int_0^1 Df\big(su_{n-1} + (1-s)u_n\big)\big(u_n - u_{n-1}\big) ds$$
$$= \sum_{i=1}^n v^i_n A\big(u_n, u_{n-1}\big) r_{i,n} = \sum_{i=1}^n \lambda_{i,n} v^i_m r_{i,n}.$$

Using the above formula, we have

$$\dot u_n(t) + \sum_i \lambda_{i,n} v^i_n r_{i,n} = 0,$$

$$\sum_i \Big(\dot v^i_n + \lambda_{i,n} v^i_n - \lambda_{i,n-1} v^i_{n-1}\Big) r_{i,n} \tag{3.5}$$
$$= -\sum_{i,j} v^i_n v^j_n \big(Dr_{i,n}\big) r_{j,n} - \sum_{i,j} v^i_n v^j_{n-1}\big(Dr_{i,n}\big) r_{j,n-1} + \sum_i \lambda_{i,n-1} v^i_{n-1}\big(r_{i,n-1} - r_{i,n}\big)$$
$$= -\sum_{i,j} v^i_n v^j_n \big(Dr_{i,n}\big) r_{j,n} - \sum_{i,j} v^i_n v^j_{n-1}\big(Dr_{i,n}\big) r_{j,n} + \sum_{i,j} v^i_n v^j_{n-1}\big(Dr_{i,n}\big)\big(r_{j,n} - r_{j,n-1}\big)$$
$$+ \sum_i \lambda_{i,n-1} v^i_{n-1}\big(r_{i,n-1} - r_{i,n}\big).$$

If we assume the condition (1.4), the functions $(Dr_{i,n}) r_{j,n}$ and $r_{i,n} - r_{i,n-1}$ are zero when $u_n - u_{n-1}$ and $u_{n-1} - u_{n-2}$ are parallel to $r_{i,n} = r_{i,n-1}$. Thus we have

$$r_{j,n} \bullet r_{i,n} = \sum_{j\neq i} \alpha_{j,n} v^j_n, \tag{3.6}$$
$$r_{i,n} - r_{i,n-1} = \sum_{j\neq i} \beta_{j,n} v^j_n + \sum_{j\neq i} \gamma_{j,n-1} v^j_{n-1},$$

as in Section 2.1. Using (3.6), the expansion (3.5) thus becomes

$$\dot v^i_n + \lambda_{i,n} v^i_n - \lambda_{i,n-1} v^i_{n-1} = \sum_{j\neq k} H^i_{n,jk}(t) v^j_n v^k_n + \sum_{j \neq k} G^i_{n,jk}(t) v^j_n v^k_{n-1}, \tag{3.7}$$

where the functions $H(t)$, $G(t)$ are uniformly bounded in K_1.

To estimate the source terms in (3.7), we consider the case of two linear equations.

3.2. **Analysis of the linear case.** Consider a single linear equation

$$\dot v^i_n + \lambda v^i_n - \lambda v^i_{n-1} = 0, \qquad \lambda > 0. \tag{3.8}$$

We can find the fundamental solution to the previous equation by means of Fourier transform: defining the periodic function

$$c(t,x) \doteq \sum_n v_n(t) e^{inx}, \qquad v_n(t) = \frac{1}{2\pi} \int_0^{2\pi} c(t,x) e^{-inx} dx,$$

we have that the equation satisfied by c is

$$\begin{aligned} c_t = \sum_n \dot v_n e^{inx} &= \lambda \sum_n \big(v_{n-1} - v_n\big) e^{inx} \\ &= \lambda\big(e^{ix} - 1\big) c, \end{aligned}$$

whose general solution is

$$c(t,x) = c(0,x) \exp\Big(\lambda\big(e^{ix}-1\big)t\Big).$$

In particular the fundamental solution starting at n_0 has $c(0,x) = \exp(i n_0 x)$, so that if $n_0 = 0$ we obtain

$$v_n(t) = \begin{cases} 0 & n < 0 \\ \big(\lambda t\big)^n / n! \cdot \exp\big(-\lambda t\big) & n \geq 0 \end{cases} \tag{3.9}$$

If now we consider two equations of the form (3.8),

$$\begin{aligned} \dot z^1_n + \lambda_1 \big(z^1_n - z^1_{n-1}\big) &= 0, \\ \dot z^2_n + \mu \big(z^2_n - z^2_{n-1}\big) &= 0, \end{aligned} \tag{3.10}$$

with initial data $z^1_n(0) = \delta_{0,n}$, $z^2_n(0) = \delta_{n_0,n}$. We can compute the *transversal term*

$$\int_0^{+\infty} \sum_{n=-\infty}^{+\infty} z^1_n(t) z^2_n(t),$$

when $n_0 \neq 0$.

Denoting with $c^i(t,x)$ the Fourier transform of $z^i_n(t)$ and assuming that $\lambda_1 > \lambda_2 > 0$, we have

$$\begin{aligned} \int_0^{+\infty} \sum_{n=-\infty}^{+\infty} z^1_n(t) z^2_n(t) dt &= \int_0^{+\infty} \frac{1}{2\pi} \int_0^{2\pi} c^1(t,x) c^2(t,-x) e^{-in_0 x} dx dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{+\infty} \exp\Big(\lambda_1 \big(e^{ix}-1\big)t + \lambda_2\big(e^{-ix}-1\big)t\Big) e^{-in_0 x} dx dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-in_0 x}}{\lambda_1\big(e^{ix}-1\big) + \lambda_2\big(e^{-ix}-1\big)} dx \\ &= \frac{1}{2\pi i} \oint_\gamma \frac{z^{-n_0}}{\big(z-1\big)\big(\lambda_1 z - \lambda_2\big)} dz, \end{aligned} \tag{3.11}$$

where γ is the path represented in Figure 2.

By means of calculus of residues we have that

$$P_0(n_0) \doteq \int_0^{+\infty} \sum_{n=-\infty}^{+\infty} z^1_n(t) z^2_n(t) dx = \begin{cases} 1/(\lambda_1 - \lambda_2) \cdot \big(\lambda_1/\lambda_2\big)^{n_0} & n_0 < 0 \\ 1/(\lambda_1 - \lambda_2) & n_0 \geq 0 \end{cases} \tag{3.12}$$

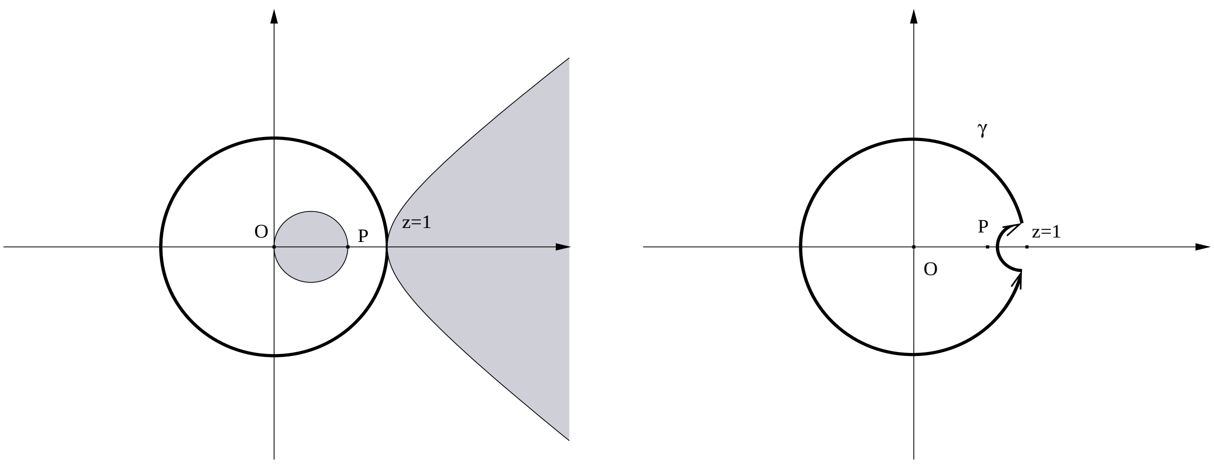

FIGURE 2. Integration path on the complex plane, where $P = \lambda_2/\lambda_1$.

3.3. **BV estimates.** Now to prove that (3.1) has a solution with uniformly bounded total variation. Define the functional

$$Q\big(u(t)\big) \doteq \sum_{i<j} \sum_{n,m=-\infty}^{+\infty} P_0(n-m)\Big\{\big|v_n^i(t)v_m^j(t)\big| + \big|v_{n-1}^i(t)v_m^j(t)\big| + \big|v_n^i(t)v_{m-1}^j(t)\big|\Big\}dxdy, \tag{3.13}$$

where P is computed using the constant of separation of speeds c instead of $\lambda_1 - \lambda_2$ and c/β instead of λ_1/λ_2. We now adapt the computation of Section 2.3 for the terms in the first sum: we have

$$\begin{aligned}
&\sum_{n,m=-\infty}^{+\infty} P_0(n-m)\left\{\frac{d|v_n^i(t)|}{dt}\big|v_m^j(t)\big| + \big|v_n^i(t)\big|\frac{d|v_m^j(t)|}{dt}\right\} \\
&\quad\le \sum_{n,m=-\infty}^{+\infty} P_0(n-m)\Big\{-\big(\lambda_{j,m}+\lambda_{i,n}\big)\big|v_n^i\big|\big|v_m^j\big| + \lambda_{i,n-1}\big|v_{n-1}^i\big|\big|v_m^j\big| + \lambda_{j,m-1}\big|v_n^i\big|\big|v_{m-1}^j\big|\Big\} \\
&\qquad\qquad + \text{Tot.Var.}(u_n)\cdot\{\text{ source terms }\} \\
&\quad= \sum_{n,m=-\infty}^{+\infty} \Big\{\lambda_{i,n}P_0(n-m-1) + \lambda_{j,m}P_0(n-m+1) - \big(\lambda_{i,n}+\lambda_{i,m}\big)P(n-m)\Big\}\big|v_n^i\big|\big|v_m^j\big| \\
&\qquad\qquad + \text{Tot.Var.}(u_n)\cdot\{\text{ source terms }\} \\
&\quad\le -\sum_{n=-\infty}^{+\infty} \big|v_n^i\big|\big|v_n^j\big| + \text{Tot.Var.}(u_n)\cdot\{\text{ source terms }\},
\end{aligned}$$

because as in the previous case one has

$$\lambda_{i,n}P_0(n-m-1) + \lambda_{j,m}P_0(n-m+1) - \big(\lambda_{i,n}+\lambda_{i,m}\big)P(n-m) \le \delta_{n,m}.$$

Since the left hand side of (3.7) is in conservation form, we conclude as in Section 2.3 that

$$\text{Tot.Var.}\big(u(0)\big) + C_0 Q\big(u(o)\big) \le \delta_1, \qquad \frac{d}{dt}\Big\{\text{Tot.Var.}(u) + C_0 Q(u)\Big\} \le 0.$$

This concludes the proof of bounded total variation.

3.4. **Stability estimates.** Finally we consider the stability estimates of (3.1). The equations for a perturbation $u + \delta h$ as $\delta \to 0$ are

$$\dot h_n(t) + Df\big(u_n\big)h_n - Df\big(u_{n-1}\big)h_{n-1} = 0. \tag{3.14}$$

Considering the projection

$$h_n(t) = \sum_i h_n^i(t) r_i(u_n),$$

we have that the equations for the components h_n^i are

$$\dot{h}_n^i + \lambda_i(u_n)h_n^i - \lambda_i(u_{n-1})h_{n-1}^i = \sum_{j \neq k} H'(n)h_{n-1}^j v_n^k + \sum_{j \neq k} G'(n)h_n^j v_n^k.$$

At this point it is clear that a functional as in Section 2.3 proves the stability of the solution. This concludes the analysis of the semidiscrete case.

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