VANISHING VISCOSITY SOLUTIONS OF HYPERBOLIC SYSTEMS WITH BOUNDARY CONDITIONS

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Abstract. We consider the parabolic system

E:party1 (0.1) $$u_t + A(\kappa(t), u)u_x = \epsilon u_{xx}, \qquad t, x > 0,$$

with Dirichlet boundary conditions. The parameter κ is a smooth function of t, and A is uniformly strictly hyperbolic. We prove that is the initial boundary data u_0, u_b and the parameter κ have sufficiently small total variation, then as $\epsilon \to 0$ the solution u^ϵ of the above system converges in L^1 to a unique solution of the corresponding hyperbolic system

E:hyppy2 (0.2) $$u_t + A(\kappa(t), u)u_x = 0, \qquad t, x > 0,$$

with a well defined notion of trace at $x = 0$. We allow the boundary $x = 0$ to have the same speed of one of the characteristic speed of the limiting hyperbolic system.

This result is achieved by a new decomposition of the solution into travelling profiles and the non characteristic part of boundary profile, the analysis of the interaction of travelling profiles with the boundary profile, the construction of boundary profile when one non linear characteristic field has a speed close to the speed of the boundary, and corresponding solution of the boundary Riemann problem (i.e. (0.2) with initial boundary data u_0, u_b constant) and the precise analysis of the trace of the solution u to (0.2) at $x = 0$.

In the last part of the paper we show how the analysis can be extended to the case when the total variation of κ is large.

A corollary of the above results is the construction of the solutions to (0.1), (0.2) in the case of oscillating boundary, i.e. $x \geq x_b(t)$, where $(t, x_b(t))$ is a smooth curve in $\mathbb{R}^2$ with speed $\sigma_b(t) = \dot{x}_b(t)$ of bounded total variation.

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S:intro05

1. Statement of the problem

The starting point of our analisys is the parabolic system with boundary

E:parasys1
$$u_t + A(u)u_x = u_{xx} \tag{1.1}$$

in $t \in \mathbb{R}^+$, $x \in (x_b(t), +\infty)$, where $x_b(t)$ is a smooth curve in the plane. In the following we will denote by $\sigma_b(t)$ its speed,

$$\sigma_b(t) \doteq \frac{dx_b(t)}{dt}.$$

We assume that the initial data and boundary Dirichlet data are small in BV norm,

E:init02
$$u(0,x) = u_0(x), \quad u(t, x_b(t)) = u_b(t), \qquad \text{Tot.Var.}(u_0),\ \text{Tot.Var.}(u_b) \leq \delta_1. \tag{1.2}$$

and that the functions u_b, u_0 are sufficiently smooth, in the sense that

E:smoth78
$$\|d^k u_0/dx^k\|_{L^1},\ \|d^k u_b/dt^k\|_{L^1} \leq C\delta_1, \quad k = 1, \dots, K, \qquad \lim_{x \to 0^+} u_0(x) = \lim_{t \to 0^+} u_b(t), \tag{1.3}$$

and for some vector $\bar{u}$

E:init93
$$\sup_x |u_0(x) - \bar{u}|,\ \sup_t |u_b(t) - \bar{u}| \leq \delta_0. \tag{1.4}$$

Here and in the following C will denote a large constant. The last condition implies that the initial and boundary data are close to a constant state (a completely different case is a BV perturbation of a large stable boundary layer). We suppose that the oscillation of the curve x_b is bounded, in the sense that

E:oscill04
$$\text{Tot.Var.}(\sigma_b(t)) \leq C\delta_0. \tag{1.5}$$

By changing coordinates $x \mapsto x - x_b(t)$ we recover the system

E:parasys21
$$u_t + A(\sigma_b(t), u)u_x = u_{xx} \tag{1.6}$$

with a drift matrix $A(\kappa, u) = A(u) - \kappa I$ depending on a scalar parameter κ. The boundary data are assigned on the line $x = 0$, and by (1.5) it follows that the time derivative of κ is integrable:

E:Atimeder1
$$\int_0^{+\infty} |\dot{\kappa}(t)| dt \leq C\delta_0. \tag{1.7}$$

In the following we will thus consider the parabolic system (1.6), with a drift matrix depending on a parameter κ (not necessarily scalar) and satisfying (1.7), and initial data $u(0,x) = u_0(x)$, $u(t,0) = u_b(t)$.

R:smoth87

Remark 1.1. We note here that the smoothness assumptions is not important when we pass to the limit. In fact, if u_0, u_b are the initial-boundary data for (1.1) satisfying (1.3), then rescaling $(t,x) \mapsto (t/\epsilon, x/\epsilon)$ we obtain the smoothness assumption

E:smoth781
$$\|d^k u_0/dx^k\|_{L^1},\ \|d^k u_b/dt^k\|_{L^1} \leq C\delta_1 \epsilon^{k-1}, \quad k = 1, \dots, K. \tag{1.8}$$

In particular, as we will see, in the hyperbolic limit we can extend the semigroup to all BV functions satisfying (1.2), (1.4).

For κ close to 0, and u in a neighborhood of radius $5\delta_0$ from $\bar{u}$, we assume that a strict hyperbolicity condition holds, i.e. for some fixed $c > 0$

E:stricthyp
$$\inf_{\kappa,u,v} \big\{\lambda_{i+1}(\kappa, u) - \lambda_i(\kappa, v)\big\} \geq c \quad i = 1, \dots, n-1, \qquad \inf_{\kappa,u} \big\{|\lambda_j(\kappa, u)|\big\} \geq c \quad j = 1, \dots, k-1, k+1, \dots, n. \tag{1.9}$$

and that the k-th eigenvalue is *boundary-characteristic*,

E:charaeq01
$$|\lambda_k(0, \bar{u})| \leq C\delta_0. \tag{1.10}$$

We will prove the following theorem:

T:parae1

Theorem 1.2. *Consider the parabolic time dependent system*

E:hooo78
$$u_t + A(\kappa(t), u)u_x = u_{xx}, \qquad t, x \geq 0, \tag{1.11}$$

with initial data u_0, u_b satisfying (1.2), (1.3), (1.4). *Assume moreover that the time dependent parameter $\kappa(t)$ is smooth and satisfies* (1.7). *Then, if δ_1, δ_0, with $\delta_1 \leq \delta_0$, are sufficiently small, the solution $u(t,x)$ exists for all $t \geq 0$ and has total variation uniformly bounded by $2\delta_0$. In particular it remains in the neighborhood of $\bar{u}$ of radius $5\delta_0$.*

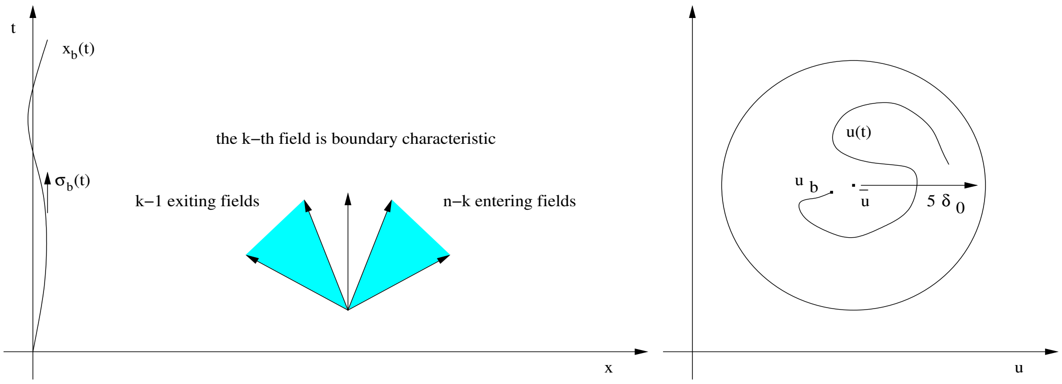

FIGURE 1. The assumption on characteristic fields and boundary speed in the neighborhood of $\bar u$ of radius $5\delta_0$. Fi:bounriem4

Moreover, if $u_1(t)$, $u_2(t)$ *are the solutions of* (1.11) *with initial boundary data* $(u_{1,0}, u_{1,b})$, $(u_{2,0}, u_{2,b})$ *and with parameters* $k_1(t)$, $k_2(t)$, *respectively, then for* $t \geq s$

E:Lipscht1
$$\begin{aligned} \|u_1(t) - u_2(s)\|_{L^1(\mathbb{R}^+)} \leq L\Big(&|t-s| + \|u_{1,0} - u_{2,0}\|_{L^1(\mathbb{R}^+)} \\ &+ \|u_{1,b} - u_{2,b}\|_{L^1(0,s)} + \text{Tot.Var.}(u)\|\kappa_1 - \kappa_2\|_{L^1(0,s)}\Big), \end{aligned} \tag{1.12}$$

where L *is constant depending only on the system* (1.11) *and the constant* δ_0.

A corollary of this theorem is the existence of a solution with uniform bounded variation for the original parabolic system (1.1), and their Lipschitz dependence w.r.t. the initial boundary data and the speed of the boundary. The assumptions for the case of oscillating boundary are resumed in figure 1: there are $k-1$ characteristic fields leaving the domain, $n-k$ characteristic fields entering the domain, and the k-th characteristic field is boundary characteristic.

One can see during the proof that we can take $\delta_0 = C\delta_1$, with C sufficiently large constant.

Next we will consider the hyperbolic limit of the equation (1.11) under the hyperbolic rescaling $(t,x) \mapsto (t/\epsilon, x/\epsilon)$. We will prove that there is a unique limit to the solution constructed in Theorem (1.2), which is a viscosity solution to the hyperbolic system with boundary

$$u_t + A(\kappa(t), u)u_x = 0.$$

In particular we construct the solution to the boundary Riemann problem

E:bururt1
$$u_0(x) = u_0, \qquad u_b(t) = u_b. \tag{1.13}$$

This solution is a self similar solution, characterized by the fact that it is the limit of the vanishing viscosity solution.

The construction of this solution relies on the possibility of contructing boundary layers also for the boundary characteristic case.

In the last part of the paper we show how this construction can be extended to the case when the parameter κ has not small total variation.

2. REGULARITY ESTIMATES

S:regulet1

The aim of this section is to obtain estimates on the higher derivative u_{xx}, u_{tx} of the parabolic system

E:system879
$$u_t + A(\kappa, u)u_x = u_{xx}, \qquad t > 0, \quad x > x_b(t), \tag{2.1}$$

the matrix $A(\kappa, u)$ being smooth, strictly hyperbolic and κ satisfying (1.7), with the a priori assumption

E:itibdcd1
$$\text{Tot.Var.}(u(t)) \leq 3\delta_0, \tag{2.2}$$

and the boundary data u_b sufficiently regular. We will also consider the equations for u_t, which is the same equation (a part the time derivative of A) satisfied by an infinitesimal perturbation h of u:

E:sys888
$$(u_t)_t + (A(\kappa, u)u_t)_x - (u_t)_{xx} = (DA(\kappa, u)u_x)u_t - (DA(\kappa, u)u_t)u_x - A_\kappa(\kappa, u)\dot\kappa u_x, \tag{2.3}$$

E:sys889

$$h_t + (A(\kappa,u)h)_x - h_{xx} = (DA(\kappa,u)u_x)h - (DA(\kappa,u)h)u_x, \tag{2.4}$$

with initial-boundary data small in the L^1 norm. Here and in the following we have used the notation

$$A_\kappa = \frac{\partial A}{\partial \kappa}, \qquad DAv = \sum_{i=1}^n \frac{\partial A}{\partial u_i} v_i, \quad v \in \mathbb{R}^n.$$

Finally we will consider the equation for the effective flux $\iota = h_x - A(\kappa,u)h$ of the perturbation h,

E:sys890

$$\begin{aligned} \iota_t + (A(\kappa,u)\iota)_x - \iota_{xx} = {} & \Big[(DA(\kappa,u)u_x)h - (DA(\kappa,u)h)u_x)\Big]_x - A_\kappa(\kappa,u)\dot\kappa h \\ & - A(\kappa,u)DA(\kappa,u)(u_x \otimes h - h \otimes u_x) + DA(\kappa,u)(u_x \otimes \iota - u_t \otimes h). \end{aligned} \tag{2.5}$$

In this case the boundary data should be estimated by means of the regularity estimates of previous equation (2.4), because what we know are the boundary data for u and for h. We will show that in all cases the L^1 norms of higher derivatives of u_x, u_t, h, ι are bounded by the L^1 norm of the solution u, h and the constant δ_0 appearing in (1.3), i.e. these norm are of the order of δ_0.

In the first part we consider the Green kernel for the scalar linear problem

$$z_t + \lambda z_x - z_{xx} = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

and in Section 2.3 we will show that if the L^1 norm of $u_x(t)$, $h(t)$ are bounded by $3\delta_0$ for $t \in [0,T]$, then also the further derivatives u_{xx}, h_x, $u_{t,x}$, ι_x have L^1 norm of the order of δ_0.

Ss:scal1

2.1. **Regularity estimates: the scalar case.**

We first observe that for a scalar equation

E:scal1

$$z_t + \lambda z_x - z_{xx} = 0, \qquad z(0,x) = z_0(x),\ z(t,0) = z_b(t), \tag{2.6}$$

the maximum principle holds, i.e. if

$$z_0'(x) \geq z_0(x) \quad \forall x \geq 0, \qquad z_b'(t) \geq z_b(t) \quad \forall t \geq 0,$$

then the solution z', z satisfy

$$z'(t,x) \geq z(t,x) \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

We split the Green kernel for (2.6) into 2 parts, depending on the initial-boundary data as follows:

(1) a single δ in $y > 0$, i.e. $z_0 = \delta(x-y)$, $z_b(t) = 0$;
(2) the jump $z_b(0) - z_0(0)$, i.e. $z_0 = 0$, $z_b = 1$;

The two parts will be denoted by $\Gamma^\lambda(t,x;y)$, $K^\lambda(t,x)$.

The explicit solution to the first case,

$$z_t + \lambda z_x - z_{xx} = 0, \qquad z_0 = \delta(x-y), \quad z_b = 0.$$

is given by

E:xpliss1

$$\begin{aligned} \Gamma^\lambda(t,x;y) &\doteq \exp\left(\frac{\lambda}{2}(x-y) - \frac{\lambda^2}{4}t\right)\big(G(t,x-y) - G(t,x+y)\big) \\ &= G^\lambda(t,x-y) - e^{-\lambda y} G^\lambda(t,x+y) \\ &= G^\lambda(t,x-y) - e^{\lambda x} G^{-\lambda}(t,x+y) \\ &= \big(1 - e^{-xy/t}\big) G^\lambda(t,x-y), \end{aligned} \tag{2.7}$$

where G^λ is the standard heat kernel with drift λ,

$$G^\lambda(t,x) \doteq \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(x-\lambda t)^2}{4t}\right).$$

The solution $K^\lambda(t,x)$ to the initial boundary value problem of the second case,

$$u_t + \lambda u_x - u_{xx} = 0, \qquad u_0 \equiv 0, \qquad u_b \equiv 1.$$

is instead

E:xpliss2

$$K^\lambda(t,x) = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x-\lambda t}{2\sqrt{t}}} e^{-y^2} dy + \frac{e^{\lambda x}}{\sqrt{\pi}} \int_{\frac{x+\lambda t}{2\sqrt{t}}}^{+\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x-\lambda t}{2\sqrt{t}}}^{+\infty} e^{-y^2} dy + \frac{e^{\lambda x}}{\sqrt{\pi}} \int_{\frac{x+\lambda t}{2\sqrt{t}}}^{+\infty} e^{-y^2} dy. \tag{2.8}$$

Ss:expp1

2.2. **Estimates on the explicit kernels.** Elementary computations give the estimates:

E:l1estimm1

$$\big\|\Gamma^\lambda_x(t)\big\|_{L^1} \leq \frac{1}{\sqrt{\pi t}}, \qquad \big\|K^\lambda_x(t)\big\|_{L^1} = 1, \qquad \big\|K^\lambda_{xx}(t)\big\|_{L^1} \leq 1 + |\lambda| + \frac{1}{\sqrt{\pi t}} \leq \frac{C}{\sqrt{t}}, \tag{2.9}$$

for small time intervals.

We now recall that for the heat kernel G^λ, the following property holds: for any $\phi, \phi' \in C^1_0(\mathbb{R})$,

$$\iint_{\mathbb{R}^2} \phi_x(x) G^\lambda(t, x-y) \phi'(y) dx dy = - \iint_{\mathbb{R}^2} \phi(x) G^\lambda(t, x-y) \phi'_y(y) dx dy.$$

We want to find then a function $\tilde\Gamma^\lambda$ such that for all $\phi, \phi' \in C^1_0(\mathbb{R}^+)$ a similar relation holds:

E:prott1

$$\iint_{\mathbb{R}^+\times\mathbb{R}^+} \phi_x(x) \Gamma^\lambda(t,x;y) \phi'(y) dx dy = - \iint_{\mathbb{R}^+\times\mathbb{R}^+} \phi(x) \tilde\Gamma^\lambda(t,x;y) \phi'_y(y) dx dy. \tag{2.10}$$

It is clear that the above dual kernel is given by $\Gamma^\lambda_x(t,x;y) + \tilde\Gamma^\lambda_y(t,x;y) = 0$, i.e.

E:tiga1

$$\begin{aligned}
\tilde\Gamma^\lambda(t,x;y) &\doteq \int_y^{+\infty} \Gamma^\lambda_x(t,x;z) dz = \int_y^{+\infty} \big(G^\lambda_x(t, x-z) - e^{-\lambda z} G^\lambda_x(t, x+z) \big) dz \\
&= G^\lambda(t,x-y) + e^{-\lambda y} G^\lambda(t, x+y) - \frac{\lambda e^{\lambda x}}{\sqrt{\pi}} \int_{\frac{x+y+\lambda t}{2\sqrt{t}}}^{+\infty} e^{-z^2} dz \\
&= G^\lambda(t,x-y) + e^{\lambda x} G^{-\lambda}(t, x+y) - \frac{\lambda e^{\lambda x}}{\sqrt{\pi}} \int_{\frac{x+y+\lambda t}{2\sqrt{t}}}^{+\infty} e^{-z^2} dz \\
&= (1 + e^{-xy/t}) G^\lambda(t,x-y) - \frac{\lambda e^{\lambda x}}{\sqrt{\pi}} \int_{\frac{x+y+\lambda t}{2\sqrt{t}}}^{+\infty} e^{-z^2} dz.
\end{aligned} \tag{2.11}$$

Note that, directly form the above formula or from

$$K^\lambda(t,x) = 1 - \int_0^{+\infty} \Gamma^\lambda(t,x;y) dy,$$

one obtains the relation

E:basma1

$$\tilde\Gamma^\lambda(t,x;0) + K^\lambda_x(t,x) = 0. \tag{2.12}$$

In fact, $\tilde\Gamma^\lambda(t,x;y) = w_x(t,x;y)$, where w is the solution to the boundary value problem

E:tilgwwb1

$$w_t + \lambda w_x - w_{xx} = 0, \qquad w(t,0;y) = 0, \ w(0,x;y) = \begin{cases} 0 & x < y \\ 1 & x \geq y \end{cases} \tag{2.13}$$

Note also that $\tilde\Gamma^\lambda > 0$. In fact, by maximum principle applied to w it follows that $w_x(t,0;y) \geq 0$, and $\tilde\Gamma^\lambda$ is the solution to the boundary value problem

E:tilgmmb1

$$\tilde\Gamma^\lambda_t + \lambda \tilde\Gamma^\lambda_x - \tilde\Gamma^\lambda_{xx} = 0, \qquad \tilde\Gamma(t=0) = \delta(x-y), \ \tilde\Gamma(x=0) = w_x(t,0;y) \geq 0. \tag{2.14}$$

In particular,

E:conser1

$$\int_{\mathbb{R}^+} \Gamma^\lambda(t,x;y) dx \leq \int_{\mathbb{R}^+} \tilde\Gamma^\lambda(t,x;y) dx = 1. \tag{2.15}$$

A generalization of the estimates (2.9) gives the following result:

E:l1estim2

$$\int_0^{+\infty} \big|\tilde\Gamma^\lambda_x(t,x;y)\big| dy \leq \frac{C}{\sqrt{t}}. \tag{2.16}$$

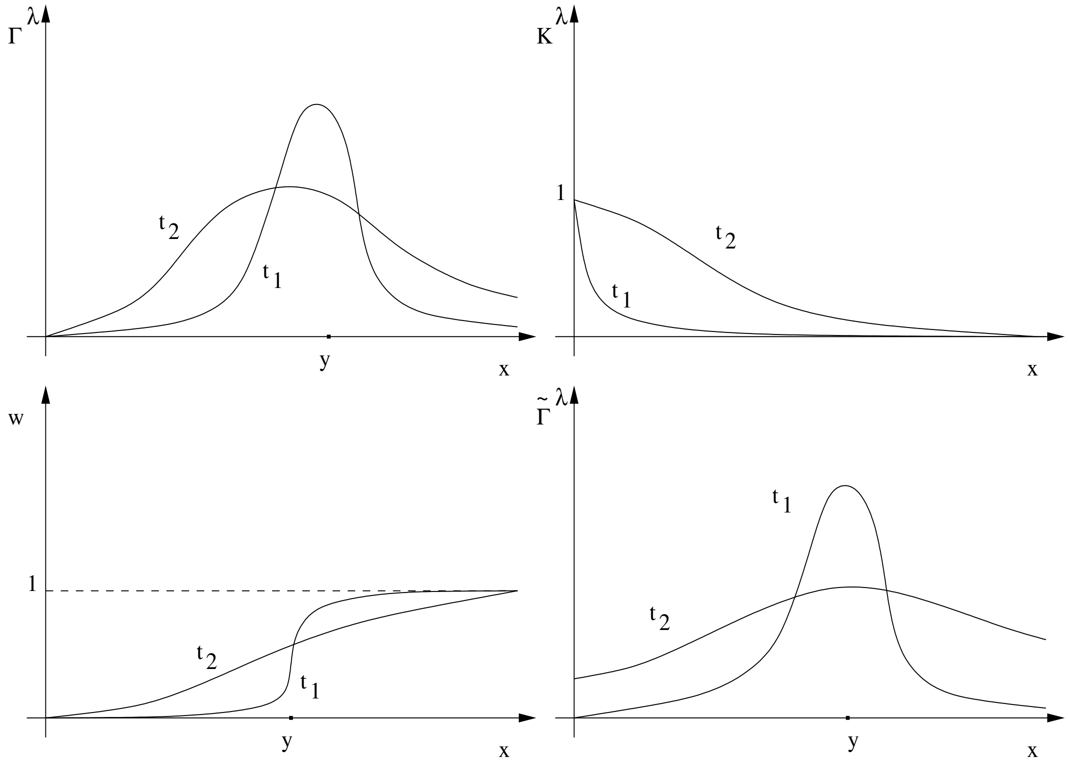

Figure 2. The four functions Γ^λ, K^λ, w and $\tilde\Gamma^\lambda$. Fi:straight1

In fact, by differentiating (2.11), we obtain

E:tuazia16

$$\begin{aligned}\tilde\Gamma^\lambda_x(t,x;y) &= G^\lambda_x(t,x-y) + e^{\lambda x} G^{-\lambda}_x(t,x+y) - \frac{\lambda^2 e^{\lambda x}}{\sqrt{\pi}} \int_{\frac{x+y+\lambda t}{2\sqrt{t}}}^{+\infty} e^{-z^2} dz \\ &= G^\lambda_x(t,x-y) + e^{-\lambda y} G^\lambda_x(t,x+y) + 2\lambda e^{-\lambda y} G^\lambda(t,x+y) - \frac{\lambda^2 e^{\lambda x}}{\sqrt{\pi}} \int_{\frac{x+y+\lambda t}{2\sqrt{t}}}^{+\infty} e^{-z^2} dz. \end{aligned} \tag{2.17}$$

Depending now on the sign of λ, one consider one of the two expression of (2.17), obtaining in both cases

E:tiga2

$$\big|\tilde\Gamma^\lambda_x(t,x;y)\big| \le \begin{cases} \big|G^\lambda(t,x-y)\big| + |G^{|\lambda|}_x(t,x+y)| + \lambda^2 e^{\lambda x} & \lambda < 0 \\ \big|G^\lambda(t,x-y)\big| + |G^{|\lambda|}_x(t,x+y)| + \lambda\big|K_x(t,x)\big| & \lambda \ge 0 \end{cases} \tag{2.18}$$

It is thus clear that in both cases

$$\big\|\tilde\Gamma^\lambda_x(t,x)\big\|_{L^1} \le 2\big\|G_x\big\|_{L^1} + |\lambda| + 1 \le \frac{C}{\sqrt{t}}$$

when t is sufficiently small, so that (2.16) holds.

Finally, we will prove that Γ^λ, $\tilde\Gamma^\lambda$ are essentially the heat kernel G^λ plus a term due to the boundary, and integrable for $t > 0$. It is clear that, since the two kernels are positive, that

E:first98

$$\Gamma^\lambda(t,x;y) \le G^\lambda(t,x-y), \qquad \tilde\Gamma^\lambda(t,x;y) \le 2G^\lambda(t,x-y) + |\lambda| e^{-|\lambda x|}. \tag{2.19}$$

From the definition of Γ^λ it follows immediately that

E:stiti1

$$\big|\Gamma^\lambda_x(t,x;y)\big| \le \Big|(1-e^{-xy/t}) G^\lambda_x(t,x-y) - y e^{-xy/t}/t \cdot G^\lambda(t,x-y)\Big| \le \big|G^\lambda_x(t,x-y)\big| + \frac{y}{2t\sqrt{\pi t}} e^{-xy/t}. \tag{2.20}$$

Similarly for $\tilde{\Gamma}^\lambda$

E:stiti2

$$\begin{aligned}\big|\tilde{\Gamma}^\lambda_x(t,x;y)\big| &\leq \left|(1+e^{-xy/t})G^\lambda_x(t,x-y) - ye^{-xy/t}/t\cdot G^\lambda(t,x-y) - \frac{\lambda^2 e^{\lambda x}}{\sqrt{\pi}}\int_{\frac{x+y+\lambda t}{2\sqrt{t}}}^{+\infty} e^{-z^2}dz\right| \\ &\leq 2\big|G^\lambda_x(t,x-y)\big| + \frac{y}{t}G^\lambda(t,x+y) + \begin{cases}\lambda^2 e^{\lambda x} & \lambda \leq 0 \\ CG^\lambda(t,x+y) & \lambda > 0\end{cases} \end{aligned} \tag{2.21}$$

where the last term follows from

$$\lambda^2 e^{\lambda x}\int_{\frac{x+y+\lambda t}{2\sqrt{t}}}^{+\infty} e^{-z^2}dz \leq Ce^{-\lambda y}G^\lambda(t,x+y),$$

for $\lambda > 0$.

R:fabietto1

Remark 2.1. The two kernel Γ^λ, $\tilde{\Gamma}^\lambda$ are useful in two different situations.

The kernel Γ^λ is used when we know the boundary condition of the equations (2.3), (2.4), to write the solution in integral representation by Duhamel formula. In fact, for the time derivative of u and the perturbation we know the boundary values $\dot{u}_b$, h_b, respectively. Conversely, $\tilde{\Gamma}^\lambda$ is need when we know the boundary condition of the solution u and we want to estimate u_x: this is the case of (2.1). In fact, the relation (2.10) allows us to compute the integral representation of the solution by means of the boundary data of the integral solution.

We observe also the different behavior of the two kernels $\Gamma^\lambda(t,x;y)$, $\tilde{\Gamma}^\lambda(t,x;y)$ as $y \to 0$. The kernel Γ^λ tends to 0, while $\tilde{\Gamma}^\lambda$ tends to the derivative of $-K$.

Ss:prrooot1

2.3. Regularity estimates for u, h.

We begin with the regularity estimate of u for (2.1). The solution to the system

E:system8710

$$u_t + A(\kappa,u)u_x - u_{xx} = 0, \quad u(0,x) = u_0(x),\ u(t,0) = u_b(t), \quad t,x \geq 0, \tag{2.22}$$

can be written as

E:duhamel1

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}} \Gamma^{A_0}(\delta t,x;y)u(t-\delta t,y)dy + \int_0^{\delta t}\int_{\mathbb{R}^+}\Gamma^{A_0}(s,x;y)(A_0 - A(\kappa,u))u_y(t-s,y)dyds \\ &+ \int_0^{\delta t} K^{A_0}(s,x)\dot{u}_b(t-s)ds + K^{A_0}(\delta t,x)u_b(t-\delta t), \end{aligned} \tag{2.23}$$

where $A_0 = A(0,\bar{u})$, and

$$\Gamma^{A_0} \doteq \sum_i \Gamma^{\lambda_i(\bar{u})} r_i(\bar{u}) \otimes l_i(\bar{u}), \qquad K^{A_0} \doteq \sum_i K^{\lambda_i(\bar{u})} r_i(\bar{u}) \otimes l_i(\bar{u}), \qquad \dot{u}_b = \frac{du_b}{dt}.$$

Observe that the regularity estimates (2.9) hold also for Γ^{A_0}, K^{A_0}:

E:l1estimm21

$$\big\|\Gamma^{A_0}_x(t)\big\|_{L^1} \leq \frac{1}{\sqrt{\pi t}}, \qquad \big\|K^{A_0}_x(t)\big\|_{L^1} = 1, \qquad \big\|K^{A_0}_{xx}(t)\big\|_{L^1} \leq \frac{C}{\sqrt{t}}, \tag{2.24}$$

for t small and C large, in a suitable norm. Observe moreover that the assumptions that u has total variation of order δ_0 and κ satisfies (1.7) imply that

$$\sup_t |A(\kappa(t),u(t)) - A_0| \leq C\delta_0.$$

Differentiating (2.23) once we obtain

$$\begin{aligned} u_x(t,x) = &\int_{\mathbb{R}^+} \Gamma^{A_0}_x(\delta t,x;y)u(t-\delta t,y)dy + \int_0^{\delta t}\int_{\mathbb{R}^+}\Gamma^{A_0}_x(s,x;y)(A_0 - A(\kappa,u))u_y(t-s,y)dyds \\ &+ \int_0^{\delta t} K^{A_0}_x(s,x)\dot{u}_b(t-s)ds + K^{A_0}_x(\delta t,x)u_b(t-\delta t), \end{aligned}$$

and using (2.12) and integrating by parts we can write it as

E:duhamel08

$$
\begin{aligned}
u_x(t,x) = {} & \int_{\mathbb{R}^+} \tilde{\Gamma}^{A_0}(\delta t, x; y) u_y(t-\delta t, y) dy + \int_0^{\delta t} \int_{\mathbb{R}^+} \tilde{\Gamma}^{A_0}(s,x;y) \big((A_0 - A(\kappa,u)) u_y(t-s,y) \big)_y dy ds \\
& + \big(\tilde{\Gamma}^{A_0}(\delta t, x; 0) + K_x^{A_0}(\delta t, x) \big) u_b(t-\delta t) \\
& + \int_0^{\delta t} \big(-\tilde{\Gamma}^{A_0}(s,x;0)(A(\kappa,u_b) - A_0) u_x(t,0) + K_x^{A_0}(s,x) \dot{u}_b(t-s) \big) ds \\
= {} & \int_{\mathbb{R}^+} \tilde{\Gamma}^{A_0}(\delta t, x; y) u_y(t-\delta t, y) dy + \int_0^{\delta t} \int_{\mathbb{R}^+} \tilde{\Gamma}^{A_0}(s,x;y) \big((A_0 - A(\kappa,u)) u_y(t-s,y) \big)_y dy ds \\
& + \int_0^{\delta t} K_x^{A_0}(s,x;0) \big(u_t(t-s,0) - (A(\kappa,u_b) - A_0) u_x(t-s,0) \big) ds.
\end{aligned}
\tag{2.25}
$$

Note that if the system is linear, the last term will contains only the oscillations of the boundary $\dot{u}_b = u_t(t,0)$ in $[t-\delta t, t]$. The eventual initial jump

$$
\lim_{x \to 0^+} u(t-\delta t, x) - \lim_{s \to 0^+} u_b(t - \delta t + s)
$$

is collected into $\tilde{\Gamma}^{A_0}(\delta t, x; 0)$. Differentiating (2.25) again we get

$$
\begin{aligned}
u_{xx}(t,x) = {} & \int_{\mathbb{R}^+} \tilde{\Gamma}_x^{A_0}(\delta t, x; y) u_y(t-\delta t, y) dy + \int_0^{\delta t} \int_{\mathbb{R}^+} \tilde{\Gamma}_x^{A_0}(s,x;y) \big((A(\kappa,u) - A_0) u_y(t-s,y) \big)_y dy ds \\
& + \int_0^{\delta t} K_{xx}^{A_0}(s,x) \big(\dot{u}_b(t-s) + (A(\kappa,u_b) - A_0) u_x(t-s,0) \big) ds.
\end{aligned}
$$

Taking the L^1 norm of both sides, noting that

$$
\big| (A(\kappa,u_b) - A_0) u_x(t,0) \big| \leq C\delta_0 \big\| u_{xx}(t) \big\|_{L^1},
$$

we obtain

$$
\begin{aligned}
\big\| u_{xx}(t) \big\|_{L^1} \leq {} & \frac{C}{\sqrt{\delta t}} \sup_s \| u_x(s) \|_{L^1} + C\sqrt{\delta t} \| A - A_0 \|_{L^\infty} \sup_{t-\delta t \leq s \leq t} \| u_{xx}(s) \|_{L^1} \\
& + C \int_0^{\delta t} \frac{1}{\sqrt{s}} \big(|\dot{u}_b(t-s)| + \delta_0 \| u_{xx}(t-s) \|_{L^1} \big) ds \\
\leq {} & \frac{C}{\sqrt{\delta t}} \delta_0 + C\sqrt{\delta t}\delta_0 \sup_{t-\delta t \leq s \leq t} \| u_{xx}(s) \|_{L^1} + C\sqrt{\delta t}\delta_0 \big(1 + \sup_{t-\delta t \leq s \leq t} \| u_{xx}(s) \|_{L^1} \big).
\end{aligned}
$$

Choosing $\sqrt{\delta t} = \min\{1, C/\delta_0\} = 1$ for $\delta_0 \ll 1$, and noting that for $t \in [0, \delta t]$ the estimate $\| u_{xx}(t) \|_{L^1} \leq C\delta_1$ follows from the regularity of the initial data, we obtain the estimate

E:regues1

$$
\| u_{xx}(t) \|_{L^1} \leq C\delta_0, \tag{2.26}
$$

for all $t \in [0,T]$ such that $\| u_x(t) \|_{L^1} \leq 3\delta_0$.

R:proof1

Remark 2.2. Note that the estimate depends on the total variation of u and the total variation of u_b, not of their squared values as in [2]. Differently from the boundary free case, one may have that $u_{xx} \simeq u_x$, as can be shown by considering the asymptotic solution of (2.8) for $\lambda < 0$:

$$
\lim_{t \to +\infty} K(t,x) = e^{\lambda x}, \quad \| (e^{\lambda x}_{xx} \|_{L^1} = -\lambda.
$$

A similar computation can be used for u_{tx}: since in this case we know the boundary data (which is the time derivative of u_b), there is no need to pass to the dual kernel $\tilde{\Gamma}$. The function u_t satisfies the equation

E:systemut879

$$
(u_t)_t + A(\kappa,u)(u_t)_x - (u_t)_{xx} + (DA(\kappa,u)u_t)u_x + A_\kappa(\kappa,u)\dot{\kappa}u_x = 0, \tag{2.27}
$$

and has bounded L^1 norm by the assumption on u_x and the estimate (2.26). We can write $u_t(t,x)$ by Duhamel formula as

E:duhamel176

$$
\begin{aligned}
u_t(t,x) = & \int_{\mathbb{R}^+} \Gamma^{A_0}(\delta t, x; y) u_t(t-\delta t, y) dy - \int_0^{\delta t}\int_{\mathbb{R}^+} \Gamma^{A_0}(s,x;y) A_\kappa(\kappa,u)\dot\kappa u_x(t-s,y) dyds \\
& + \int_0^{\delta t}\int_{\mathbb{R}^+} \Gamma^{A_0}(s,x;y)\Big((A_0 - A(\kappa,u)) u_{tx}(t-s,y) + (DA(\kappa,u)u_t) u_x(t-s,y)\Big) dyds \\
& + \int_0^{\delta t} K^{A_0}(s,x) \frac{d^2}{dt^2} u_b(t-s) ds + K^{A_0}(\delta t, x)\dot u_b(t-\delta t).
\end{aligned} \tag{2.28}
$$

Differentiating (2.28) once we obtain

$$
\begin{aligned}
u_{tx}(t,x) = & \int_{\mathbb{R}^+} \Gamma_x^{A_0}(\delta t, x; y) u_t(t-\delta t, y) dy - \int_0^{\delta t}\int_{\mathbb{R}^+} \Gamma_x^{A_0}(s,x;y) A_t(\kappa,u)\dot\kappa u_x(t-s,y) dyds \\
& + \int_0^{\delta t}\int_{\mathbb{R}^+} \Gamma_x^{A_0}(s,x;y)\Big((A_0 - A(\kappa,u)) u_{tx}(t-s,y) + (DA(\kappa,u)u_t) u_x(t-s,y)\Big) dyds \\
& + \int_0^{\delta t} K_x^{A_0}(s,x) \frac{d^2}{dt^2} u_b(t-s) ds + K_x^{A_0}(\delta t, x)\dot u_b(t-\delta t),
\end{aligned}
$$

As before

$$
\big| (A(\kappa,u_b) - A_0) u_{tx}(t,0) + (DA(\kappa,u_b)u_t) u_t(t,0)\big| \leq C\delta_0 \big\|u_{tx}(t)\big\|_{L^1}.
$$

By means of (2.9) we conclude for $\delta t \leq t \leq T$ that

$$
\begin{aligned}
\big\|u_{tx}(t)\big\|_{L^1} \leq & \frac{C}{\sqrt{\delta t}} \sup_s \|u_t(s)\|_{L^1} + C\sqrt{\delta t}\|A_\kappa \dot\kappa\|_{L^\infty}\delta_0 + C\sqrt{\delta t}\delta_0 \sup_{t-\delta t \leq s \leq t} \|u_{tx}(s)\|_{L^1} \\
& + C\int_0^{\delta t} \frac{1}{\sqrt{s}} \left| \frac{d^2}{dt^2} u_b(t-s)\right| ds + C\|\dot u_b\|_{L^\infty} \\
\leq & \frac{C}{\sqrt{\delta t}}\delta_0 + C\sqrt{\delta t}\delta_0 \sup_{t-\delta t\leq s\leq t} \|u_{tx}(s)\|_{L^1} + C\sqrt{\delta t}\delta_0 + C\delta_0.
\end{aligned}
$$

Choosing $\sqrt{\delta t} = \min\{1, \mathcal{O}(1)\delta_1\} = 1$ for $\delta_0 \ll 1$, and noting that for $0 \leq t \leq \delta t$ by standard techniques one has $\|u_{tx}(t)\|_{L^1} \leq \mathcal{O}(1)\delta_0$, we can write

E:regues43

$$
\|u_{tx}(t)\|_{L^1} \leq C\delta_0. \tag{2.29}
$$

for $t \in [0,T]$, the maximal time interval such that $\|u_x(t)\|_{L^1} \leq 3\delta_0$.

Since the equation for a perturbation h is the same equation satisfied by u_t, the only difference being the term $A_\kappa(t,u)\dot\kappa u_x$, we obtain also the estimate

E:regues44

$$
\|h_x(t)\|_{L^1} \leq C\delta_0, \qquad t \in [0,T], \tag{2.30}
$$

if the L^1 norm of $h(t) \leq 3\delta_0$ for $t \leq T$. At this point we can proceed with the estimate of h_{xx}, by following the same procedure used to estimate u_{xx}, i.e. by passing to the kernel $\tilde\Gamma^{A_0}$. With similar computations, one can show that

$$
\|h_{xx}\|_{L^1} \leq C\delta_0,
$$

so that it follows

E:regues45

$$
\|\iota_x\|_{L^1} \leq C\delta_0. \tag{2.31}
$$

We note that the estimates of u_{tx} and ι_x at the boundary means that u_x, h_x have boundary conditions which are smooth in time and bounded. Thus we can proceed with the same arguments as before and obtain regularity estimates for u_{xxx}, h_{xxx}, and so on up to the regularity of A, κ.

The above estimates can be collected by saying that if the total variation of u is sufficiently small in $[0,T]$ and the initial-boundary data are sufficiently smooth and close to a constant $\bar u$, then all the higher derivatives of u exists and are of the same order of $\|u_x\|_{L^1}$. In particular we obtain also the estimates of the L^∞ norm of the derivatives, which are all of the order δ_0.

S:deco1

3. A decomposition for ODE

We consider the ODE

E:odeexa1

$$\dot z = A(z), \qquad z = (z^e, z^0, z^-) \in \mathbb{R}^i \otimes \mathbb{R}^j \otimes \mathbb{R}^k, \tag{3.1}$$

$A(z) \in C^r(\mathbb{R}^{i+j+k})$, and we assume the following:

(1) the matrix $DA(0)$ has $i+j$ eigenvalues with positive real part and k eigenvalues with real part strictly less than 0;
(2) in a small neighborhood of $z=0$, there exists a smooth i dimensional hypersurface E of equilibria, parameterized as
$$z = h(s_1, \dots, s_i), \qquad h(0) = 0,$$
where h is a smooth function with $\mathrm{rank}(Dh) = i$. We can assume that $E = \{z^0 = 0, z^- = 0\}$, i.e. it coincides with the subspace of the variables z^e.

A classical application of the stable manifold gives the existence of an invariant manifold $\mathcal{C}^s$ of dimension k, which contains all the orbits converging exponentially fast to 0. Since here we know the existence of a i-dimensional set of equilibria, a natural question is whether we can extend this manifold to include all the orbits of (3.1) which converges with uniform exponential speed to some equilibrium in $E = \{z = h(s), s \in \mathbb{R}^i\}$.

We look thus for the following set:

all the trajectories which converge exponentially fast to the equilibria $E = \{z = h(s), s \in \mathbb{R}^i\}$.

We will call this manifold the *center uniformly-stable (CUS) manifold* $\mathcal{C}^{cus}$ of the set E, to emphasize the uniform convergence to 0 of the variables z^0, z^-. The difference with the stable manifold is that here we do not specify the equilibrium point to which the trajectory should converge: this will enlarge the manifold to dimension $i+k$, while the stable manifold has only dimension k.

We will thus prove the following theorem:

T:cusecx1

Theorem 3.1. *Consider the system* (3.1)*, with $A(z) \in C^r$ and satisfying assumptions 1), 2). Then, in a neighborhood of $z = 0$, there exists an invariant $i+k$ dimensional manifold $\mathcal{C}^{cus}$ of class C^r, characterized uniquely by the fact that the orbits on it converge with uniform exponential speed to some equilibria $z = (z^e, 0, 0) \in E$. The manifold $\mathcal{C}^{cus}$ is tangent in each equilibria $z^e \in E$ to the eigenspace $R^-(z^e)$ generated by the k most negative eigenvalues of $A(z^e, 0, 0)$. In particular, it can be parameterized by (z^e, z^-),*

E:invmn809

$$E = \Big\{ (z^e, z^-, \phi(z^e, z^-)), |z^e|, |z^-| \ll 1 \Big\}. \tag{3.2}$$

Observe that the eigenspace $R^-(z^e)$ exists certainly because near $z=0$ there is a spectral gap among the k most negative eigenvalues and the others.

R:firtuu8

Remark 3.2. Note that assumption 1) is verified on the center stable manifold $\mathcal{C}^{cs}$ of any equilibrium point $\bar u$. Note moreover that by time reversal one can prove the same results on the center unstable manifold, obtaining in this way a center uniformly unstable manifold $\mathcal{C}^{cuu}$.

It is important to observe also that the set E does not need to contains all equilibria close to $z=0$. We only require the existence of a smooth manifold of equilibria E to obtain this invariant manifold. Of course this manifold does not contain all the orbit which decay exponentially to an equilibrium, as fig. 3 shows.

The proof of the above theorem follows as a corollary from the following Hadamar-Perron theorem [8]:

T:hadper1

Theorem 3.3. *Let $f_m : \mathbb{R}^n \mapsto \mathbb{R}^n$, $m \in \mathbb{Z}$, be a C^r diffeomorphism, $r \geq 1$ of the form*

$$f_m(x,y) = (A_m x + \alpha_m(x,y), B_m y + \beta_m(x,y)), \qquad (x,y) \in \mathbb{R}^k \times \mathbb{R}^{n-k},$$

such that $A_m : \mathbb{R}^k \mapsto \mathbb{R}^k$, $B_m : \mathbb{R}^{n-k} \mapsto \mathbb{R}^{n-k}$ and, for some $\lambda < \mu$, $\|A_m^{-1}\| \leq 1/\mu$, $\|B_m\| \leq \lambda$, and moreover $\alpha_m(0) = 0$, $\beta_m(0) = 0$.

There exists a $\gamma_0 = \gamma_0(\lambda, \mu)$ such that for all $\gamma \in (0, \gamma_o)$ there is a $\delta = \delta(\lambda, \mu, \gamma)$ such that, if $\|\alpha_m\|_{C^1}, \|\beta_m\|_{C^1} \leq \delta$ for all $m \in \mathbb{Z}$, there is

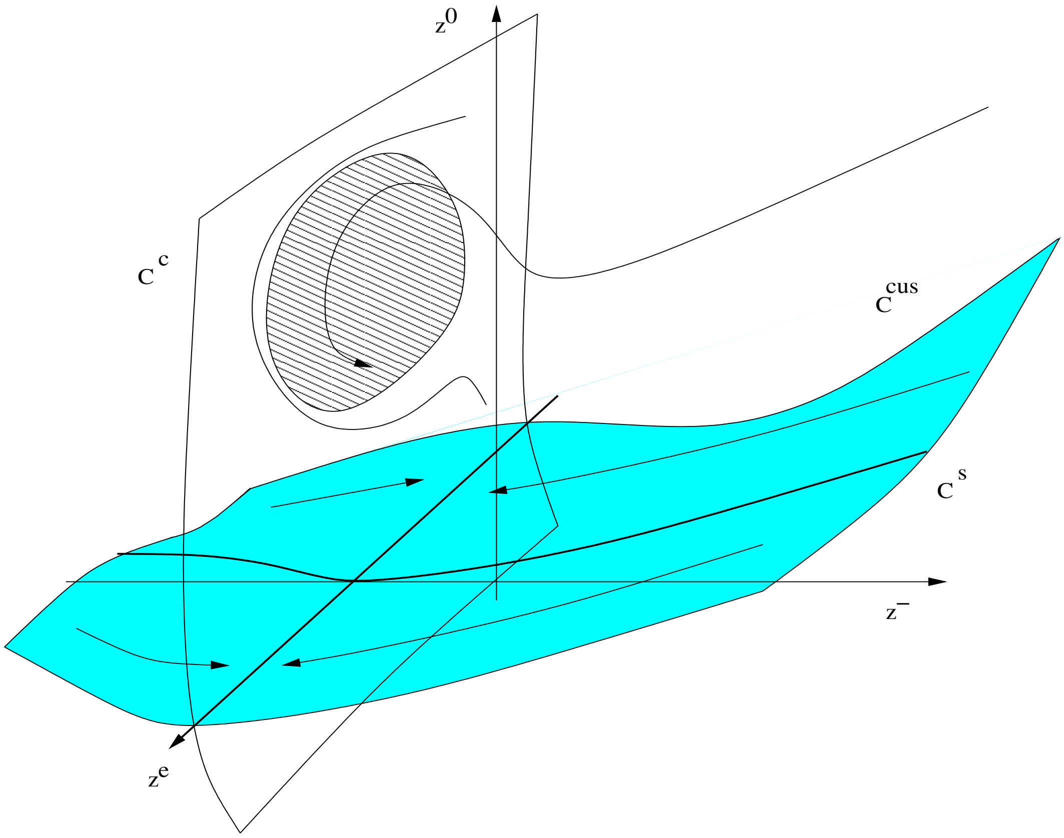

FIGURE 3. The center manifold $\mathcal{C}^c$ and the center uniformly stable manifold $\mathcal{C}^{cus}$. The dashed region contains other equilibria, not the ones in $\{z^0 = 0, z^- = 0\}$. Moreover there maybe other orbits converging to an equilibrium in E, but with speed much lower than on $\mathcal{C}^{cus}$. Fi:bounriem1

- *a unique family* $\{W^+_m\}_{m\in\mathbb{Z}}$ *of* k*-dimensional* C^1 *manifolds of the form*

$$W^+_m = \Big\{(x, \phi^+_m(x)), x \in \mathbb{R}^k\Big\},$$

- *a unique family* $\{W^-_m\}_{m\in\mathbb{Z}}$ *of* k*-dimensional* C^1 *manifolds of the form*

$$W^-_m = \Big\{(\phi^-_m(y), y), y \in \mathbb{R}^{n-k}\Big\},$$

with $\sup_m \|D\phi^\pm_m\| < \gamma$ *and such that*

(1) *the families* $\{W^\pm_m\}_{m\in\mathbb{Z}}$ *are invariant for* f_m*, i.e.* $f_m(W^\pm_m) = W^\pm_{m+1}$*;*
(2) *the maps* f_m *have a defined behavior on* $\{W^\pm_m\}_{m\in\mathbb{Z}}$*,*

$$\|f_m(z)\| < (1+\gamma)(\lambda + \delta(1+\gamma))\|z\|, \qquad z \in W^-_m,$$

$$\|f^{-1}_{m-1}(z)\| < (\mu/(1+\gamma) - \delta)^{-1}\|z\|, \qquad z \in W^+_m;$$

(3) *if* $\nu \in (\lambda, \mu)$*, and* $\|f_{m+L-1} \circ \ldots \circ f_m(z)\| < C\nu^L\|z\|$ *for all* $L \geq 0$ *and some* C*, then* $z \in W^-_m$*. Similarly, if* $\|f^{-1}_{m-L} \circ \ldots \circ f^{-1}_{m-1}(z)\| < C\nu^{-L}\|z\|$ *for all* $L \geq 0$ *and some* C*, then* $z \in W^+_m$*.*

Moreover the families $\{W^\pm_m\}_{m\in\mathbb{Z}}$ *depend continuously in* C^1 *topology from the family of maps* $\{f_m\}_{m\in\mathbb{Z}}$*.*

Finally, if $\gamma, \delta \ll 1$ *and* $\lambda < 1$ *(*$\mu > 1$*), then* $\{W^-_m\}_{m\in\mathbb{Z}}$ *(*$\{W^+_m\}_{m\in\mathbb{Z}}$*) is* C^r *and depends continuously in the* C^r *topology from the family of maps* $\{f_m\}_{m\in\mathbb{Z}}$*.*

This theorem is essentially the one given in [8], where we used the remarks given in the same book at the end of its proof to improve the C^r continuous dependence, see also the existence of stable and unstable manifolds for hyperbolic flows [8]. We observe that the family of smooth manifolds is the one for which we know that the behavior of the orbits is exponential decreasing, namely $\{W^-_m\}_{m\in\mathbb{Z}}$ if $\lambda < 1$ and $\{W^+_m\}_{m\in\mathbb{Z}}$ if $\mu > 1$.

We now prove Theorem 3.1:

Proof. Since the region E is positively invariant under the flow, and the manifold we are looking for contains orbit which remains in a neighborhood if $z = 0$, by considering a smooth cutoff function, we can assume that the function $A(z)$ can be written as

E:newfrte1
$$A(z) = (A^{e0}(z), A^-(z)) = (DA^{e0}(0)(z^e, z^0) + \alpha(z), DA^-(0)z^- + \beta(z)), \tag{3.3}$$

with α, β quadratic near $z = 0$ and with C^1 norm sufficiently small. Let $\psi : \mathbb{R}^{i+j+k} \mapsto \mathbb{R}^{i+j+k}$ be the flow map with 1 time step, which clearly satisfies

E:newfrte2
$$\psi(z) = \Big(e^{DA^{e0}(0)}(z^e, z^0) + \alpha'(z), e^{DA^-(0)}z^- + \beta'(z)\Big), \tag{3.4}$$

with α', β' quadratic and sufficiently small.

For any fixed $z \in E$ close to 0 there is a spectral gap among the k most negative and the other $i + j$ eigenvalues. It follows that for some $\lambda < \mu$, $\lambda < 1$, we have the estimates $\|e^{-DA^{e0}(0)}\| \leq 1/\mu$, $\|e^{DA^-(0)}\| \leq \lambda$, and moreover $\psi(z^e, 0, 0) = z^e$.

Fixed any $\bar z = (\bar z^e, 0, 0) \in E$, we can thus write the map ψ as $\psi_{\bar z}(z) = \psi(z) - \psi(\bar z)$, and apply Theorem 3.3 to the family of maps $\{f_m(z) = \psi_{\bar z}\}_{m\in\mathbb{Z}}$, to obtain a unique C^r smooth k dimensional manifold $W^-(\bar z)$, parameterized by $R^-(z^e)$. Since we are close to 0, it is clear that we can parameterize $W^-(\bar z)$ by $(0, 0, z^-)$, i.e. $W^-(\bar z) = \{(\phi^-(\bar z^e, z^-), z^-)\}$, with $(\phi(\bar z^-, 0), 0) = \bar z$.

The last thing to prove is that the bundle $\cup_{z\in E} W^-(z)$ is actually a manifold parameterized by $(z^e, 0, z^-)$ near $z = 0$. This clearly occurs because the map $(z^e, 0, z^-) \mapsto (\phi(z^e, z^-), z^-)$ is smooth and invertible in a neighborhood of $z = 0$. $\square$

R:behaeq1
Remark 3.4. Using the above invariant manifolds, i.e. the center-unstable manifold and the uniformly stable manifold, we can rewrite the equations (3.1) as follows. Define the new variable $\tilde z = (\tilde z^e, \tilde z^0, \tilde z^-)$ by

E:newch01
$$\left\{\begin{array}{rcl} \tilde z^e &=& z^e \\ \tilde z^0 &=& z^0 - \mathcal{C}^{cus}(z^e, z^-) \\ \tilde z^- &=& z^- - \mathcal{C}^c(z^e, z^0) \end{array}\right. \tag{3.5}$$

Note that this transformation is locally invertible and smooth. At this point it is clear that $\tilde z^0 = 0$ defines the uniformly stable manifold, while $\tilde z^- = 0$ defines the center manifold, so that one can check that in these coordinates the equations become

E:odenew09
$$\left\{\begin{array}{rcl} d\tilde z^e/dt &=& A^e \tilde z^0 + g(\tilde z) \\ d\tilde z^0/dt &=& (A^0 + h^0(\tilde z))\tilde z^0 \\ d\tilde z^-/dt &=& (A^- + m^-(\tilde z))\tilde z^- \end{array}\right. \tag{3.6}$$

where $g(0) = h^0(0) = m^-(0) = 0$. One can thus verify that the trajectories in $\mathcal{C}^{cus}$ are uniformly exponentially decaying to an equilibrium.

Another observation is that a similar proof can be used to find the center uniformly stable bundle for a positively invariant region E in a sufficiently small neighborhood of $z = 0$. In this case, depending on E, it can be a smooth manifold or only an invariant region. The same results hold for unstable manifolds of negatively invariant regions.

We observe moreover that one can obtain the invariant manifolds converging to E with a prescribed exponential speed λ, if λ belongs to some spectral gap of the eigenvalues of the linear part DA. This will be used later on.

Ss:applpara1
3.1. Application to parabolic systems with boundary.

Consider now the parabolic PDE depending on a parameter

E:parabi1
$$u_t + A(\kappa, u)u_x = u_{xx}, \qquad (t, x) \in \big\{t, x > 0\big\}, \qquad u \in \mathbb{R}^n, \tag{3.7}$$

$A(\kappa, u)$ smooth and strict hyperbolic for all κ close to 0. The equation for the boundary layer with speed $\sigma = 0$ is in first order form

E:boupara1
$$\left\{ \begin{array}{rcl} u_x &=& p \\ p_x &=& A(\kappa, u)p \\ \kappa_x &=& 0 \end{array} \right. \tag{3.8}$$

For u, κ close to $\bar{u}$, 0 respectively, we assume that there are $k-1$ eigenvalues of $A(\kappa, u)$ strictly less than 0, the k-th eigenvalue is close to 0 and the other $n-k$ are strictly greater than 0. Without any loss of generality we assume that $\lambda_k(u = \bar{u}, \kappa = 0) = 0$.

It is clear that the set $\{p = 0\}$ is a $n+1$ dimensional set of equilibria, so that by means of Theorem 3.1, there exists an invariant manifold $\mathcal{C}^{cus}$ of dimension $n+k$, which can be written as

E:parainv1
$$p = \tilde{R}_s(\kappa, u, p_s)p_s, \tag{3.9}$$

where $p_s = (p_1, \dots, p_{k-1})$, $\tilde{R}_s$ smooth function: in fact, the equilibrium manifold is $p = 0$, hence the function ϕ of (3.2) vanishes when $p_s = 0$. This manifold contains all the orbits converging to the equilibria $\{p = 0\}$ with uniformly exponential speed.

Since we are considering the boundary-characteristic case, the idea is to find generalized eigenvectors (or eigenspaces, in the case of boundary layers) which can describe a travelling profile of the k-th family, a non characteristic boundary profile or a characteristic boundary profile. An important requirement is that the equations for these profiles should be in conservation form, so that we can expect BV bounds on all three cases. We thus consider

E:boupara10
$$\left\{ \begin{array}{rcl} u_x &=& p \\ p_x &=& A(\kappa, u)p \\ \kappa_x &=& 0 \end{array} \right. \tag{3.10}$$

By the assumptions on A there are $k-1$ eigenvalues strictly negative, 1 eigenvalue close to 0, $n-k$ strictly positive eigenvalues and $n+1$ null eigenvalues.

The first step is to reduce the equations on the center stable manifold $\mathcal{C}^{cs}$ of an equilibrium, let us say $(u = \bar{u}, p = 0, \kappa = 0)$. This manifold can be written as

E:centerst1
$$p = R_{cs}(\kappa, u, v_{cs})v_{cs}, \qquad \langle R_{cs}(\kappa, u, v_{cs}), R_{cs}(\kappa, u, v_{cs})\rangle = I \in \mathbb{R}^{k \times k}. \tag{3.11}$$

where $R_{cs} \in \mathbb{R}^{n \times k}$ satisfies the fundamental matrix relation (due to invariance of the center stable manifold)

E:commRcs1
$$R_{cs}A_{cs} + DR_{cs}(R_{cs}v_{cs}) + R_{cs,v}(A_{cs}v_{cs}) = A(\kappa, u)R_{cs}, \quad A_{cs} = \langle R_{cs}, A(\kappa, u)R_{cs}\rangle. \tag{3.12}$$

On the center-stable manifold we thus obtain the reduced system

E:bouparedu1
$$\left\{ \begin{array}{rcl} u_x &=& R_{cs}v_{cs} \\ v_{cs,x} &=& A_{cs}(\kappa, u, v_{cs})v_{cs} \\ \kappa_x &=& 0 \end{array} \right. \tag{3.13}$$

It is clear that the above system has by construction $k-1$ strictly negative eigenvalues, one eigenvalue close to 0, and $n+1$ null eigenvalues. Moreover the manifold of equilibria $\{p = 0\}$ becomes now $\{v_{cs} = 0\}$, its dimension $n+1$.

At this point we write the center manifold of (3.13),

E:reducenma1
$$v_{cs} = v_k r_k(\kappa, u, v_k), \qquad \langle r_k, r_k\rangle = 1, \tag{3.14}$$

satisfying

E:commrc1
$$r_k\tilde{\lambda}_k + Dr_k(R_{cs}r_kv_k) + r_{k,v}\tilde{\lambda}_kv_k = A_{cs}r_k, \quad \tilde{\lambda}_k = \langle r_k, A_{cs}(\kappa, u, v_kr_k)r_k\rangle. \tag{3.15}$$

and the uniformly stable manifold,

E:redustama1
$$v_{cs} = R_s(\kappa, u, v_s)v_s, \qquad \langle R_s, R_s\rangle = I \in \mathbb{R}^{(k-1)\times(k-1)}, \tag{3.16}$$

E:commrs1
$$R_s\tilde{A}_s + DR_s(R_{cs}R_sv_s) + R_{s,v}(\tilde{A}_sv_s) = A_{cs}R_s, \quad \tilde{A}_s = \langle R_s, A_{cs}(\kappa, u, R_sv_s)R_s\rangle. \tag{3.17}$$

It is clear that the two vectors

E:previousvect1

$$\begin{aligned}\tilde r_k(\kappa,u,v_k) &= R_{cs}(\kappa,u,v_k r_k(\kappa,u,v_k)) r_k(\kappa,u,v_k),\\ \tilde R_s(\kappa,u,v_s) &= R_{cs}(\kappa,u,R_s(\kappa,u,v_s)v_s) R_s(\kappa,u,v_s),\end{aligned} \tag{3.18}$$

are the tangent vectors to the center and uniformly stable manifold of the original system (3.10). In fact

$$\begin{aligned}D\tilde R_s \tilde R_s v_s + \tilde R_{s,v}\tilde A_s v_s &= DR_{cs}(R_{cs}R_s v_s)R_s + R_{cs,v}(DR_s(R_{cs}R_s v_s)v_s)R_s + R_{cs}DR_s(R_{cs}R_s v_s)\\ &\quad + R_{cs,v}(R_s\tilde A_s v_s + R_{s,v}(\tilde A_s v_s)v_s)R_s + R_{cs}R_{s,p}\tilde A_s v_s\\ &= DR_{cs}(R_{cs}R_s)R_s v_s - \sigma R_{cs,v}R_s v_s v_s\\ &\quad + R_{cs,v}(R_s\tilde A_s + DR_s(R_{cs}R_s v_s) - R_{s,v}\tilde A_s v_s)v_s R_s\\ &\quad + R_{cs}(DR_s(R_{cs}R_s v_s) + R_{s,v}\tilde A_s v_s)\\ &= DR_{cs}(R_{cs}R_s)R_s v_s - R_{cs,v}A_{cs}R_s v_s v_s\\ &\quad + R_{cs}A_{cs}R_s - R_{cs}R_s\tilde A_s\\ &= A\tilde R_s - \tilde R_s \tilde A_s,\end{aligned}$$

and similarly for $\tilde r_k$.

Up to now we have obtained 3 functions R_{cs}, R_s, r_k, which describe the center-stable, uniformly stable and center manifold respectively, and the reduced equations on these invariant manifolds. The next step is to diagonalize the ODE on the center stable manifold as in Remark 3.4, equation (3.6).

On the center-stable manifold, we decompose v_{cs} as in (3.5), which in our case becomes

E:newch03

$$v_{cs} = R_s(\kappa,u,v_s)v_s + v_k\tilde r_k(\kappa,u,v_k), \tag{3.19}$$

corresponding to the decomposition of the original vector $p \in \mathbb{R}^n$

E:gendeco05

$$\begin{aligned}p &= R_{cs}(\kappa,u,R_s v_s + v_k r_k)R_s(\kappa,u,v_s)v_s + R_{cs}(\kappa,u,R_s v_s + v_k r_k) r_k(\kappa,u,v_k)v_k\\ &= \hat R_s(\kappa,u,v_s,v_k)v_s + \hat r_k(\kappa,u,v_s,v_k)v_k.\end{aligned} \tag{3.20}$$

Substituting (3.20) in (3.8), by direct differentiation we obtain

$$\begin{aligned}&(R_s + R_{s,v}\cdot v_s)(v_{s,x} - \tilde A_s v_s) + (r_k + r_{k,v}v_k)(v_{k,x} - \tilde\lambda_k v_k)\\ &\qquad + DR_s(R_{cs}r_k v_k)v_s + Dr_k(R_{cs}R_s v_s)v_k = 0,\end{aligned}$$

so that if $(L_s(\kappa,u,v_s,v_k), l_k(\kappa,u,v_s,v_k))$ is the dual base of

$$\Big(R_s(\kappa,u,v_s) + (R_{s,v}(\kappa,u,v_s)\cdot)v_s, r_k(\kappa,u,v_k) + r_{k,v}(\kappa,u,v_k)v_k\Big),$$

and defining

E:truefluxS1

$$\begin{aligned}\hat A_s(\kappa,u,v_s,v_k) &= \tilde A_s(\kappa,u,v_s) - \langle L_s, DR_s(R_{cs}r_k v_k) + Dr_k(R_{cs}R_s\cdot)v_k\rangle\\ &= \tilde A_s(\kappa,u,v_s) + \mathcal{O}(1)v_k,\end{aligned} \tag{3.21}$$

E:truefluxC1

$$\begin{aligned}\hat\lambda_k(\kappa,u,v_s,v_k) &= \tilde\lambda_k(\kappa,u,v_k) - \langle l_k, DR_s(R_{cs}r_k\cdot)v_s + Dr_k(R_{cs}R_s v_s)\rangle\\ &= \tilde\lambda_k(\kappa,u,v_k) + \mathcal{O}(1)v_s,\end{aligned} \tag{3.22}$$

we obtain

E:finalsyt01

$$\left\{\begin{array}{rcc} u_x &=& R_{cs}(\kappa,u,R_s v_s + r_k v_k)(R_s(\kappa,u,v_s)v_s + r_k(\kappa,u,v_k)v_k)\\ v_{s,x} &=& \hat A_s(\kappa,u,v_s,v_k)v_s\\ v_{k,x} &=& \hat\lambda_k(\kappa,u,v_s,v_k)v_k\\ \kappa_x &=& 0\end{array}\right. \tag{3.23}$$

These are the diagonalized equations on the center stable manifold. Note that trivially on has

$$\hat A_s(\kappa,u,v_s,0) = \tilde A_s(\kappa,u,v_s), \qquad \hat\lambda_k(\kappa,u,0,v_k) = \tilde\lambda_k(\kappa,u,v_c),$$

so that the $k-1$ eigenvalues of $\hat A_s$ and the generalized eigenvalue $\hat\lambda_k$ are close to the original k most negative eigenvectors of $A(u)$. In particular the eigenvalues of $\hat A_s$ are strictly negative and $\hat\lambda_k$ is close to 0 for (κ,u,v_s,v_k) close to $(0,\bar u,0,0)$.

Together with these eigenvectors, we recall from [3] that there are the i-th generalized eigenvectors and eigenvalues, for $i \neq k$, obtained with the following procedure. Consider the center manifold for the ODE

E:ipara1
$$\left\{ \begin{array}{ccc} u_x & = & p \\ p_x & = & (A(\kappa, u) - \sigma_i I)p \\ \kappa_x & = & 0 \\ \sigma_{i,x} & = & 0 \end{array} \right. \tag{3.24}$$

near the equilibrium point $(\bar{u}, 0, 0, \lambda_i(\bar{u}))$. This manifold has dimension $n+2$, and can be written as

E:icenma1
$$p = v_i \tilde{r}_i(\kappa, u, v_i, \sigma_i), \qquad \langle \tilde{r}_i, \tilde{r}_i \rangle = 1. \tag{3.25}$$

By direct substitution in (3.24) it follows

E:estiri1
$$\tilde{r}_i(\kappa, u, 0, \sigma_i) = r_i(u), \qquad \tilde{r}_{i,\sigma} = \mathcal{O}(1) v_i. \tag{3.26}$$

As above, related to $\tilde{r}_i$ there is the generalized eigenvalue $\tilde{\lambda}_i$,

E:ieffedre1
$$\tilde{\lambda}_i(\kappa, u, v_i, \sigma_i) = \langle \tilde{r}_i(\kappa, u, v_i, \sigma_i), A(\kappa, u) \tilde{r}_i(\kappa, u, v_i, \sigma_i) \rangle, \tag{3.27}$$

which generate the commutation relation

E:commuri1
$$(A(\kappa, u) - \tilde{\lambda}_i) \tilde{r}_i = v_i D\tilde{r}_i \tilde{r}_i + v_i \tilde{r}_{i,v} (\tilde{\lambda}_i - \sigma_i), \tag{3.28}$$

and moreover

E:closetr1
$$\tilde{\lambda}_i(\kappa, u, 0, \sigma_i) = \lambda_i(\kappa, u), \qquad \tilde{\lambda}_{i,\sigma} = \mathcal{O}(1) v_i^2. \tag{3.29}$$

It is clear that the $\tilde{r}_k$ constructed before corresponds to the $\tilde{r}_k$ constructed here when $\sigma_k = 0$, up to the uniqueness of the center manifold, i.e.

$$\tilde{r}_k(\kappa, u, v_k, 0) = R_{cs}\Big(\kappa, u, v_k r_k(\kappa, u, v_k)\Big) r_k(\kappa, u, v_k).$$

At this point we have the full set of vector (matrix) valued functions we need for the decomposition of Section 4. We define two vector (matrix) valued functions, which we call again $\hat{R}_b$, $\hat{r}_k$, by

E:decovess1
$$\begin{aligned} (\kappa, u, v_s, v_k) \mapsto \hat{R}_b(\kappa, u, v_s, v_k) &\doteq R_{cs}\Big(\kappa, u, R_s(\kappa, u, v_s) v_s + v_k r_k(\kappa, u, v_k)\Big) R_s(\kappa, u, v_s) \\ &= \hat{R}_s(\kappa, u, v_s, v_k), \end{aligned} \tag{3.30}$$

E:decovecc1
$$\begin{aligned} (\kappa, u, v_s, v_k, \sigma_k) \mapsto \hat{r}_k(\kappa, u, v_b, v_k, \sigma_k) &\doteq \tilde{r}_k(\kappa, u, v_k, \sigma_k) \\ &+ \bigg(R_{cs}\Big(\kappa, u, R_s(\kappa, u, v_s) v_s + v_k r_k(\kappa, u, v_k)\Big) r_k(\kappa, u, v_k) - \tilde{r}_k(\kappa, u, v_k, 0) \bigg) \end{aligned} \tag{3.31}$$

which satisfy the following relations

E:rel01
$$\hat{R}_b(\kappa, u, v_s) = \tilde{R}_s(\kappa, u, v_s), \qquad \hat{r}_k(\kappa, u, 0, v_k, \sigma_k) = \tilde{r}_k(\kappa, u, v_k, \sigma_k), \tag{3.32}$$

E:sigmdee1
$$\hat{r}_{k,\sigma_k} = \mathcal{O}(1)|v_k|. \tag{3.33}$$

Note that the vector $\hat{r}_k$ is essentially equal to $\tilde{r}_k$ with a perturbation due to the uniformly stable part of the boundary layer, which we expect (and we will prove later on) to be exponentially decreasing as we move away from the boundary. Note moreover that $\hat{r}_k$ never vanishes, because $|v_s| \ll 1$: we can thus assume that it is normalized to 1, i.e.

E:decovecc11
$$\hat{r}_k(\kappa, u, v_b, v_k, \sigma_k) = \frac{\tilde{r}_k(\kappa, u, v_k, \sigma_k) + \Big(\hat{r}_k(\kappa, u, v_s, v_k) - \tilde{r}_k(\kappa, u, v_k, 0)\Big)}{\Big|\tilde{r}_k(\kappa, u, v_k, \sigma_k) + \Big(\hat{r}_k(\kappa, u, v_s, v_k) - \tilde{r}_k(\kappa, u, v_k, 0)\Big)\Big|}. \tag{3.34}$$

When $\sigma_k = 0$, these vectors satisfy an important relation: by direct substitution,

E:commt776
$$\begin{aligned} &\Big(\hat{R}_s \hat{A}_s + D\hat{R}_s(\hat{R}_s v_s + v_k \hat{r}_k) + \hat{R}_{s,v_s} \hat{A}_s v_s + \hat{R}_{s,v_k} \hat{\lambda}_k v_k\Big) v_s + \\ &\Big(\hat{r}_k \hat{\lambda}_k + D\hat{r}_k(\hat{R}_s v_s + v_k \hat{r}_k) + \hat{r}_{k,v_s} \hat{\lambda}_k v_s + \hat{r}_{k,v_k} \hat{\lambda}_k v_k\Big) v_k = A(\kappa, u)(\hat{R}_s v_s + \hat{r}_k v_k). \end{aligned} \tag{3.35}$$

The generalized drift matrix $\hat{A}_s(\kappa, u, v_s, v_k)$ and the generalized eigenvalue $\hat{\lambda}_k(\kappa, u, v_s, v_k, \sigma_k)$ are defined in terms of (3.21), (3.22): for all κ close to 0, denote with $(L_s(\kappa, u, v_s, v_k), l_k(\kappa, u, v_s, v_k))$ the dual base (inverse matrix) of

$$\Big(R_s(\kappa, u, v_s) + R_{s,v}(\kappa, u, v_s) v_s, r_k(\kappa, u, v_k) + r_{k,v}(\kappa, u, v_k) v_k \Big).$$

Then set

E:truefluxS2

$$\begin{aligned} \hat{A}_s(\kappa, u, v_s, v_k) = {} & \tilde{A}_s(\kappa, u, v_s) \\ & - \Big\langle L_s(\kappa, u, v_s, v_k), DR_s(R_{cs} r_k v_k) + Dr_k(R_{cs} R_s() \cdot) v_k \Big\rangle, \end{aligned} \tag{3.36}$$

E:truefluxC2

$$\begin{aligned} \hat{\lambda}_k(\kappa, u, v_s, v_k, \sigma_k) = {} & \tilde{\lambda}_k(\kappa, u, v_k, \sigma_k) \\ & - \Big\langle l_k(\kappa, u, v_s, v_k), DR_s(R_{cs} r_k \cdot) v_s + Dr_k(R_{cs} R_s v_s) \Big\rangle. \end{aligned} \tag{3.37}$$

We obtain the following estimates for $\hat{A}_s$, $\hat{\lambda}_k$:

E:0drichr1

$$\hat{A}_s(\kappa, u, 0, 0) = A_s(\kappa, u), \qquad \hat{\lambda}_k(\kappa, u, 0, 0, \sigma_k) = \lambda_k(\kappa, u), \tag{3.38}$$

where $A_s(\kappa, u)$ is the $(k-1) \times (k-1)$ matrix corresponding to the projection of $A(\kappa, u)$ on the eigenspace of the strictly negative eigenvalues of $A(\kappa, u)$. By using (3.29) it follows also that

E:deriflux2

$$\hat{\lambda}_{k,\sigma_k} = \mathcal{O}(1) |v_k|^2. \tag{3.39}$$

S:compequ1

4. Equations for the components

In this section we write the equations satisfied by the decomposition of u_x in travelling profiles and boundary layer,

E:decompux01

$$u_x = \hat{R}_b(\kappa, u, v_b, v_k) v_b + v_k \hat{r}_k(\kappa, u, v_b, v_k, \sigma_k) + \sum_{i \neq k} v_i \tilde{r}_i(\kappa, u, v_i, \sigma_i), \tag{4.1}$$

E:decomput02

$$u_t = \hat{R}_b(\kappa, u, v_b, v_k) w_b + w_k \hat{r}_k(\kappa, u, v_b, v_k, \sigma_k) + \sum_{i \neq k} (w_i - \lambda_{i,0} v_i) \tilde{r}_i(\kappa, u, v_i, \sigma_i), \tag{4.2}$$

and the variable σ_i is given by

E:speedchu01

$$\sigma_i = \lambda_{i,0} - \theta\left(\frac{w_i}{v_i}\right), \qquad \theta(x) = \begin{cases} x & |x| \leq \delta_0 \\ \text{smooth connection} & \delta_0 < x \leq 3\delta_0 \\ 0 & |x| > 3\delta_0 \end{cases} \tag{4.3}$$

θ is a cutoff function, and δ_0 is a small constant. By using the parabolic equation

E:paraori1

$$u_t + A(\kappa, u) u_x = u_{xx}, \qquad u(0, x) = u_0(x), \;\; u(t, 0) = u_b(t), \tag{4.4}$$

and adding $k-1$ more conditions because we have $2(n+k-1)$ variables in $2n$ equations, one can obtain the equation for the components (v_b, v_k, v_i), (w_b, w_k, w_i), with suitable initial boundary terms (it is not important here to know their precise form), and then study the source terms of these equations. As noted in [3], the equation for the u_t variable is essential to choose appropriately the speed σ_k and σ_i $(i \neq k)$.

A further step is then to study the continuous dependence of the solution w.r.t. the initial and boundary data, i.e. to consider the equation satisfied by a perturbation $u + \epsilon h$ for $\epsilon \to 0$, and show that if the initial boundary data for the perturbation h are in L^1, then the L^1 norm of h remains bounded. This will give the continuous dependence of the solution w.r.t. the initial boundary data in the L^1 norm. Also for the perturbation one has to consider the equation for the *effective flux* $\iota = h_x - A(u) h$, which allows to define appropriately the speed for the waves in h.

Since the equations for a perturbation h and its related flux $\iota = h_x - A(u) h$ are satisfied in particular by u_x, u_t, then proving L^1 estimates for h, ι is the same task as proving L^1 estimates for u_x, u_t. The only difference is that the boundary conditions for u_x, h are different: in fact we know the boundary value u_b for the integral of u_x, while for h we have Dirichlet boundary conditions. Moreover for u_t we have Dirichlet boundary conditions given by $\dot{u}_b$, while the boundary condition for ι should be estimated from the solution h to the perturbation equation. We will remark how to handle the boundary conditions for the variables u_x, u_t every time their analysis differs from the equation of h, ι.

The equations for a perturbation h are

E:perturb1
$$h_t + (A(\kappa,u)h)_x - h_{xx} = (DAu_x)h - (DAh)u_x, \tag{4.5}$$

with initial boundary conditions

E:initpert1
$$h(0,x) = h_0(x), \qquad h(t,x_b(t)) = \bar h_b(t) \tag{4.6}$$

in L^1 and sufficiently regular, $h_0, \bar h_b \in W^{1,N}$, for some N sufficiently large. Due to linearity of the equation, we can assume that their norm is less that δ_0 in $W^{1,N}$. This simplifies the computations because the decomposition in travelling profiles will depend nonlinearly on h, ι, hence with this assumption we do not need any further rescaling. We introduce also the effective flux for the perturbation,

E:effectflux1
$$\iota = h_x - A(\kappa,u)h, \tag{4.7}$$

which satisfies the equations

E:iotaeq1
$$\begin{aligned} \iota_t + (A(\kappa,u)\iota)_x - \iota_{xx} = &- A_\kappa \dot\kappa h + \Big[DA(u_x \otimes h - h \otimes u_x) \Big]_x \\ &- A(\kappa,u) DA(u_x \otimes h - h \otimes u_x) + DA(u_x \otimes \iota - u_t \otimes h). \end{aligned} \tag{4.8}$$

The initial boundary conditions for ι are given by

E:initcodit1
$$\iota_0(x) = h_{0,x}(x) - A(\kappa(0),u_0(x))h_0(x), \qquad \bar\iota_b(t) = \lim_{x \to x_b(t)^+} h_x(t,x) - A(\kappa(t),u_b(t))\bar h_b(t). \tag{4.9}$$

Due to the assumption on h_0, we know that ι_0 is in $W^{1,N-1}$ and small. The boundary condition $\bar\iota_b$ is uniquely given by the solution h, and by Section 2 we know that it is bounded in $W^{1,N-1}_{\rm loc}$, but in this case we cannot deduce that it is integrable and small in $L^1(0,+\infty)$. This will be accomplished by studying the interaction inside the domain of the nonlinear waves of h.

R:diff08 *Remark* 4.1. While for the variable u_t the boundary conditions are given by $u_{t,b} = du_b/dt$, the boundary conditions $u_{x,b}$ for u_x are obtained indirectly by solving the equation, so that we do not know the L^1 norm of $u_{x,b}$ (even if by regularity estimates of Section 2 we know that are bounded and smooth). In this case, however, we will see that we can write all estimates in terms of the oscillations of u_b, plus some terms controlled also in this case by the initial data, the interaction of waves of u_x and the oscillations of κ.

Ss:bountr1

4.1. **Decomposition into boundary layer and travelling profiles.** We consider the following decomposition of the variable h,

E:decomp01
$$\begin{aligned} h &= \hat R_b(\kappa,u,v_b,v_k)h_b + h_k \hat r_k(\kappa,u,v_b,v_k,\zeta_k) + \sum_{i\neq k} h_i \tilde r_i(\kappa,u,v_i,\zeta_i) \\ &= \hat R_b h_b + h_k \check r_k + \sum_{i \neq k} h_i \check r_i, \end{aligned} \tag{4.10}$$

where $\hat R_b$ is the projector on the uniformly stable manifold of the boundary layer given by (3.30), $\hat r_k$ is the generalized eigenvector for the k-th characteristic field given by (3.31), $\tilde r_i$ is the generalized eigenvector of the i-th waves given by (3.25). We used the $\check r$ to distinguish the generalized eigenvector evaluated with the speed ζ_i of the perturbation h and the generalized eigenvector evaluated with speed σ_i, i.e. the speed of the component v_i of u_x. Similarly, in the following we will write

E:tildela1
$$\tilde\lambda_i = \tilde\lambda_i(\kappa,u,v_i,\sigma_i), \qquad \check\lambda_i = \tilde\lambda_i(\kappa,u,v_i,\zeta_i). \tag{4.11}$$

The same notation for the k-th characteristic field, where $\check\lambda_k$ is the k generalized eigenvalue (3.37) evaluated with speed ζ_k, while $\hat\lambda_k$ is the same function (3.37) evaluated with speed σ_k. Note that $\hat R_b$ is the same for the decomposition of u_x and h, and also $\hat A_s$ is the same, since they do not depend on the speeds ζ_i, $i = 1,\dots,n$, of the travelling profiles.

To define the speed ζ_i, $i=1,\dots,n$, we decompose the effective flux ι as

E:decomp02
$$\begin{aligned} \iota &= \hat R_b(\kappa,u,v_b,v_k)\iota_b + \iota_k \hat r_k(\kappa,u,v_b,v_k,\zeta_k) + \sum_{i\neq k}(\iota_i - \lambda_{i,0}h_i)\tilde r_i(\kappa,u,v_i,\zeta_i) \\ &= \hat R_b \iota_b + \iota_k \check r_k + \sum_{i\neq k}(\iota_i - \lambda_{i,0}h_i)\check r_i. \end{aligned} \tag{4.12}$$

Following (4.3), the variable ζ_i is given by

E:speedch01
$$\zeta_i = \lambda_{i,0} - \vartheta(\iota_i/h_i), \qquad \vartheta(x) = \begin{cases} x & |x| \leq \delta_0 \\ \text{smooth connection} & \delta_0 < x \leq 3\delta_0 \\ 0 & |x| > 3\delta_0 \end{cases} \tag{4.13}$$

As noted in [3], the speed in (4.13) is a discontinuous function of h_i, ι_i, so that the decomposition (4.10), (4.12), (4.13) is only Lipschitz continuous. Note moreover that up to now the decomposition is not assigned, due to the fact that on the left hand side we have the $2n$ dimensional vector (h, ι), but we will decompose it in $2n+2(k-1)$ components. However, one can prove that the decomposition is smooth outside a finite number of time t:

for any fixed functions (h_b, ι_b), there are perturbations m_b, n_b of h_b, ι_b, and perturbations m, n of h, ι, arbitrarily small in $W^{1,N} \cap C^N$, N arbitrary, such that outside a finite number of times t the decomposition (4.10), (4.12), (4.13) *is smooth in $\{t \in [0,T], x \geq 0\}$.*

As a consequence, once the decomposition is assigned with the procedure we consider below, we can consider an arbitrarily small perturbation to the equations of the components (h_i, ι_i) such that the decomposition is smooth outside a finite number of times.

Proof. First recall the following simple results. Let $\Xi : \mathbb{R}^k \supset K \mapsto \mathbb{R}^n$ a smooth map, K compact set, and consider a smooth set $E \subset \mathbb{R}^n$ of codimension $k' \geq 2$. For simplicity assume that E is the graph of a locally invertible map Ξ'. Then the following holds:

(1) if $k = k'$, then the maps Ξ which intersect E only a finite number of times are dense;
(2) if $k+1 \leq k'$, then the maps Ξ which never intersect E are dense.

The last result holds also if Ξ' can be approximated in C^0 by smooth maps.

We begin by decomposing the initial data only on the components (h_i, ι_i), $i = 1, \dots, n$. Since in this case the manifold where the map is only Lipschitz continuous is the set

$$\begin{aligned} E = &\bigg\{ (\kappa, h, \iota) : h = \sum_{j \neq k} h_j \tilde r_j, \iota = \sum_{j \neq k} (\iota_j - \lambda_{j,0} h_j) \tilde r_j \bigg\} \\ &\cup \bigg(\bigcup_{i \neq k} \bigg\{ (\kappa, h, \iota) : h = h_k \hat r_k + \sum_{j \neq i} h_j \tilde r_j, \iota = \iota_k \hat r_k + \sum_{j \neq i} (\iota_j - \lambda_{j,0} h_j) \tilde r_j \bigg\} \bigg), \end{aligned}$$

which has codimension 2, by a arbitrary small perturbation of the initial data we can assume that the map (4.10), (4.12), (4.13), with $h_b = \iota_b = 0$, is smooth at $t = 0$. In the following we will perturb slightly this map near $t = x = 0$: we can assume that also this perturbation does not intersect E.

The boundary condition for h, ι are thus defined by assuming that $h_i = \iota_i = 0$ for $i = 1, \dots, k-1$. Since also in this case the set

$$\begin{aligned} F = &\bigg\{ (\kappa, h, \iota) : h = \sum_{j > k} h_j \tilde r_j, \iota = \sum_{j > k} (\iota_j - \lambda_{j,0} h_j) \tilde r_j \bigg\} \\ &\cup \bigg(\bigcup_{i > k} \bigg\{ (\kappa, h, \iota) : h = h_k \hat r_k + \sum_{j > k, j \neq i} h_j \tilde r_j, \iota = \iota_k \hat r_k + \sum_{j > k, j \neq i} (\iota_j - \lambda_{j,0} h_j) \tilde r_j \bigg\} \bigg), \end{aligned}$$

has codimension 2, we can assume that the decomposition of the boundary data is smooth. In particular, the decomposition (4.10), (4.12), (4.13) will be smooth near $\{t = 0\} \cup \{x = 0\}$.

As we will see below, the variables h_b, ι_b satisfy an equation of the form

$$h_{b,t} + (\hat A_b(t,x) h_b)_x - h_{b,xx} = 0, \quad \iota_{b,t} + (\hat A_b(t,x) \iota_b)_x - \iota_{b,xx} = 0,$$

with $\hat A_b$ depending on the other components (see (3.36)), hence at least Lipschitz continuous. Since the eigenvalues of $\hat A_b$ are strictly negative, independently on the decomposition, we have that h_b, ι_b are exponentially decreasing (this will be verified later on). The same result holds if in the right hand side there is an exponentially decreasing source ϕ_b, ψ_b. Write

$$h_{b,t} + \hat A_b h_{b,x} - h_{b,xx} = \hat A_{b,x} h_b + \phi_b = \breve A_{b,x} h_b + (\hat A_{b,x} - \breve A_{b,x}) h_b + \phi_b = \breve A_{b,x} h_b,$$

where $\check{A}_b$ is a C^2 function converging to $\hat{A}_b$ in $C^{0,1}_{\mathrm{loc}}$, and similarly for ι_b. The source term ϕ_b is integrable and arbitrary small in $[0,T] \times \mathbb{R}^+$, because h_b is exponentially decreasing and $\check{A}_b$ converges to $\hat{A}_b$. Since it is clear that an arbitrary small term in the right hand side of h is not essential in the proof of L^1 bounds, we will neglect it in the following.

At this point we are left with the map $(\kappa, h_b, h_1, \dots, h_n, \iota_b, \iota_1, \dots, \iota_n) \mapsto (\kappa, h, \iota)$, with h_b, ι_b smooth in $\mathbb{R}^+ \times \mathbb{R}^+$. We describe the set E where the map (4.10), (4.12), (4.13) is only Lipschitz continuous. For any fixed h_b, ι_b, the set E is given by

$$
\begin{aligned}
E = & \bigg\{ (\kappa, h, \iota) : h = \hat{R}_b h_b + \sum_{j \neq k} h_j \tilde{r}_j, \iota = \hat{R}_b \iota_b + \sum_{j \neq k} (\iota_j - \lambda_{j,0} h_j) \tilde{r}_j \bigg\} \\
& \cup \bigg(\bigcup_{i \neq k} \bigg\{ (\kappa, h, \iota) : h = \hat{R}_b h_b + h_k \hat{r}_k + \sum_{j \neq i} h_j \tilde{r}_j, \iota = \hat{R}_b \iota_b + \iota_k \hat{r}_k + \sum_{j \neq i} (\iota_j - \lambda_{j,0} h_j) \tilde{r}_j \bigg\} \bigg),
\end{aligned}
$$

which has codimension 2. Moreover, the set of points where E is only Lipschitz is given by

$$
\begin{aligned}
F = & \bigg(\bigcup_{i \neq k} \bigg\{ (\kappa, h, \iota) : h = \hat{R}_b h_b + \sum_{j \neq k,i} h_j \tilde{r}_j, \iota = \hat{R}_b \iota_b + \sum_{j \neq k,i} (\iota_j - \lambda_{j,0} h_j) \tilde{r}_j \bigg\} \\
& \cup \bigg(\bigcup_{i,j \neq k, j > i} \bigg\{ (\kappa, h, \iota) : h = \hat{R}_b h_b + h_k \hat{r}_k + \sum_{\ell \neq i,j} h_\ell \tilde{r}_\ell, \iota = \hat{R}_b \iota_b + \iota_k \hat{r}_k + \sum_{\ell \neq i,j} (\iota_\ell - \lambda_{\ell,0} h_\ell) \tilde{r}_\ell \bigg\} \bigg),
\end{aligned}
$$

which has codimension 4. As a consequence, by a slight perturbation in C^2 we can assume that h, ι never intersect F. It follows that since $E \setminus F$ is regular, we can perturb h, ι is such a way that h, ι intersect $E \setminus F$ only a finite number of times in every compact set of $\mathbb{R}^2$. By the fact that h, ι are in L^1, the conclusion follows. □

The above result is the generalization to the boundary case of the following Lemma, which was stated in [3]:

L:regul01 **Lemma 4.2.** *Let $(h(t), \iota(t))$ be the solutions of (4.5), (4.8) in L^1 for $t \in [0,T]$ and consider the decomposition given by*

E:decomplem01 (4.14)
$$
\left\{
\begin{array}{rcl}
h & = & \sum_{i=1}^n h_i \tilde{r}_i(u, v_i, \zeta_i) \\
\iota & = & \sum_{i=1}^n (\iota_i - \lambda_{i,0} h_i) \tilde{r}_i(u, v_i, \zeta_i) \\
\zeta_i & = & \lambda_{i,0} - \vartheta(\iota_i / h_i)
\end{array}
\right.
$$

with u, v given smooth function. Then for all ϵ there exists a perturbation $(m,n) \in C^2$ of the right hand side of (4.5), (4.8), such that its $L^1 \cap C^2$ norm is less that ϵ and $(h, \iota)(t,x)$ solution to the perturbed equations belong to

$$
E = \bigcup_i \bigg\{ (h, \iota) \in \mathbb{R}^{2n} : h = \sum_{j \neq i} h_j \tilde{r}_j, \iota = \sum_{j \neq i} (\iota_j - \lambda_{j,0} h_j) \tilde{r}_j \bigg\}
$$

only a finite number $N(\epsilon)$ of times.

Under the hypotheses of regularity, we now compute the derivatives of h, ι. In this task, we need only to obtain the coefficients of the principal derivatives, and the coefficients of the derivative of $\check{r}_k$ w.r.t. ζ_k. In fact, the derivative of ζ_i is of the form

$$
\zeta_{i,x} = \theta \left(\frac{\iota_i}{h_i} \right) \left(\frac{\iota_{i,x}}{h_i} - \frac{\iota_i h_{i,x}}{h_i^2} \right),
$$

so that it can be very large. The idea is then to collect these unbounded terms, and to study them explicitly. Next, one shows that the remaining terms are uniformly Lipschitz and quadratic, so that from the fact that it vanishes on particular solutions one deduce the general form of the source term.

We first differentiate w.r.t. t,

E:htder1 (4.15)
$$
\begin{aligned}
h_t = & \hat{R}_b h_{b,t} + D\hat{R}_b u_t h_b + \hat{R}_{b,v_b} v_{b,t} h_b + \hat{R}_{b,v_k} v_{k,t} h_b + \hat{R}_{b,\kappa} \dot{\kappa} h_b \\
& + h_{k,t} (\check{r}_k + h_k \zeta_{k,h} \check{r}_{k,\sigma}) + h_k (D\check{r}_k u_t + v_{b,t} \check{r}_{k,v_b} + \hat{r}_{k,\kappa} \dot{\kappa} + v_{k,t} \check{r}_{i,v_k}) + h_k \iota_{k,t} \zeta_{k,\iota} \check{r}_{k,\sigma} \\
& + \sum_{i \neq k} h_{i,t} (\check{r}_i + h_i \zeta_{i,h} \check{r}_{i,\sigma}) + \sum_{i \neq k} h_i (D\check{r}_i u_t + v_{i,t} \check{r}_{i,v} + \check{r}_{i,\kappa} \dot{\kappa}) + \sum_{i \neq k} h_i \iota_{i,t} \zeta_{i,\iota} \check{r}_{i,\sigma},
\end{aligned}
$$

E:iotatder1
$$
\begin{aligned}
\iota_t = {} & \hat R_b \iota_{b,t} + D\hat R_b u_t \iota_b + \hat R_{b,v_b} v_{b,t}\iota_b + \hat R_{b,v_k} v_{k,t}\iota_b + \hat R_{b,\kappa}\dot\kappa\iota_b \\
& + \iota_{k,t}(\check r_k + \iota_k\zeta_{k,\iota}\check r_{k,\sigma}) + \iota_k(D\check r_k u_t + v_{b,t}\check r_{k,v_b} + \check r_{k,\kappa}\dot\kappa + v_{k,t}\check r_{i,v_k}) + \iota_k h_{k,t}\zeta_{k,h}\check r_{k,\sigma} \\
& + \sum_{i\neq k}\iota_{i,t}(\check r_i + \iota_i\zeta_{i,\iota}\check r_{i,\sigma}) + \sum_{i\neq k}\iota_i(D\check r_i u_t + v_{i,t}\check r_{i,v} + \check r_{i,\kappa}\dot\kappa) + \sum_{i\neq k}\iota_i h_{i,t}\zeta_{i,h}\check r_{i,\sigma}.
\end{aligned} \tag{4.16}
$$

Next, we compute the second derivative w.r.t. to x, collecting all the term containing derivatives of $\hat R_b$, $\check r$ not w.r.t. ζ in the nonlinear terms ϕ, ψ. We recall that the principal terms are the second derivatives of v_i, h_i, ι_i. It is easy to see that this term is a second order polynomial on the components of h, ι, u_x, u_t, with coefficients depending on the parameter κ, the solution u and the components v_i, w_i, h_i, ι_i.

One obtains

E:hxxder1
$$
\begin{aligned}
h_{xx} = {} & \hat R_b h_{b,xx} + \hat R_{b,v_b} v_{b,xx} h_b + \hat R_{b,v_k} v_{k,xx} h_b \\
& + h_{k,xx}(\check r_k + h_k\zeta_{k,h}\check r_{k,\sigma}) + h_k(v_{b,xx}\check r_{k,v_b} + v_{k,xx}\check r_{k,v_k}) + h_k\iota_{k,xx}\zeta_{k,\iota}\check r_{k,\sigma} \\
& + h_k\zeta_{k,x}(D\check r_{k,\sigma}u_x + v_{b,x}\check r_{k,v_b\sigma} + v_{k,x}\check r_{i,v_k\sigma}) + h_k\theta''(\iota_k/h_k)_x^2\check r_{k,\sigma} + h_k(\theta'(\iota_k/h_k)_x)^2\check r_{k,\sigma\sigma} \\
& + \sum_{i\neq k} h_{i,xx}(\check r_i + h_i\zeta_{i,h}\check r_{i,\sigma}) + h_i v_{i,xx}\check r_{i,v} + h_i\iota_{i,xx}\zeta_{i,\iota}\check r_{i,\sigma} \\
& + \sum_{i\neq k} h_i\zeta_{i,x}(D\check r_{i,\sigma}u_x + v_{i,x}\check r_{i,v\sigma}) + h_i\theta''(\iota_i/h_i)_x^2\check r_{i,\sigma} + h_i(\theta'(\iota_i/h_i)_x)^2\check r_{i,\sigma\sigma} \\
& + \phi(\kappa,u,u_x,u_t,u_{xx},u_{tx},h,h_x,\iota,\iota_x),
\end{aligned} \tag{4.17}
$$

E:iotaxxder1
$$
\begin{aligned}
\iota_{xx} = {} & \hat R_b \iota_{b,xx} + \hat R_{b,v_b} v_{b,xx}\iota_b + \hat R_{b,v_k} v_{k,xx}\iota_b \\
& + \iota_{k,xx}(\check r_k + \iota_k\zeta_{k,\iota}\check r_{k,\sigma}) + \iota_k(v_{b,xx}\check r_{k,v_b} + v_{k,xx}\check r_{k,v_k}) + \iota_k h_{k,xx}\zeta_{k,h}\check r_{k,\sigma} + \iota_k(\theta'(\iota_k/h_k)_x)^2\check r_{k,\sigma\sigma} \\
& + \iota_k\zeta_{k,x}(D\check r_{k,\sigma}u_x + v_{b,x}\check r_{k,v_b\sigma} + v_{k,x}\check r_{i,v_k\sigma}) + h_k(\theta''(\iota_k/h_k) + 2\theta')(\iota_k/h_k)_x^2\check r_{k,\sigma} \\
& + \sum_{i\neq k}\iota_{i,xx}(\check r_i + \iota_i\zeta_{i,\iota}\check r_{i,\sigma}) + \iota_i v_{i,xx}\check r_{i,v} + \iota_i h_{i,xx}\zeta_{i,h}\check r_{i,\sigma} \\
& + \sum_{i\neq k}\iota_i\zeta_{i,x}(D\check r_{i,\sigma}u_x + v_{i,x}\check r_{i,v\sigma}) + h_i(\theta''(\iota_i/h_i) + 2\theta')(\iota_i/h_i)_x^2\check r_{i,\sigma} + \iota_i(\theta'(\iota_i/h_i)_x)^2\check r_{i,\sigma\sigma} \\
& + \psi(\kappa,u,u_x,u_t,u_{xx},u_{tx},h,h_x,\iota,\iota_x),
\end{aligned} \tag{4.18}
$$

with ϕ, ψ quadratic functions of the components of u_x, u_t and h, ι and their derivatives: in fact pointwise

E:formphi1
$$
\begin{aligned}
\phi(t,x) = {} & \sum_{i,j} a_{ij}(t,x)h_i v_j + \sum_{i,j} b_{ij}(t,x)h_i v_{j,x} + \sum_{i,j} c_{ij}(t,x)h_{i,x}v_j + \sum_{i,j} d_{ij}(t,x)\iota_i v_j \\
& + \sum_{i,j} e_{ij}(t,x)\iota_i v_{j,x} + \sum_{i,j} f_{ij}(t,x)\iota_{i,x}v_j + \sum_i g_i(t,x)h_{i,x}v_{i,x} + \sum_i m_i(t,x)\iota_{i,x}v_{i,x} \\
& + \sum_{i,j} n_{ij}(t,x)h_i w_j + \sum_{i,j} o_{ij}(t,x)\iota_i w_j + p(t,x)h_{k,x}v_{b,x} + q(t,x)h_{b,x}v_{k,x} \\
& + r(t,x)\iota_{k,x}v_{b,x} + s(t,x)\iota_{b,x}v_{k,x},
\end{aligned} \tag{4.19}
$$

and similarly for ψ, with $i,j = b,1,\dots,n$. The coefficients are uniform Lipschitz continuous.

R:majorder1 *Remark* 4.3. The above expression for ϕ, ψ can be obtained by considering the following table:

	h_i	$h_{i,x}$	ι_i	$\iota_{i,x}$
v_j	for any i, j	for any i, j	for any i, j	for any i, j
$v_{j,x}$	for any i, j	only for $i=j$ or (b,k)	for any i, j	only for $i=j$ or (b,k)
w_j	for any i, j	never	for any i, j	never
$w_{j,x}$	never	never	never	never

In the intersection of every column and row, we write if there is a term containing the product of the component of the perturbation in the column with the terms of the solution in the row. For example, the term $h_{i,x}v_{j,x}$ means that an x-derivative is assigned to h_i and the other derivatives generates $\check r_{i,v}v_{i,x}$:

thus only for $i = j$ these terms appear, or the couples $v_{b,x}h_{k,x}$, $h_{b,x}v_{k,x}$, i.e. the boundary and the characteristic field (which appears both in $\hat{R}_b$, $\check{r}_k$). Moreover, since we never have the term $u_{t,x}$, then there will be no terms containing $w_{i,x}$, or $h_{i,x}w_j$, and so on.

To study the terms $(A(u)h)_x$, $(A(u)\iota)_x$, (3.12), (3.17) and (3.21) give the estimate

$$\begin{aligned} A(\kappa,u)\hat{R}_b(\kappa,u,v_s,v_k)h_b &= R_{cs}(\kappa,u,R_sv_s+r_kv_k)A_{cs}(\kappa,u,R_sv_s+r_kv_k)R_s(\kappa,u,v_s)h_b+\phi'(\kappa,u,u_x,u_t,h) \\ &= R_{cs}(\kappa,u,R_sv_s+r_kv_k)A_{cs}(\kappa,u,R_sv_s)R_s(\kappa,u,v_s)h_b+\phi'(\kappa,u,u_x,u_t,h) \\ &= R_b(\kappa,u,R_sv_s+r_kv_k)R_s(\kappa,u,v_s)\tilde{A}_s(\kappa,u,v_b)h_b+\phi'(\kappa,u,u_x,h) \\ &= \hat{R}_b(\kappa,u,v_s,v_k)\hat{A}_s(\kappa,u,v_b,v_k)h_b+\phi'(\kappa,u,u_x,h), \end{aligned}$$

where ϕ' denotes as above a second order polynomial in the components of u_x, h. The same for

$$A(\kappa,u)\hat{R}_b(\kappa,u,v_s,v_k)\iota_b = \hat{R}_b(\kappa,u,v_s,v_k)\hat{A}_s(\kappa,u,v_b,v_k)\iota_b+\psi'(\kappa,u,u_x,h).$$

Similarly, one obtains

$$\begin{aligned} A(\kappa,u)\hat{r}_k(\kappa,u,v_b,v_k,\zeta_k)h_k &= \hat{r}_k(\kappa,u,v_b,v_k,\zeta_k)\hat{\lambda}_k(\kappa,u,v_b,v_k,\sigma_k)h_k+\phi'(\kappa,u,u_x,h), \\ A(\kappa,u)\hat{r}_k(\kappa,u,v_b,v_k,\zeta_k)\iota_k &= \hat{r}_k(\kappa,u,v_b,v_k,\zeta_k)\hat{\lambda}_k(\kappa,u,v_b,v_k,\sigma_k)\iota_k+\psi'(\kappa,u,u_x,h), \end{aligned}$$

with ϕ a second order polynomial in the components of u_x, h, and

$$\begin{aligned} A(\kappa,u)\tilde{r}_i(\kappa,u,v_k,\zeta_i)h_i &= \tilde{r}_k(\kappa,u,v_i,\zeta_i)\tilde{\lambda}_k(\kappa,u,v_i,\sigma_i)h_i+\phi'(\kappa,u,u_x,h), \\ A(\kappa,u)\tilde{r}_i(\kappa,u,v_k,\zeta_i)\iota_i &= \tilde{r}_k(\kappa,u,v_i,\zeta_i)\tilde{\lambda}_k(\kappa,u,v_i,\sigma_i)\iota_i+\psi'(\kappa,u,u_x,h). \end{aligned}$$

As before, we denote with $\check{\lambda}_k$, $\check{\lambda}_i$ the drift speed computed with the speed ζ of the perturbation, and with $\hat{\lambda}_k$, $\tilde{\lambda}_i$ the effective drift computed with the speed σ of the solution u. In the previous equalities we have substituted the speed of h_i, ι_i with the speed of v_i, w_i. Note that

E:diffet01
$$(\check{\lambda}_k-\hat{\lambda}_k)h_k = \mathcal{O}(1)|v_k|^2h_k(\zeta_k-\sigma_k), \quad (\check{\lambda}_i-\tilde{\lambda}_i)h_i = \mathcal{O}(1)v_i^2h_i(\zeta_i-\sigma_i), \tag{4.20}$$

by (3.37) the derivative w.r.t. σ_k appears only in the term $\tilde{\lambda}_k$ (no problem for $i \neq k$ because of (3.29)).

Observe that if we differentiate the terms ϕ', ψ' w.r.t. ζ or σ, then we have always $\zeta_{i,x}$ multiplied either by ι_i or h_i. This is a consequence of the decomposition we assumed: the speed ζ_i appears only in the terms $h_k\hat{r}_k(u,v_k,\zeta_k)$, $\iota_k\hat{r}_k(u,v_k,\zeta_k)$, $h_i\tilde{r}_i(u,v_i,\zeta_i)$, $\iota_i\tilde{r}_i(u,v_i,\zeta_i)$. Since a simple computation

$$h_i\zeta_{i,x} = \theta\left(\frac{\iota_i}{h_i}\right)\big(\iota_{i,x}-\iota h_{i,x}/h_i\big), \qquad \iota_i\zeta_{i,x} = \frac{\iota_i}{h_i}(h_i\zeta_{i,x})$$

shows that we obtain a Lipschitz function, we associate it to the quadratic rest part ϕ', ψ'. The same observations hold for the derivative of σ: in fact in this case it follows from (4.20).

One see as a consequence that we can write for the x derivative of the convective terms $A(u)h$, $A(u)\iota$

E:commut02
$$\begin{aligned} (A(\kappa,u)h)_x = {} & \hat{R}_b(\hat{A}_bh_b)_x+\hat{R}_{b,v_b}(\hat{A}_bv_b)_xh_b+\hat{R}_{b,v_k}(\hat{\lambda}_kv_k)_xh_b+(\hat{\lambda}_kh_k)_x(\check{r}_k+h_k\zeta_{k,h}\check{r}_{k,\sigma}) \\ & +h_k((\hat{A}_bv_b)_x\check{r}_{k,v_b}+h_k(\hat{\lambda}_kv_k)_x\check{r}_{k,v_k})+h_k(\hat{\lambda}_k\iota_k)_x\zeta_{k,\iota}\check{r}_{k,\sigma} \\ & +\sum_{i\neq k}(\tilde{\lambda}_ih_i)_x(\check{r}_i+h_i\zeta_{i,h}\check{r}_{i,\sigma})+\sum_{i,\neq k}h_i(\tilde{\lambda}_iv_i)_x\check{r}_{i,v}+\sum_{i\neq k}h_i(\tilde{\lambda}_i\iota_i)_x\zeta_{i,\iota}\check{r}_{i,\sigma} \\ & +\phi'(\kappa,u,u_x,u_{xx},h,h_x,\iota,\iota_x), \end{aligned} \tag{4.21}$$

E:commut03
$$\begin{aligned} (A(\kappa,u)\iota)_x = {} & \hat{R}_b(\hat{A}_b\iota_b)_x+\hat{R}_{b,v_b}(\hat{A}_bv_b)_x\iota_b+\hat{R}_{b,v_k}(\hat{\lambda}_kv_k)\iota_b+(\hat{\lambda}_k\iota_k)_x(\hat{r}_k+\iota_k\zeta_{k,\iota}\check{r}_{k,\sigma}) \\ & +\iota_k((\hat{A}_bv_b)_x\check{r}_{k,v_b}+\iota_k(\hat{\lambda}_kv_k)_x\check{r}_{k,v_k})+\iota_k(\hat{\lambda}_kh_k)_x\zeta_{k,h}\check{r}_{k,\sigma} \\ & +\sum_{i\neq k}(\tilde{\lambda}_i\iota_i)_x(\check{r}_i+\iota_i\zeta_{i,\iota}\check{r}_{i,\sigma})+\sum_{i,\neq k}\iota_i(\tilde{\lambda}_iv_i)_x\check{r}_{i,v}+\sum_{i\neq k}\iota_i(\tilde{\lambda}_ih_i)_x\zeta_{i,h}\check{r}_{i,\sigma} \\ & +\psi'(\kappa,u,u_x,u_{xx},h,h_x,\iota,\iota_x), \end{aligned} \tag{4.22}$$

where the functions ϕ', ψ' are quadratic in (v,w,h,ι). The term $\hat{A}_b$, $\hat{\lambda}_k$, $\tilde{\lambda}_i$ are the effective drifts for the variables v_b, v_k, v_i, respectively.

R:whytilde1 *Remark* 4.4. The idea here is that we want the drift of our perturbation h_i to be the same as the drift of the original solution v_i. This is clearly possible because the difference of the drift depends only on the difference in speed $\sigma_i - \zeta_i$, and it is multiplied by the factor $v_i h_i$ or $v_i \iota_i$.

Repeating the analysis of Remark 4.3, we can generate the following table for the form of the quadratic terms in ϕ', ψ': taking into account that we have only one derivative, but that we are differentiating also σ_k, σ_i, we can write

	h_i	$h_{i,x}$	ι_i	$\iota_{i,x}$
v_j	for any i, j	for any i, j	for any i, j	for any i, j
$v_{j,x}$	for any i, j	never	for any i, j	never
w_j	for $i=j$ or (k,k)	for $i=j$ or (k,k)	for $i=j$ or (k,k)	for $i=j$ or (k,k)
$w_{j,x}$	for $i=j$ or (k,k)	never	for $i=j$ or (k,k)	never

In fact the terms with w_i appearing in σ_i are multiplied by h_i.

Substituting (4.15), (4.17), (4.21) into (4.5), and (4.16), (4.18), (4.22) into (4.8), we obtain

E:hequ01
$$\begin{aligned}
&\hat R_b(h_{b,t} + (\hat A_b h_b)_x - h_{b,xx}) + (\check r_k + h_k \zeta_{k,h} \check r_{k,\sigma})(h_{k,t} + (\hat\lambda_k h_k)_x - h_{k,xx}) \\
&\quad + h_k \zeta_{k,\iota} \check r_{k,\sigma}(\iota_{k,t} + (\hat\lambda_k \iota_k)_x - \iota_{k,xx}) + \sum_{i\neq k} (\check r_i + h_i \zeta_{i,h} \check r_{i,\sigma})(h_{i,t} + (\tilde\lambda_i h_i)_x - h_{i,xx}) \\
&\quad + \sum_{i\neq k} h_i \zeta_{i,\iota} \check r_{i,\sigma}(\iota_{i,t} + (\tilde\lambda_i \iota_i)_x - \iota_{i,xx}) + ((\hat R_{b,v_b}\cdot) h_b + \check r_{k,v_b} h_k)(v_{b,t} + (\hat A_b v_b)_x - v_{b,xx}) \\
&\quad + (\hat R_{b,v_k} h_b + \check r_{k,v_k} h_k)(v_{k,t} + (\hat\lambda_k v_k)_x - v_{k,xx}) + \sum_{i\neq k} \check r_{i,v} h_i (v_{i,t} + (\tilde\lambda_i v_i)_x - v_{i,xx}) \\
&\quad = \phi(\kappa, u, v, v_x, w, w_x, h, h_x, \iota, \iota_x) + \sum_{i\neq k} \mathcal O(1) v_i h_i (\zeta_{i,x})^2 \\
&\quad + \mathcal O(1)(|v_b| + |v_k|) h_k (\zeta_{k,x})^2 + \mathcal O(1) \left(|h_b| + \sum_{i=1}^n |h_i| \right) \dot\kappa,
\end{aligned} \tag{4.23}$$

E:iotaeq02
$$\begin{aligned}
&\hat R_b(\iota_{b,t} + (\hat A_b \iota_b)_x - \iota_{b,xx}) + (\check r_k + \iota_k \zeta_{k,\iota} \check r_{k,\sigma})(\iota_{k,t} + (\hat\lambda_k \iota_k)_x - \iota_{k,xx}) \\
&\quad + \iota_k \zeta_{k,h} \check r_{k,\sigma}(h_{k,t} + (\hat\lambda_k h_k)_x - h_{k,xx}) + \sum_{i\neq k} (\check r_i + \iota_i \zeta_{i,\iota} \check r_{i,\sigma})(\iota_{i,t} + (\tilde\lambda_i \iota_i)_x - \iota_{i,xx}) \\
&\quad + \sum_{i\neq k} \iota_i \zeta_{i,h} \check r_{i,\sigma}(h_{i,t} + (\tilde\lambda_i h_i)_x - h_{i,xx}) + (\hat R_{b,v_b} \cdot \iota_b + \check r_{k,v_b} \cdot \iota_k)(v_{b,t} + (\hat A_b v_b)_x - v_{b,xx}) \\
&\quad + (\hat R_{b,v_k} \iota_b + \check r_{k,v_k} \iota_k)(v_{k,t} + (\hat\lambda_k v_k)_x - v_{k,xx}) + \sum_{i\neq k} \check r_{i,v} \iota_i (v_{i,t} + (\tilde\lambda_i v_i)_x - v_{i,xx}) \\
&\quad = \psi(\kappa, u, v, v_x, w, w_x, h, h_x, \iota, \iota_x) + \sum_{i\neq k} \mathcal O(1) v_i \iota_i (\zeta_{i,x})^2 \\
&\quad + \mathcal O(1)(|v_b| + |v_k|) \iota_k (\zeta_{k,x})^2 + \mathcal O(1) \left(|\iota_b| + \sum_{i=1}^n |\iota_i| \right) \dot\kappa,
\end{aligned} \tag{4.24}$$

where ϕ, ψ have the quadratic form

E:formphi2

$$\begin{aligned}
\phi(t,x) = & \sum_{i,j} a_{ij}(t,x) h_i v_j + \sum_{i,j} b_{ij}(t,x) h_i v_{j,x} + \sum_{i,j} c_{ij}(t,x) h_{i,x} v_j + \sum_{i,j} d_{ij}(t,x) \iota_i v_j \\
& + \sum_{i,j} e_{ij}(t,x) \iota_i v_{j,x} + \sum_{i,j} f_{ij}(t,x) \iota_{i,x} v_j + \sum_i g_i(t,x) h_{i,x} v_{i,x} + \sum_i m_i(t,x) \iota_{i,x} v_{i,x} \\
& + \sum_{i,j} n_{ij}(t,x) h_i w_j + \sum_{i,j} o_{ij}(t,x) \iota_i w_j + p(t,x) h_{k,x} v_{b,x} + q(t,x) h_{b,x} v_{k,x} \\
& + \sum_i r_{ij}(t,x) w_i h_{i,x} + \sum_i s_{ij}(t,x) w_{i,x} h_i + \sum_i r'_{ij}(t,x) w_i \iota_{i,x} + \sum_i s'_{ij}(t,x) w_{i,x} \iota_i \\
& + t(t,x) \iota_{k,x} v_{b,x} + t'(t,x) \iota_{b,x} v_{k,x},
\end{aligned} \tag{4.25}$$

and similarly for ψ. We note here that we collected into ϕ, ψ also the right hand side of (4.5), (4.8), which is quadratic and contains only products of terms of different families $i \neq j$, or (b,i). We note that the above equations (4.23), (4.24) do not define by themselves which are the equations satisfied by the components $(h_b, h_i, \iota_b, \iota_i)$. In fact, as noted before, there are $2(n+k-1)$ variables in n equations. We will address this equations in Section 4.3.

R:eqfrovw1

Remark 4.5. In the case we are writing the equations for the components of u_x, u_t, we reduce (4.23), (4.24) to the simpler expression

E:vequ01

$$\begin{aligned}
& (\hat R_b + (\hat R_{b,v_b}\cdot) v_b + \hat r_{k,v_b} v_k)(v_{b,t} + (\hat A_b v_b)_x - v_{b,xx}) \\
& + (\hat R_{b,v_k} v_b + \hat r_k + \hat r_{k,v_k} v_k + v_k \sigma_{k,v} \hat r_{k,\sigma})(v_{k,t} + (\hat\lambda_k v_k)_x - v_{k,xx}) + v_k \sigma_{k,w} \hat r_{k,\sigma}(w_{k,t} + (\hat\lambda_k w_k)_x - w_{k,xx}) \\
& + \sum_{i\neq k} (\tilde r_i + v_i \tilde r_{i,v} + v_i \sigma_{i,v} \tilde r_{i,\sigma})(v_{i,t} + (\tilde\lambda_i v_i)_x - v_{i,xx}) + \sum_{i \neq k} v_i \sigma_{i,w} \tilde r_{i,\sigma}(w_{i,t} + (\tilde\lambda_i w_i)_x - w_{i,xx})
\end{aligned}$$

$$= \phi(\kappa, u, v, v_x, w, w_x) + \sum_i \mathcal{O}(1) v_i^2 (\sigma_{i,x})^2 + \mathcal{O}(1) \left(|v_b| + \sum_{i=1}^n |v_i| \right) \dot\kappa, \tag{4.26}$$

E:weq02

$$\begin{aligned}
& \hat R_b (w_{b,t} + (\hat A_b w_b)_x - w_{b,xx}) + ((\hat R_{b,v_b}\cdot) w_b + \hat r_{k,v_b} w_k)(v_{b,t} + (\hat A_b v_b)_x - v_{b,xx}) \\
& + (\hat r_k + w_k \sigma_{k,w} \hat r_{k,\sigma})(w_{k,t} + (\hat\lambda_k w_k)_x - w_{k,xx}) \\
& + (\hat R_{b,v_k} w_b + \hat r_{k,v_k} w_k + w_k \sigma_{k,v} \hat r_{k,\sigma})(v_{k,t} + (\hat\lambda_k v_k)_x - v_{k,xx}) \\
& + \sum_{i \neq k} (\tilde r_i + w_i \sigma_{i,w} \tilde r_{i,\sigma})(w_{i,t} + (\tilde\lambda_i w_i)_x - w_{i,xx}) + \sum_{i\neq k} (w_i \tilde r_{i,v} + w_i \sigma_{i,v} \tilde r_{i,\sigma})(v_{i,t} + (\tilde\lambda_i v_i)_x - v_{i,xx}) \\
& = \psi(\kappa, u, v, v_x, w, w_x) + \sum_i \mathcal{O}(1) v_i w_i (\sigma_{i,x})^2 + \mathcal{O}(1) \left(|v_b| + \sum_{i=1}^n |v_i| \right) \dot\kappa.
\end{aligned} \tag{4.27}$$

As we know from Section 3.1, the matrix made by the vectors of the components (v_i, w_i), $i = 1, \dots, n$, i.e.

E:invert885

$$\begin{bmatrix}
\hat R_{b,v_k} v_b + \hat r_k + \hat r_{k,v_k} v_k + v_k \sigma_{k,v} \hat r_{k,\sigma} & \tilde r_i + v_i \tilde r_{i,v} + v_i \sigma_{i,v} \tilde r_{i,\sigma} & v_k \sigma_{k,w} \hat r_{k,\sigma_k} & v_i \sigma_{i,w} \tilde r_{i,\sigma} \\
\hat R_{b,v_k} w_b + \hat r_{k,v_k} w_k + w_k \sigma_{k,v} \hat r_{k,\sigma} & w_i \tilde r_{i,v} + w_i \sigma_{i,v} \tilde r_{i,\sigma} & \hat r_k + w_k \sigma_{k,w} \hat r_{k,\sigma_k} & \tilde r_i + w_i \sigma_{i,w} \tilde r_{i,\sigma}
\end{bmatrix} \tag{4.28}$$

is invertible, so that we can assume that the equations satisfied by each components are

where ω_k, ω_i and ϖ_k, ϖ_i are the decomposition of the source terms for the decomposition of u_x, u_t, denoted by ω, ϖ, respectively, on the base of the matrix (4.28), and with suitable initial boundary conditions (we will describe below how to choose them). For a more precise discussion on the decomposition of the initial boundary data and source terms, see Section 4.3.

We assume that the source terms ϕ_k, ϕ_i, ψ_k, ψ_i are integrable:

E:integr08

$$\int_0^{+\infty} \int_{\mathbb{R}} |\phi_k(t,x)| dx dt, \int_0^{+\infty} \int_{\mathbb{R}} |\phi_i(t,x)| dx dt \leq C \delta_0^2. \tag{4.29}$$

If with this assumption we can prove that the source terms for h, ι are integrable in the plane and satisfy the same estimates (4.29) for initial data with L^1 norm of the order δ_1, then a posteriori we obtain the BV estimates and the stability estimates.

Using (4.7), we can obtain the relation among the components ι_i and h_i, $h_{i,x}$:

E:iotahrel1

$$\begin{aligned}
&\hat{R}_b(\iota_b - h_{b,x} + \hat{A}_b h_b) + (\iota_k - h_{k,x} + \hat{\lambda}_k h_k)\check{r}_k + ((\hat{R}_b \cdot)h_b + \check{r}_{k,v_b} h_k)(w_b - v_{b,x} + \hat{A}_b v_b) \\
&\quad + (\hat{R}_{b,v_k} h_b + \check{r}_{k,v_k} h_k)(w_k - v_{k,x} + \hat{\lambda}_k v_k) + \sum_{i \neq k} (\iota_i - h_{i,x} + \tilde{\lambda}_i h_i)\check{r}_i + \check{r}_{i,v} h_i (w_i - v_{i,x} + \check{\lambda}_i v_i) \\
&= h_k \zeta_{k,x} \check{r}_{k,\sigma} + \sum_{i \neq k} h_i \zeta_{i,x} \check{r}_{i,\sigma} + \varphi(\kappa, u, v, w, h, \iota),
\end{aligned} \tag{4.30}$$

where φ is quadratic in the components h, ι, v, w. We note that also in this case we have the freedom of choosing how to decompose the right hand side. A particular case is when only the i-th waves, $i = 1, \dots, n$, in h, v are present, where we can make use of (3.28): in fact one obtains that

E:prtcas18

$$\begin{aligned}
\iota_i \hat{r}_i = \iota &= h_{i,x} \hat{r}_i + h_i v_i D\hat{r}_i \tilde{r}_i + h_i v_{i,x} \hat{r}_{i,v} + h_i \hat{r}_{i,\sigma} \zeta_{i,x} - h_i A(u) \hat{r}_i \\
&= (h_{i,x} - \tilde{\lambda} h_i)\hat{r}_i + \mathcal{O}(1) h_i v_i (\zeta_i - \sigma_i) + \mathcal{O}(1) h_i (v_{i,x} + (\sigma_i - \tilde{\lambda}_i) v_i) + \mathcal{O}(1) v_i h_i \zeta_{i,x}.
\end{aligned} \tag{4.31}$$

Ss:formuu1

4.2. **Explicit form for the source terms.** We now analyze the structure of the source terms ϕ, ψ. Since we know that these terms are uniformly Lipschitz and quadratic, the idea is to study the conditions for which the terms are 0, i.e. when the left hand side of (4.23), (4.24) is 0, and thus to write explicitly the most general form of all the quadratic terms which vanishes on these surfaces. The reason why we choose the generalized eigenvalues $\hat{A}_b$, $\hat{\lambda}_k$, $\tilde{\lambda}_i$ also for the perturbation, instead of $\check{\lambda}_k$, $\check{\lambda}_i$, follows from the analysis below.

In the following cases the right hand side of (4.23), (4.24) is 0:

(1) a travelling profile of the i-th family, $i \neq k$, with speed $\sigma_i = -w_i/v_i = \zeta_i = -\iota_i/h_i$. The other waves are 0;
(2) a travelling profile of the k-th family, with speed $\sigma_k = -w_k/v_k = \zeta_k = -\iota_k/h_k$. The speed of the boundary is constant and the other waves are 0;
(3) a boundary layer with both v_b and v_k, travelling with speed $0 = -w_b/v_b = -\iota_b/h_b$. The other waves are 0.

In all three cases, the parameter κ is constant. We verify this statement in the case of i waves. The proof for the other cases follows similarly.

From the assumption that σ_i, ζ_i are constant, the speed terms of the right hand side of (4.23), (4.24) vanish, and since the cutoff functions θ, ϑ are not acting, we have that $w_i = -\sigma_i v_i$ and $w_{i,x} = -\sigma_i v_{i,x}$. By means of $u_t + A(\kappa, u) u_x - u_{xx} = 0$ and (4.31) with $\iota_i = w_i$, $h_i = v_i$ it follows that v_i satisfies

E:effectflu065

$$v_{i,x} = (\tilde{\lambda}_i - \sigma_i) v_i = \tilde{\lambda}_i v_i + w_i. \tag{4.32}$$

Differentiating w.r.t. x and noting that $v_{i,t} = w_{i,x}$, we obtain that

$$v_{i,t} + (\tilde{\lambda}_i v_i)_x - v_{i,xx} = 0.$$

Now $u_{tt} + (A(\kappa, u) u_t)_x - u_{txx} = 0$ for a single travelling profile, because the left hand side of the equation for u_t (which has the same form of (4.5), with u_t in place of h) is 0 in this case. By the proportionality of v_i, w_i, it follows that also w_i satisfies

$$w_{i,t} + (\tilde{\lambda}_i w_i)_x - w_{i,xx} = 0.$$

Thus the source terms ω_i, ϖ_i for the equations of v_i, w_i vanish. At this point, since $\zeta_i = \sigma_i = -\iota_i/h_i = -w_i/v_i$, it follows that also h_i, ι_i satisfy the same equations of v, w:

$$h_{i,t} + (\tilde{\lambda}_i h_i)_x - h_{i,xx} = 0, \qquad \iota_{i,t} + (\tilde{\lambda}_i \iota_i)_x - \iota_{i,xx} = 0.$$

Note that repeating the computations leading to (4.32) in all three cases considered above, it follows that

E:ientioh1

$$\iota_b = h_{b,x} - \hat{A}_b h_b, \quad \iota_k = h_{k,x} - \hat{\lambda}_k h_k, \quad \iota_i = h_{i,x} - \tilde{\lambda}_i h_i. \tag{4.33}$$

R:cutorr1

Remark 4.6. When the cutoff function $\vartheta(\iota_i/h_i)$ is active, we have that only two case may happen (see (4.30) and (4.31)):

(1) $h_{i,x}$ is much greater than h_i;
(2) there is a j-th wave h_j, ι_j and derivatives, with $j \neq i$ or $j = b$, greater than $(\delta_0/4)h_i$.

In the opposite situation, i.e. when $\vartheta(\iota_i/h_i) = \iota_i/h_i$, either $h_{i,x}$ is small or there is a wave of a different family of size comparable with h_i. This will help in simplifying the form of the source terms because we can substitute h_i with $h_{i,x}$ or vice versa, depending on the case. The same observation holds for $\theta(w_i/v_i)$.

We now show which is the form of the source term, by imposing that ϕ, ψ should vanish in the situations considered above.

Sss:ithwave1

4.2.1. *Waves of the i-th family, case 1).* In this case only the terms containing i-th components can be present, with $i \neq k$, and the right hand side of (4.23), (4.24) vanishes when

$$w_i + \sigma_i v_i = w_{i,x} + \sigma_i v_{i,x} = 0, \qquad \iota_i + \zeta_i h_i = \iota_{i,x} + \zeta_i h_{i,x} = 0, \qquad \zeta_i = \sigma_i.$$

Thus the term we can expect contains as a factor all the above terms. As a consequence, recalling that the source is quadratic w.r.t. the components h, ι, v, w, and linear in h, ι, we obtain that the only possible form is

E:itsouc1

$$\begin{aligned} \mathfrak{I}_i(t,x) \doteq\, & \mathcal{O}(1)(|v_i| + |v_{i,x}| + |w_i| + |w_{i,x}|)(|\iota_i + \zeta_i h_i| + |\iota_{i,x} + \zeta_i h_{i,x}|) \\ & + \mathcal{O}(1)(|h_i| + |h_{i,x}| + |\iota_i| + |\iota_{i,x}|)(|w_i + \sigma_i v_i| + |w_{i,x} + \sigma_i v_{i,x}|) \\ & + \mathcal{O}(1)(|v_i| + |v_{i,x}| + |w_i| + |w_{i,x}|)(|h_i| + |h_{i,x}| + |\iota_i| + |\iota_{i,x}|)(\zeta_i - \sigma_i). \end{aligned} \tag{4.34}$$

We can further simplify the above expression by using Remark 4.6: in fact, when $\iota_i + \zeta_i h_i \neq 0$, i.e. ϑ is not the identity, then we can estimate h_i with $h_{i,x}$. Note that also ι_i can be estimated with $h_{i,x}$. The opposite situation happens when ϑ is not acting, i.e. we can substitute $h_{i,x}$ and ι with h_i. The same observation holds for v_i, w_i: if θ is the identity, we replace w_i, $v_{i,x}$ with v_i, while in the other case we replace w_i, v_i with $v_{i,x}$.

We thus can simplify the above terms by

E:itsouc2

$$\begin{aligned} \mathfrak{I}_i(t,x) \doteq\, & \mathcal{O}(1)(|v_{i,x}|^2 + |w_{i,x}|^2 + |\iota_{i,x}|^2 + |h_{i,x}|^2)\chi\Big\{|w_i/v_i| \geq \delta_0, |\iota_i/h_i| \geq \delta_0\Big\} \\ & + \mathcal{O}(1)|h_i|(|v_{i,x}| + |w_{i,x}|)\chi\Big\{|w_i/v_i| \geq 2\delta_0, |\iota_i/h_i| \leq \delta_0\Big\} \\ & + \mathcal{O}(1)(|h_{i,x}| + |\iota_{i,x}|)|v_i|\chi\Big\{|w_i/v_i| \leq \delta_0, |\iota_i/h_i| \geq 2\delta_0\Big\} \\ & + \mathcal{O}(1)\Big(|v_i||\iota_{i,x} + \zeta_i h_{i,x}| + |h_i||w_{i,x} + \sigma_i v_{i,x}| \\ & \qquad + |v_i||h_i|(\zeta_i - \sigma_i)\Big)\chi\Big\{|w_i/v_i| \leq \delta_0, |\iota_i/h_i| \leq \delta_0\Big\}. \end{aligned} \tag{4.35}$$

The function $\chi\{E\}$ is the characteristic function of the set E.

Sss:kthwave1

4.2.2. *Waves of the k-th family, case 2).* As in the previous point, we have

E:ktsouc2

$$\begin{aligned} \mathfrak{I}_k(t,x) \doteq\, & \mathcal{O}(1)(|v_{k,x}|^2 + |w_{k,x}|^2 + |\iota_{k,x}|^2 + |h_{k,x}|^2)\chi\Big\{|w_k/v_k| \geq \delta_0, |\iota_k/h_k| \geq \delta_0\Big\} \\ & + \mathcal{O}(1)|h_k|(|v_{k,x}| + |w_{k,x}|)\chi\Big\{|w_k/v_k| \geq 2\delta_0, |\iota_k/h_k| \leq \delta_0\Big\} \\ & + \mathcal{O}(1)(|h_{k,x}| + |\iota_{k,x}|)|v_k|\chi\Big\{|w_k/v_k| \leq \delta_0, |\iota_k/h_k| \geq 2\delta_0\Big\} \\ & + \mathcal{O}(1)\Big(|v_k||\iota_{k,x} + \zeta_k h_{k,x}| + |h_k||w_{k,x} + \sigma_k v_{k,x}| \\ & \qquad + |v_k||h_k|(\zeta_k - \sigma_k)\Big)\chi\Big\{|w_k/v_k| \leq \delta_0, |\iota_k/h_k| \leq \delta_0\Big\}. \end{aligned} \tag{4.36}$$

Sss:boundrge1

4.2.3. *Boundary layers.* In this case two waves are present, and set where the right hand side of (4.23), (4.24) vanishes can be written as (recall that the speed of the boundary is 0)

$$w_b + \sigma_b v_b = w_b = w_{b,x} = 0, \quad \iota_b = \iota_{b,x} = 0, \quad w_k = w_{k,x} = 0, \quad \iota_k = \iota_{k,x} = 0.$$

First we note that the structure of the terms which have only waves of the characteristic family h_k, ι_k, v_k, w_k have been considered in the previous point 4.2.2, so that their structure is given by (4.36): here we will consider only term with v_k, w_k multiplied by h_b, ι_b, and vice versa. We thus can write

E:bdrsouc1
$$\begin{aligned} \int_b(t,x) \doteq{}& \mathcal{O}(1)(|v_b|+|v_{b,x}|+|v_k|+|v_{k,x}|)(|\iota_b|+|\iota_{b,x}|)+\mathcal{O}(1)(|v_b|+|v_{b,x}|)(|\iota_k|+|\iota_{k,x}|) \\ &+\mathcal{O}(1)(|h_b|+|h_{b,x}|+|h_k|+|h_{k,x}|)(|w_b|+|w_{b,x}|)+\mathcal{O}(1)(|h_b|+|h_{b,x}|)(|w_k|+|w_{k,x}|). \end{aligned} \tag{4.37}$$

Finally we have to consider all the terms which are 0 in cases 1), 2), 3): these are all the quadratic products of waves of different families, i.e.

- waves of the families $i \neq j$, $i,j = 1,\dots,n$,

E:trnsv08
$$\begin{aligned} \sqcup_1 \doteq{}& \sum_{i\neq j} \mathcal{O}(1)(|v_i|+|v_{i,x}|+|w_i|+|w_{i,x}|)(|h_j|+|\iota_j|) \\ &+\sum_{i\neq j}\mathcal{O}(1)(|v_i|+|w_i|)(|h_{j,x}|+|\iota_{j,x}|); \end{aligned} \tag{4.38}$$

- waves of the families $i \neq k$ and the boundary term,

E:trnsv09
$$\begin{aligned} \sqcup_2 \doteq{}& \sum_{i\neq k} \mathcal{O}(1)(|v_i|+|v_{i,x}|+|w_i|+|w_{i,x}|)(|h_b|+|\iota_b|) \\ &+\sum_{i\neq k}\mathcal{O}(1)(|v_i|+|w_i|)(|h_{b,x}|+|\iota_{b,x}|) \\ &+\sum_{i\neq k}\mathcal{O}(1)(|h_i|+|h_{i,x}|+|\iota_i|+|\iota_{i,x}|)(|v_b|+|w_b|) \\ &+\sum_{i\neq k}\mathcal{O}(1)(|h_i|+|h_{i,x}|+|\iota_i|)(|v_{b,x}|+|w_{b,x}|). \end{aligned} \tag{4.39}$$

Writing the above terms, we notice that due to Remark 4.4 the terms

$$(|v_{i,x}|+|w_{i,x}|)(|h_{i,x}|+|\iota_{i,x}|), \quad (|v_{i,x}|+|w_{i,x}|)(|h_{b,x}|+|\iota_{b,x}|), \quad (|v_{b,x}|+|w_{b,x}|)(|h_{i,x}|+|\iota_{i,x}|)$$

cannot appear. We thus obtain that the form of the source term is the following:

E:sourceform1
$$\phi(t,x), \psi(t,x) = \int_b(t,x)+\int_k(t,x)+\sum_{i\neq k}\int_i(t,x)+\sqcup_1(t,x)+\sqcup_2(t,x). \tag{4.40}$$

R:iothre8
Remark 4.7. Using the same technique, one can obtain the form of the term φ in (4.30), namely

E:varphiex18
$$\begin{aligned} \varphi ={}& \mathcal{O}(1)\Big\{(|v_b|+|v_k|)|\iota_b|+|v_b||\iota_k|+(|h_b|+|h_k|)|w_b|+|h_b||w_k|\Big\} \\ &+\sum_{i=1}^n \mathcal{O}(1)\bigg[(|v_{i,x}|^2+|w_{i,x}|^2+|\iota_{i,x}|^2+|h_{i,x}|^2)\chi\Big\{|w_i/v_i|\geq\delta_0, |\iota_i/h_i|\geq\delta_0\Big\} \\ &+|h_i|(|v_{i,x}|+|w_{i,x}|)\chi\Big\{|w_i/v_i|\geq 2\delta_0, |\iota_i/h_i|\leq\delta_0\Big\}+(|h_{i,x}|+|\iota_{i,x}|)|v_i|\chi\Big\{|w_i/v_i|\leq\delta_0, |\iota_i/h_i|\geq 2\delta_0\Big\} \\ &+|v_i||h_i|(\zeta_i-\sigma_i)\chi\Big\{|w_i/v_i|\leq\delta_0, |\iota_i/h_i|\leq\delta_0\Big\}\bigg]+\sum_{i\neq j}\mathcal{O}(1)(|v_i|+|w_i|)(|h_j|+|\iota_j|) \\ &+\sum_{i\neq k}\mathcal{O}(1)\Big\{(|v_i|+|w_i|)(|h_b|+|\iota_b|)+(|h_i|+|\iota_i|)(|v_b|+|w_b|)\Big\}. \end{aligned} \tag{4.41}$$

Note that the presence of derivatives is due only to the fact that in the regions where the cutoff functions θ, ϑ are active we can estimate v, w, h, ι with $v_{i,x}$, $h_{i,x}$, respectively. In particular, the relation among w, v has a simplified term φ:

E:varphiex19
$$\begin{aligned} \varphi ={}& \mathcal{O}(1)\Big\{(|v_b|+|v_k|)|w_b|+|v_b||w_k|\Big\}+\sum_{i=1}^n\mathcal{O}(1)|v_i||w_i+\sigma_i v_i| \\ &+\sum_{i\neq j}\mathcal{O}(1)|v_i||v_j|+\sum_{i\neq k}\mathcal{O}(1)|v_i||v_b|. \end{aligned} \tag{4.42}$$

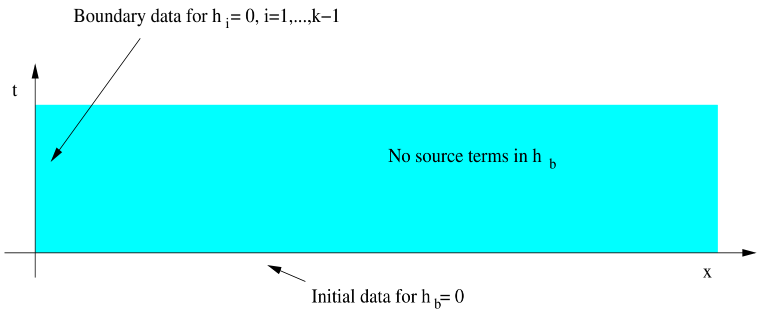

FIGURE 4. The choice of the initial boundary data and source terms. Fi:bounriem3

Ss:equsepr1

4.3. **Initial-boundary data decomposition.** We now assign the initial boundary data for each component h_i, ι_i. We recall that the two set of vectors

E:twobase1
$$(\hat{R}_b, \check{r}_k, \check{r}_{k+1}, \dots, \check{r}_n), \qquad (\check{r}_1, \dots, \check{r}_n), \tag{4.43}$$

are a base of $\mathbb{R}^n$: this means that we find the components of a vector $\nu \in \mathbb{R}^n$ w.r.t. both bases. In fact, for $\kappa = 0$, $u = \bar{u}$, $v = 0$ the above bases correspond to the eigenvectors of $A(0, \bar{u})$.

Observe first that the boundary conditions for h are uniquely defined by the boundary data $\bar{h}_b$ for h, while for ι it is determined by the solution to the equations. By the regularity estimates of Section 2, the boundary condition $\bar{\iota}_b$ for ι is smooth and bounded if the total variation of h is small.

Since we have $2(n+k-1)$ variables $(h_b, h_1, \dots, h_n)$, $(\iota_b, \iota_1, \dots, \iota_n)$ in $2n$ equations, we have the freedom to decompose the initial boundary data (h_0, ι_0), $(\bar{h}_b, \bar{\iota}_b)$ and the source term

E:sourcompl1
$$\begin{aligned} \Phi(t,x) = {} & \phi(t,x) + \sum_{i \neq k} \mathcal{O}(1) v_i h_i (\zeta_{i,x})^2 + \mathcal{O}(1)(|v_b| + |v_k|) h_k (\zeta_{k,x})^2 \\ & + \mathcal{O}(1) \left(|h_b| + \sum_{i=1}^n |h_i| \right) \dot{\kappa} + \mathcal{O}(1)\omega(t,x) + \mathcal{O}(1)\varpi(t,x), \\ \Psi(t,x) = {} & \psi(t,x) + \sum_{i \neq k} \mathcal{O}(1) v_i \iota_i (\zeta_{i,x})^2 + \mathcal{O}(1)(|v_b| + |v_k|) \iota_k (\zeta_{k,x})^2 \\ & + \mathcal{O}(1) \left(|\iota_b| + \sum_{i=1}^n |\iota_i| \right) \dot{\kappa} + \mathcal{O}(1)\omega(t,x) + \mathcal{O}(1)\varpi(t,x), \end{aligned} \tag{4.44}$$

into one of the two bases (4.43). Here are the guidelines (fig. 4) for the decomposition:

(1) the boundary data $h_{i,b}(t)$, $\iota_{i,b}(t)$ for h_i, ι_i, $i = 1, \dots, k-1$ is 0:

E:boundleav89
$$h_{i,b}(t) = h_i(t,0) = 0, \qquad \iota_{i,b}(t) = \iota_i(t,0) = 0. \tag{4.45}$$

Thus we obtain that

E:decobound9
$$\left\{ \begin{array}{rcl} \bar{h}_b(t) & = & \hat{R}_b(\kappa, u_b, v_{b,b}, v_{k,b}) h_{b,b} + \hat{r}_k(\kappa, u_b, v_{b,b}, v_{k,b}, \zeta_k) h_{k,b} + \sum_{i>k} h_{i,b} \tilde{r}_i(\kappa, u_b, v_{i,b}, \zeta_i) \\ \bar{\iota}_b(t) & = & \hat{R}_b(\kappa, u_b, v_{b,b}, v_{k,b}) \iota_{b,b} + \hat{r}_k(\kappa, u_b, v_{b,b}, v_{k,b}, \zeta_k) \iota_{k,b} + \sum_{i>k} \iota_{i,b} \tilde{r}_i(\kappa, u_b, v_{i,b}, \zeta_i) \end{array} \right. \tag{4.46}$$

with $\zeta_i = \lambda_i(\bar{u}) - \vartheta(\iota_i/h_i)$. A standard computation shows that the above map

$$\Big(h_{b,b}, h_{k,b}, \dots, h_{n,b}, \iota_{b,b}, \iota_{k,b}, \dots, \iota_{n,b} \Big) \mapsto (\bar{h}_b, \bar{\iota}_b)$$

is uniformly Lipschitz and invertible, so that we obtain the boundary data $(h_{b,b}, h_{i,b}, \iota_{b,b}, \iota_{i,b})$ for the remaining $2n$ variables $(h_b, h_i, \iota_b, \iota_i)$, $i = k, \dots, n$. Note that we know from regularity estimates that the boundary data are bounded by $C\delta_1$ in $C^{0,1}$,

E:regulbdco89
$$\|h_{b,b}\|_{C^{0,1}} + \|\iota_{b,b}\|_{C^{0,1}} + \sum_{i=k}^n \Big(\|h_{i,b}\|_{C^{0,1}} + \|\iota_{i,b}\|_{C^{0,1}} \Big) \leq C\delta_1. \tag{4.47}$$

However the L^1 norm in $(0,T)$ of the boundary data have to be estimated.

R:onlyLip1 *Remark* 4.8. As we explain in the discussion before Lemma 4.2, the boundary data for all components are assumed to be smooth. However, only the Lipschitz norm (or C^1 norm) of these components can be estimated by the C^1 norm of the boundary data. This is sufficient for our needs, because we have only to estimate first derivatives of the components.

(2) For a fixed $\delta > 0$, in the region $[\delta, +\infty)$ we decompose the initial data h_0, ι_0 for h, ι by considering

E:decoinitd9
$$\left\{ \begin{array}{rcl} h_0(x) &=& \hat{r}_k(\kappa, u_0, 0, v_{k,0}, \zeta_k) h_{k,0} + \sum_{i\neq k} h_{i,0} \tilde{r}_i(\kappa, u_0, v_{i,0}, \zeta_i) \\ \iota_0(x) &=& \hat{r}_k(\kappa, u_0, 0, v_{k,0}, \zeta_k) \iota_{k,0} + \sum_{i\neq k} \iota_{i,0} \tilde{r}_i(\kappa, u_0, v_{i,0}, \zeta_i) \end{array} \right. \tag{4.48}$$

with $\zeta_i = \lambda_i(\bar{u}) - \vartheta(\iota_i/h_i)$, which can be verified to be invertible for v small. Thus the initial data for the boundary layer variables h_b, ι_b is 0 for $x \geq \delta$, while for the other component h_i, ι_i, $i = 1, \dots, n$, is determined by inverting the map (4.48). As for (4.47), these components are uniformly Lipschitz continuous:

E:regulbdco90
$$\sum_{i=1}^{n} \Big(\|h_{i,0}\|_{C^{0,1}} + \|\iota_{i,0}\|_{C^{0,1}} \Big) \leq C\delta_1. \tag{4.49}$$

Let υ_i, ς_i, $i = 1, \dots, k-1$ be smooth function, with C^N norm less that $C\delta_1$, connecting the initial data $h_{i,0}$, $\iota_{i,0}$ to 0 for the components leaving the domain, i.e. $i = 1, \dots, k-1$. Thus for $0 \leq x \leq \delta$ use the decomposition

E:decoinit10
$$\left\{ \begin{array}{rcl} h_0(t) - \sum_{i<k} \upsilon_i \tilde{r}_i(\kappa, u_0, v_{i,0}, \zeta_i) &=& \hat{R}_b(\kappa, u_0, v_{b,0}, v_{k,0}) h_{b,0} + \hat{r}_k(\kappa, u_0, v_{b,0}, v_{k,0}, \zeta_k) h_{k,0} \\ && + \sum_{i>k} h_{i,0} \tilde{r}_i(\kappa, u_0, v_{i,0}, \zeta_i) \\ \iota_0(t) - \sum_{i<k} \varsigma_i \tilde{r}_i(\kappa, u_0, v_{i,0}, \zeta_i) &=& \hat{R}_b(\kappa, u_0, v_{b,0}, v_{k,0}) \iota_{b,0} + \hat{r}_k(\kappa, u_0, v_{b,0}, v_{k,0}, \zeta_k) \iota_{k,0} \\ && + \sum_{i>k} \iota_{i,0} \tilde{r}_i(\kappa, u_0, v_{i,0}, \zeta_i) \end{array} \right. \tag{4.50}$$

with $\zeta_i = \lambda_i(\bar{u}) - \vartheta(\iota_i/h_i)$. Clearly the initial data for h_b, ι_b have compact support. Moreover, the assumptions on the initial data (1.2), (1.3) and Lemma 4.2, imply that $(h_{b,0}, h_{i,0})$, $(\iota_{b,0}, \iota_{i,0})$, $i = 1, \dots, n$, belong to $C^{0,1}$ with norm less than $\mathcal{O}(1)\delta_1$. Without any loss of generality, we assume that the L^1 norm of all components is less that δ_0, if δ_1 is sufficiently small,

E:initda8918
$$\sum_{i=1}^{n} \Big(\|h_{i,0}\|_{W^{1,1}} + \|\iota_{i,0}\|_{W^{1,1}} \Big) \leq C\delta_1 \leq \delta_0. \tag{4.51}$$

The same observations of Remark 4.8 hold here.

(3) Finally, the source term is decomposed only in the waves components. We thus construct the variables h_b, ι_b by solving

$$h_{b,t} + (\hat{A}_b h_b)_x - h_{b,xx} = 0, \qquad \iota_{b,t} + (\hat{A}_b \iota_b)_x - \iota_{b,xx} = 0.$$

Thus, from (4.23), (4.24), the source terms ϕ_i, ψ_i, $i = 1, \dots, n$ are given by

E:sourcet1
$$(\phi_i, \psi_i)_{i=1}^k = \begin{bmatrix} \check{r}_i + h_i \zeta_{i,h} \check{r}_{k,\sigma} & h_i \zeta_{i,\iota} \check{r}_{i,\sigma} \\ \iota_i \zeta_{k,h} \check{r}_{k,\sigma} & \check{r}_i + \iota_i \zeta_{i,\iota} \check{r}_{i,\sigma} \end{bmatrix}^{-1} (\Phi, \Psi). \tag{4.52}$$

Clearly, if we assume that the source term is less than $\mathcal{O}(1)\delta_0^2$ in $L^1(\{0 \leq t \leq T, x \geq 0\})$, so ϕ_i, ψ_i are.

In Remark 4.5 we have supposed that the initial boundary data for v, w and the right hand side of (4.26), (4.27) have been decomposed by the same scheme.

We thus have the following set of equations:

(1) $2(k-1)$ scalar equations with source for the components h_i, ι_i, $i = 1, \dots, k-q$ leaving the domain:

E:elaveq1
$$\left\{ \begin{array}{rcl} h_{i,t} + (\tilde{\lambda}_i(t,x) h_i)_x - h_{i,xx} &=& \phi_i(t,x) \\ \iota_{i,t} + (\tilde{\lambda}_i(t,x) \iota_i)_x - \iota_{i,xx} &=& \psi_i(t,x) \\ (h_i(0,x), \iota_i(0,x)) &=& (h_{i,0}(x), \iota_{i,0}(x)) \\ (h_i(t,0), \iota_i(t,0)) &=& (0,0) \end{array} \right. \tag{4.53}$$

with $\tilde{\lambda} \leq c < 0$, and initial data in $L^1 \cap C^{0,1}$ with norm $\mathcal{O}(1)\delta_1$;

(2) 2 scalar equations with source for the characteristic fields h_k, ι_k:

E:elaveq2

$$\left\{\begin{array}{ccc} h_{k,t} + (\hat{\lambda}_k(t,x) h_k)_x - h_{k,xx} & = & \phi_k(t,x) \\ \iota_{k,t} + (\hat{\lambda}_k(t,x)\iota_k)_x - \iota_{k,xx} & = & \psi_k(t,x) \\ (h_k(0,x), \iota_k(0,x)) & = & (h_{k,0}(x), \iota_{k,0}(x)) \\ (h_k(t,0), \iota_k(t,0)) & = & (h_{k,b}(x), \iota_{k,b}(x)) \end{array}\right. \tag{4.54}$$

with $|\hat{\lambda}_k| \leq \mathcal{O}(1)\delta_0$ and initial data in $L^1 \cap C^{0,1}$ with norm $\mathcal{O}(1)\delta_1$. The boundary data has Lipschitz norm of the order of δ_0, but its L^1 norm has to be estimated;

(3) $2(n-k)$ scalar equations with source for the fields entering the domain h_i, ι_i, $i = k+1, \dots, n$:

E:elaveq3

$$\left\{\begin{array}{ccc} h_{i,t} + (\tilde{\lambda}_i(t,x) h_i)_x - h_{i,xx} & = & \phi_i(t,x) \\ \iota_{i,t} + (\tilde{\lambda}_i(t,x)\iota_i)_x - \iota_{i,xx} & = & \psi_i(t,x) \\ (h_i(0,x), \iota_i(0,x)) & = & (h_{i,0}(x), \iota_{i,0}(x)) \\ (h_i(t,0), \iota_i(t,0)) & = & (h_{i,b}(x), \iota_{i,b}(x)) \end{array}\right. \tag{4.55}$$

with $\tilde{\lambda}_i \geq c > 0$ and initial data in $L^1 \cap C^{0,1}$ with norm $\mathcal{O}(1)\delta_1$. The boundary data has Lipschitz norm of the order of δ_0, but its L^1 norm has to be estimated;

(4) 2 $(k-1) \times (k-1)$ systems of the form for the boundary layer components h_b, ι_b:

E:elaveq4

$$\left\{\begin{array}{ccc} h_{b,t} + (\hat{A}_b(t,x) h_b)_x - h_{b,xx} & = & 0 \\ \iota_{b,t} + (\hat{A}_b(t,x)\iota_b)_x - \iota_{b,xx} & = & 0 \\ (h_b(0,x), \iota_b(0,x)) & = & (h_{b,0}(x), \iota_{b,0}(x)) \\ (h_b(t,0), \iota_b(t,0)) & = & (h_{b,b}(t), \iota_{b,b}(t)) \end{array}\right. \tag{4.56}$$

and $\hat{A}_b$ negative definite. The initial data $(h_{b,0}, \iota_{b,0})$ has compact support in $[0,\delta]$ and boundary data with Lipschitz norm of the order of δ_0. Its L^1 norm has to be estimated.

In the next sections we will consider the various cases above with different techniques.

S:scaletr1

5. Estimates for scalar equations

Aim of this section is twofold: first we prove an estimate for the L^1 norm of a scalar equation with source. Next, we give some estimates on the derivatives of the boundary conditions, knowing only that the boundary condition is L^1. This will allow to prove that the boundary data for (h_i, ι_i), $i = k, \dots, n$, is in L^1. We need also to study the oscillations of the derivative $h_{b,x}$ of the boundary data. Finally we will consider the differences which will arise when analyzing (v, w) instead of (h, ι).

Ss:TVest875

5.1. **Estimate of the L^1 norm.** We start with some estimates for a scalar conservation law of the form

E:model1

$$z_t + (\lambda(t,x) z)_x - z_{xx} = s(t,x), \tag{5.1}$$

with initial data z_0 and boundary data z_b. We first estimate the L^1 norm of the solution. We have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^+} |z(t,x)| dx &= \int_{\mathbb{R}^+} \operatorname{sgn}(z)(z_x - \lambda z)_x dx + \int_{\mathbb{R}^+} \operatorname{sgn}(z) s(t,x) dx \\ &\leq -\operatorname{sgn}(z_b)(z_{b,x} - \lambda z_b) + \int_{\mathbb{R}^+} |s(t,x)| dx \\ &\leq -|z_b|_x + \lambda |z_b| + \int_{\mathbb{R}^+} |s(t,x)| dx. \end{aligned}$$

The quantity $z_{b,x} - \lambda z_b$ can be thought as the oscillations of the integrated variable $Z_x = z$. It follows that we can estimate the L^1 norm at time T by

E:L1estsc1

$$\|z(T)\|_{L^1(\mathbb{R}^+)} \leq \|z_0\|_{L^1(\mathbb{R}^+)} + \int_0^T \left(-\operatorname{sgn}(z_b(t))\Big(z_{b,x}(t) - \lambda(t, x_b(t)) z_b(t)\Big)dt + \int_0^T \int_{\mathbb{R}^+} |s(t,x)| dx dt. \tag{5.2}$$

In the following we will assume that the initial data has L^1 norm less than $C\delta_1$, the boundary data has norm of the order of $C^2\delta_1$, and the source is of the order of δ_0^2, with $\delta_1 \leq \delta_0$:

E:initssd1

$$\|z_0\|_{L^1} \leq C\delta_1, \quad \|z_b\|_{L^1} \leq C^2\delta_1, \quad \|s\|_{L^1((0,T)\times\mathbb{R}^+)} \leq C\delta_0^2. \tag{5.3}$$

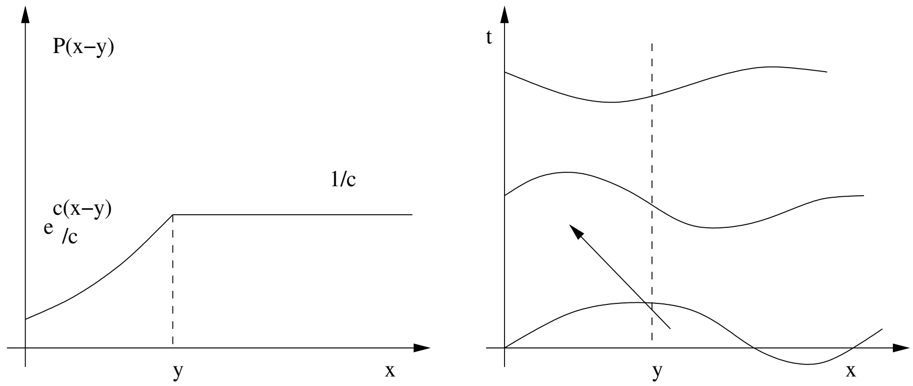

FIGURE 5. The weight function P and a pictorial explanation of formula (5.9) for the case $\lambda < -c$. Fi:bounriem6

Ss:shaper1 5.2. **Estimates for the characteristic fields leaving the domain.** Here we consider again a scalar equation of the form (5.1), but we assume that the characteristic speed λ is strictly negative:

E:speeexit1
$$\lambda(t,x) \leq c < 0, \tag{5.4}$$

where $c \gg \delta_0$. Thus, the integral curves of $\dot{x} = \lambda(t,x)$ are leaving the domain. Moreover, in connection with case 1), equation (4.53), we assume that the boundary condition $z_b(t)$ is 0: in a different language, this means that the integral variable $Z_x = z$ satisfies Neumann boundary condition.

Since for regular functions one has

$$\lim_{x\to 0^+} \mathrm{sgn}(z(t,x)) z_x(t,x) = |z_{b,x}|,$$

then by (5.2) it follows

E:redu01
$$\int_{\mathbb{R}^+} |z_{b,x}(t)|dt \leq \|z_0\|_{L^1} + \int_0^T \int_{\mathbb{R}^+} |s(t,x)|dxdt \leq C(\delta_1 + \delta_0^2). \tag{5.5}$$

The next estimate is the oscillations of the solutions along all vertical curves of the form $x(t) = y$, with y fixed constant in $[0,+\infty)$. Consider the functional (fig. 5)

E:approfut1
$$Q_y(z) \doteq \int_{\mathbb{R}^+} P(x-y)|z(x)|dx, \tag{5.6}$$

with

E:Pfunlmin0
$$P(x) = \begin{cases} e^{cx}/c & x \leq 0 \\ 1/c & x \geq 0 \end{cases} \tag{5.7}$$

By differentiating (5.6) w.r.t. t, it follows that (remember that $z_b = 0$)

E:qderlm0
$$\begin{aligned} \frac{dQ}{dt} &= \int_{\mathbb{R}^+} P(x-y)(\mathrm{sgn}(z)(z_x - \lambda z)_x + s(t,x))dx \\ &\leq \int_{\mathbb{R}^+} (P'' + \lambda P')|z(x)|dx - e^{-cy}|z_{b,x}|/c + \int_{\mathbb{R}^+} P(x-y)s(t,x)dx \\ &\leq -|z(y)| - e^{-cy}|z_{b,x}(t)|/c + \frac{1}{c}\int_{\mathbb{R}^+} |s(t,x)|dx. \end{aligned} \tag{5.8}$$

The above inequality yields

E:int+y1
$$\int_0^T |z(t,y)|dt \leq \|z_0\|_{L^1} + \frac{1}{c}\int_0^T \int_{\mathbb{R}^+} |s(t,x)|dx \leq \delta_1 + \mathcal{O}(1)\delta_0^2 \leq C(\delta_1 + \delta_0^2). \tag{5.9}$$

Note that the boundary data $z_{b,x}$ enters the equation with an exponential decaying rate.

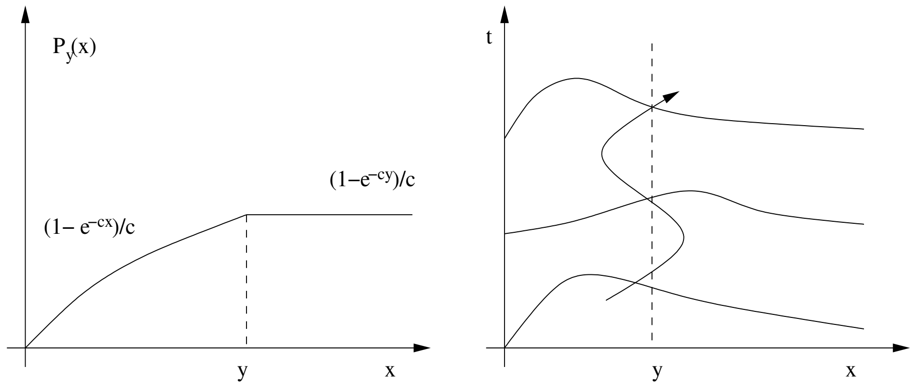

FIGURE 6. The weight function P_y and a pictorial explanation of formula (5.15) for the case $\lambda \simeq 0$. Fi:bounriem7

We now can also estimate the flux of x derivative along the vertical line $x(t) = y$. If Γ^{λ_0} is the Green kernel for the equation (5.1) with $\lambda_0 \leq c < 0$, we can write

E:Greenrep08
$$z_x(t) = \Gamma^{\lambda_0}_x(\tau) * z(t-\tau) + \int_0^\tau \Gamma^{\lambda_0}_x(\tau - s) * ((\lambda - \lambda_0) z(t+s-\tau))_x ds \\ + \int_0^\tau \Gamma^{\lambda_0}_x(\tau - s) * s(t+s-\tau) ds. \tag{5.10}$$

Assuming $\lambda - \lambda_0$ of the order δ_0 and the total variation of λ is less than $\mathcal{O}(1)\delta_0$, and defining

$$K(t) \doteq \sup_{y \geq 0} \left\{ \int_0^t |z_x(s,y)| ds \right\},$$

it follows that from (5.10) (neglecting an initial layer of thickness τ, for which the result trivially holds by the results of Section 2)

$$K(t) \leq \mathcal{O}(1)\delta_0 + \mathcal{O}(1)\sqrt{\tau}\delta_0 K(t) + \mathcal{O}(1)\delta_0^2.$$

We have used the integral estimate

$$\int_0^\tau \Gamma^{\lambda_0}_x(s,x) ds \leq \mathcal{O}(1),$$

valid if $\lambda_0 \leq c < 0$. We thus conclude for τ of the order 1 and δ_0 sufficiently small,

E:int+y2
$$\int_0^{+\infty} |z_x(t,y)| \leq C(\delta_1 + \delta_0^2). \tag{5.11}$$

This bound holds uniformly on vertical curves at every distance form the boundary. The constant C depends only on the separation in speed c when δ_0 is small enough. A more precise analysis can show that $K(t) \leq \delta_0$ for $\delta_1 \leq \delta_0 \ll 1$.

Ss:lambdaeq0

5.3. **Estimates for the characteristic field with speed close to $\sigma_b = 0$.** In this section we consider (5.1) with speed λ close to the speed of the boundary:

E:speechrh1
$$\lambda(t,x) = \mathcal{O}(1)\delta_0, \qquad \text{Tot.Var.}(\lambda(t)) \leq \mathcal{O}(1)\delta_0. \tag{5.12}$$

Our aim is to prove that there is a relation among the L^1 norm of the boundary data z_b and the L^1 norm of its derivative $z_{b,x}$. Consider the functional (fig. 6)

E:funsim1
$$Q(z) \doteq \int_{\mathbb{R}^+} P_y(x)|z(x)|dx, \tag{5.13}$$

where the weight function P is given by

E:Psime01
$$P_y(x) = \begin{cases} (1-e^{-dx})/c & x < y \\ (1-e^{-dy})/c & x \geq y \end{cases} \tag{5.14}$$

with d a small positive constant, larger than δ_0. By differentiating we obtain thus (note that $P_y(0)=0$)

$$\begin{aligned}\frac{dQ}{dt} &= \int_{\mathbb{R}^+} P_y(x)(\mathrm{sgn}(z)(z_x-\lambda z)_x + s(t,x))dx \\ &\leq \int_{\mathbb{R}^+} (P''+\lambda P')|z(x)|dx + |z_b(t)| - e^{-dy}|z(y)| + \int_{\mathbb{R}^+} P(x-y)s(t,x)dx \\ &\leq -e^{-dy}|z(y)| + |z_b(t)| + \frac{1}{c}\int_{\mathbb{R}^+} s(t,x)dx.\end{aligned}$$

This implies that for $\delta_0 \ll 1$

E:estier87

$$\int_0^T |z(t,y)|dt \leq e^{dy}(\delta_1 + C\delta_1 + C\delta_0^2) \leq C(\delta_1+\delta_0^2)e^{dy}, \tag{5.15}$$

i.e. the flux of z at distance k from the boundary is of the order of the initial-boundary oscillations multiplied by an exponential term. For the linear case one can see easily that by considering the solution to the scalar equation, the integral actually grows like $log(1+x)$, so that one cannot expect to have a small integral (5.15).

We now proceed to the estimate of the x derivative. In fact, if $\Gamma=\Gamma^0$ is the Green kernel with speed 0, multiply

E:Greenrep09

$$\begin{aligned}z_x(t) &= \Gamma_x(\tau)*z(t-\tau) + \int_0^\tau \Gamma_x(\tau-s)*(\lambda z(t+s-\tau))_x ds \\ &\quad + \int_0^\tau \Gamma_x(\tau-s)*s(t+s-\tau)ds + \int_0^\tau K_x(s,x)\dot z_b(t-s)ds + K_x(\tau,x)z_b(t-\tau),\end{aligned} \tag{5.16}$$

by e^{-dx}, and observe the estimates ($\Gamma(t,x;y) = G(t,x-y) - G(t,x+y)$ in this case)

$$\int_0^{+\infty} e^{-dx}\Gamma(t,x;y)e^{dy}dy, \int_0^{+\infty} e^{-dx}|\Gamma_x(t,x;y)|e^{dy}dy \leq \mathcal{O}(1)e^{d^2t}, \quad e^{-dx}K_x(t,x) \leq \mathcal{O}(1),$$

Using the same technique above. i.e. taking the supremum of the oscillations w.r.t. x weighted with e^{-dx}, one obtains that

E:oscichh1

$$\int_0^T |e^{-dy}z_x(t,y)|dt \leq C(\delta_1+\delta_0^2). \tag{5.17}$$

It follows that the oscillations at a distance of y from the boundary are controlled by e^{dy}: in particular the oscillations of $z_{b,x}$ are of the order of $\delta_1+\delta_0^2$. We note finally that the constant d can be chosen of the order of δ_0, hence it is much smaller than c, the separation of speed from strict hyperbolicity.

Using (5.17), from (5.2) it follows that

$$\|z(t)\|_{L^1} \leq C(\delta_1+\delta_0^2).$$

Ss:lambdgre1

5.4. **Estimates for the characteristic fields entering the domain.** The final case we consider is (5.1) with drift speed $\lambda \geq c > 0$. By repeating the computations of the above section (with $d=\|\lambda\|_{L^\infty}$), one can prove that

E:bouderc31

$$\int_0^T e^{-dy}|z_x(t,y)|dt \leq C(\delta_1+\delta_0^2). \tag{5.18}$$

Consider then the functional (fig. 7)

E:funsim12

$$Q(z) \doteq \int_{\mathbb{R}^+} P(x-y)|z(x)|dx, \tag{5.19}$$

where the weight function P is given by

E:Psime010

$$P(x) = \begin{cases} 1/c & x<0 \\ e^{-cx}/c & x\geq 0\end{cases} \tag{5.20}$$

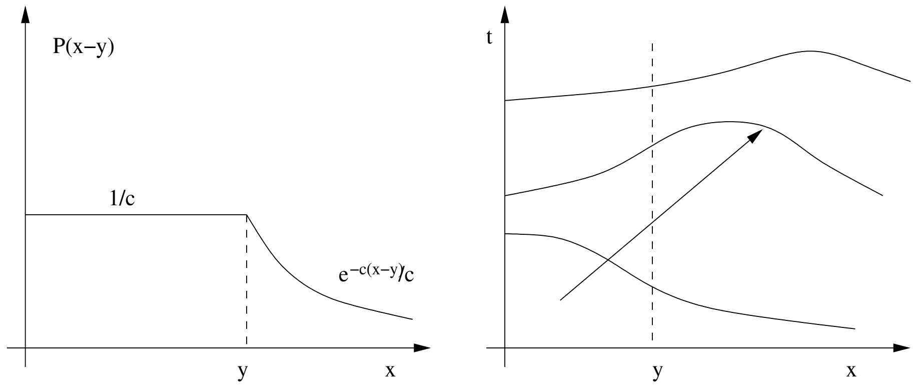

FIGURE 7. The weight function P and a pictorial explanation of formula (5.21) for the case $\lambda > c$. Fi:bounriem8

By differentiating we obtain thus

$$\begin{aligned}\frac{dQ}{dt} &= \int_{\mathbb{R}^+} P(x-y)(\mathrm{sgn}(z)(z_x - \lambda z)_x + s(t,x))dx \\ &\leq \int_{\mathbb{R}^+} (P'' + \lambda P')|z(x)|dx + \frac{1}{c}(|z_b|_x - \lambda |z_b|) - |z(y)| + \int_{\mathbb{R}^+} P(x-y)|s(t,x)|dx \\ &\leq \frac{1}{c}(|z_b|_x - \lambda|z_b|) - |z(y)| + \frac{1}{c}\int_{\mathbb{R}^+} P(x-y)|s(t,x)|dx.\end{aligned}$$

This implies that

E:estier88 (5.21)
$$\int_0^T |z(t,k)|dt = C(\delta_1 + \delta_0^2),$$

i.e. the flux at distance k from the boundary is bounded. A similar procedure to the regularity estimates above shows that for all $y \geq 0$

E:estier98 (5.22)
$$\int_0^T |z_x(t,y)|dt \leq C(\delta_1 + \delta_0^2),$$

without any exponential growing weight as in (5.17).

R:vcase1 *Remark* 5.1. One can repeat the above computations for the components v_i entering the domain, i.e. $i = k+1, \dots, n$, by assuming that the oscillations of the boundary data $v_{i,b}$ are in L^1: if

E:oscbdre (5.23)
$$\int_0^T \Big| v_{i,b,x}(t) - \tilde{\lambda}_i(t,0)v_{i,b}(t)\Big| dt \leq C\delta_1,$$

then

E:oscderw1 (5.24)
$$\int_0^T |v_i(t,y)|dt, \int_0^T |v_{i,x}(t,y)|dt \leq C(\delta_1 + \delta_0^2).$$

Note that while for the characteristic fields leaving the domain the same estimates of Section 5.2 hold, for the k-th characteristic fields in general v_k is not integrable in time. We finally observe that the components of u_t satisfy the same estimates of the components of h.

Sss:bbbeee1

5.5. **Estimates for the vector variables h_b, ι_b.** The last situation we are going to consider is the $(k-1)\times(k-1)$ system (4.56):

E:model2 (5.25)
$$Z_t + (\hat{A}_b(t,x)Z)_x - Z_{xx} = 0,$$

with initial data Z_0 with compact support and negative definite matrix flux $\hat{A}_b$. Since the boundary data is bounded by δ_0 in L^∞ and the initial data has compact support, one can easily show that

E:expdec9 (5.26)
$$|Z(t,x)| \leq \mathcal{O}(1)\delta_0 e^{cx},$$

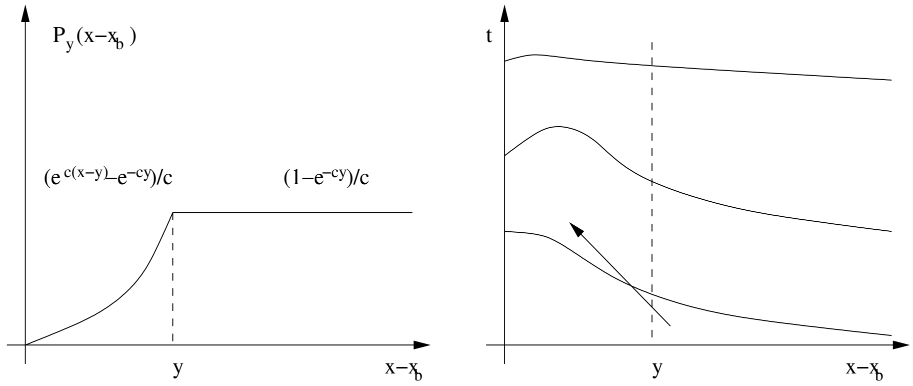

FIGURE 8. The weight function P_y and a pictorial explanation of formula (5.29). Fi:bounriem9

where $c = \sup_{t,x,i} \lambda_i(t,x) < 0$, where λ_i are the eigenvalues of $\hat{A}$. Note that the constant c is the constant of strict hyperbolicity (1.9).

By writing (5.25) as

E:scutre1

$$z_{i,t} + (a_{b,ii}(t,x) z_i)_x - z_{i,xx} = -\sum_{j \neq i} (a_{b,ij} z_j)_x, \tag{5.27}$$

and by using the same functional $Q_y(t) = \int P_y |Z|$ of Section 5.2 with weight function $P_y(x)$ given by (fig. 8)

E:Pfunlmin2

$$P_y(x) = \begin{cases} (e^{c(x-y)} - e^{-cy})/c & x \leq 0 \\ (1 - e^{-cy})/c & x \geq 0 \end{cases} \tag{5.28}$$

it can be shown that

E:oscillet1

$$\int_0^T |Z(t,y)| dt \leq C(\delta_1 + \delta_0^2) e^{-cy}, \qquad \int_0^T |Z_x(t,y)| dt \leq C(\delta_1 + \delta_0^2) e^{-cy}. \tag{5.29}$$

In fact in this case the initial data has compact support, so that $Q_y(0) \leq \mathcal{O}(1)\delta_1 e^{-cy}$.

R:beppe1

Remark 5.2. The same observation of Remark 5.1 for the variable v_b holds here: v_b is not integrable in time, but it is still exponentially decreasing.

Ss:bourder1

5.6. **Estimates of the boundary conditions for the components of h, ι.** We conclude this section by showing that the boundary conditions for h, ι are integrable and small. We recall that since the map

E:mabbfr1

$$\left\{ \begin{array}{rcl} \bar{h}_b &=& \hat{R}_b(\kappa, u_b, v_{b,b}, v_{k,b}) h_b + \hat{r}_k(\kappa, u_b, v_{b,b}, v_{k,b}, \zeta_k) h_k + \sum_{i>k} h_i \tilde{r}_i(\kappa, u_b, v_{i,b}, \zeta_i) \\ \bar{\iota}_b &=& \hat{R}_b(\kappa, u_b, v_{b,b}, v_{k,b}) \iota_b + \hat{r}_k(\kappa, u_b, v_{b,b}, v_{k,b}, \zeta_k) \iota_k + \sum_{i>k} (\iota_i - \lambda_{i,0} h_i) \tilde{r}_i(\kappa, u_b, v_{i,b}, \zeta_i) \end{array} \right. \tag{5.30}$$

with $\zeta_i = \lambda_{i,0} - \vartheta(\iota_i/h_i)$, is Lipschitz continuous and invertible for v small, then the boundary conditions for h_b, ι_b, h_i, ι_i, $i = k, \dots, n$ are defined by inverting the above map. Recall that for h_i, ι_i, $i = 0, \dots, k-1$ the boundary conditions are set to be 0.

Using the first equation of (5.30), and the fact that the matrix $(\hat{R}_b, \hat{r}_k, \dots, \tilde{r}_n)$ is a base in $\mathbb{R}^n$ with inverse uniformly bounded, then the components h_i are in L^1. By assuming that the boundary data $\bar{h}_b$ is small enough, we have that the L^1 norm of $h_{b,b}$, $h_{i,b}$, $i = k, \dots, n$, has L^1 norm in $[0, +\infty)$ less than $C\delta_1$.

To estimate $\bar{\iota}_b$, we use (4.30), which can be rewritten as

E:iotahrel2

$$\begin{aligned} \iota &= h_x - A(u) h \\ &= \hat{R}_b (h_{b,x} - \hat{A}_b h_b) + (h_{k,x} - \hat{\lambda}_k h_k) \check{r}_k + \sum_{i>k} (h_{i,x} - \tilde{\lambda}_i h_i) \check{r}_i \\ &\quad + \mathcal{O}(1)\delta_0 \left(|h_b| + |h_{b,x}| + \sum_{i \geq k} (|h_i| + |h_{i,x}|) \right) + \mathcal{O}(1)\delta_0 \sum_i |\iota_{i,x}|, \end{aligned} \tag{5.31}$$

where we observe that the only term where h does not appear is in the derivative of ζ. Integrating in $[0,T]$, we obtain that

$$\int_0^T |\iota(t)|dt \leq \mathcal{O}(1)\int_0^T |h(t)|dt + \mathcal{O}(1)\delta_0 \sum_i \int_0^T |\iota_{i,x}(t)|dt.$$

A standard argument (using the fact that from the results of the above sections the L^1 norm of the x derivative of the boundary data is of the same order of the L^1 norm of the boundary data) shows that for δ_0 sufficiently small, then the above integral is less than $C(\delta_1 + \delta_0^2)$. By assuming $\bar{h}_b$, $\bar{\iota}_b$ sufficiently small in L^1, we can suppose that the L^1 norm of the boundary data for all components h_i, ι_i is less than δ_0.

R:vwvarit16 *Remark* 5.3. We describe how to estimate the oscillations of the components v, $v_{i,x} - \tilde{\lambda}_i v_i$. Repeating the above computations with w instead of h, we obtain that the quantities w_b, w_k, $w_i - \lambda_{i,0}v_i$, $i \neq k$, are in L^1, with L^1 norm of the order of $C\delta_1$.

We can write (4.30) with v, w instead of h, ι as

E:iotahrel3

$$\begin{aligned} w &= v_x - A(u)v \\ &= (\hat{R}_b + \hat{R}_{b,v_b}v_b + \hat{r}_{k,v_b}v_k)(v_{b,x} - \hat{A}_b v_b) \\ &\quad + (v_{k,x} - \hat{\lambda}_k v_k)(\hat{r}_k + v_k \hat{r}_{k,v_k} + \hat{R}_{b,v_k}v_b) + \sum_{i<k} v_{i,x}(\tilde{r}_i + v_i \tilde{r}_{i,v}) \\ &\quad + \sum_{i>k}(v_{i,x} - \tilde{\lambda}_i v_i)(\tilde{r}_i + v_i \tilde{r}_{i,v}) + \mathcal{O}(1)\delta_0 \sum_{i\neq k}(|w_i| + |w_{i,x}|) \\ &\quad + \mathcal{O}(1)\delta_0 \sum_{i>k}|v_i| + \mathcal{O}(1)\delta_0(|w_k| + |w_{k,x}|) + \mathcal{O}(1)|v_k||w_k + \sigma_k v_k|. \end{aligned} \tag{5.32}$$

For the fields not boundary characteristic we know the boundedness of their L^1 norm in t, while for the boundary characteristic we have

E:brutal1

$$|w_k + \sigma_k v_k| \simeq |w_k|, \tag{5.33}$$

if the function θ is chosen appropriately. Recall that for the characteristic fields leaving the domain the boundary data is 0, while its x derivative is of the order of $\delta_1 + \delta_0^2$.

Integrating (5.32) in $[0,T]$ one obtain

E:border1

$$\int_0^T |v_{b,x} - \hat{A}_b v_b|dt + \int_0^T |v_{k,x} - \hat{\lambda}_k v_k|dt + \sum_{i>k}\int_0^T |v_{i,x} - \tilde{\lambda}_i v_i|dt \leq \mathcal{O}(1)(\delta_1 + \delta_0^2), \tag{5.34}$$

where we used the results of this section to estimate the integrals of the components v_i entering the domain in terms of the oscillation of the boundary data

$$\sum_{i>k}\int_0^T |v_{i,b}(t)|dt \leq C\sum_{i>k}\int_0^T |v_{i,bx}(t) - \tilde{\lambda}_i v_{i,b}(t)|dt. \tag{5.35}$$

We thus have that the oscillations of v have boundary data with bounded oscillations. Note that for the non boundary characteristic components of w, the same results hold, while for the boundary characteristic one since $\lambda_{k,0} = 0$, it follows directly form formula (4.2).

S:intesc1

6. Interaction functionals

In this section we show how to estimate the source terms given by (4.44). A part of Φ, Ψ is integrable and of order δ_0^2: the part due to the oscillations of the constant κ, which is multiplied by the components of u_x, u_t, and the source terms ω, ϖ of v, w. The remaining part is divided into 3 parts:

(1) non transversal terms, given by (4.34), (4.36);
(2) transversal terms, given by (4.38), (4.39);
(3) boundary terms, given by (4.37).

One of the main novelties of this paper is the estimation of the last terms. The first two are estimated similarly to [3]: we introduce three functional, whose decay controls the corresponding source terms. The boundary terms are essentially energy terms, i.e. the amount of waves on the boundary layer travelling with speed different from $\sigma_b = 0$ is integrable and of order δ_0^2.

Ss:diffew55

6.1. **Interactions of waves of different families.**

We start by considering the terms of different families. In fact, once these terms have been estimated, the non transversal terms are essentially related to estimates for a scalar equation.

We thus consider two equations

E:trnmodel1
$$z_{1,t} + (\lambda_1(t,x) z_1)_x - z_{1,xx} = s_1(t,x), \qquad x \geq 0, \tag{6.1}$$

E:trnmodel2
$$z_{2,t} + (\lambda_2(t,x) z_2)_x - z_{2,xx} = s_2(t,x), \qquad x \geq K, \tag{6.2}$$

with the assumptions that the oscillations of the boundary data for both is bounded in L^1 by δ_0, the initial data has L^1 norm less than δ_0 and the speed are strictly separated,

E:strihyp1
$$\lambda_2 - \lambda_1 \geq c > 0. \tag{6.3}$$

We do not require that the boundary curves coincides, i.e. $K = 0$.

Consider the functional

E:Qtran89
$$Q(t) \doteq \int_0^{+\infty} \int_K^{+\infty} P(x-y) |z_1(t,x)| |z_2(t,y)| dx dy, \tag{6.4}$$

with P given by

E:Ptranbg1
$$P(x) = \begin{cases} e^{cx/2}/c & x \leq 0 \\ 1/c & x > 0 \end{cases} \tag{6.5}$$

Differentiating w.r.t. t we obtain

$$\begin{aligned}
\frac{dQ}{dt} &= \int_0^{+\infty} \int_K^{+\infty} P(x-y) (|z_{1,t}(x)||z_2(y)| + |z_1(x)||z_{2,t}(y)|) dx dy \\
&\leq \int_0^{+\infty} \int_K^{+\infty} P(x-y) (|s_1(x)||z_2(y)| + |z_1(x)||s_2(y)|) dx dy \\
&\quad - (|z_1|_x - \lambda_1 |z_1|)_{x=0} \int_K^{+\infty} P(-y) |z_2(y)| dy - (|z_2|_x - \lambda_2 |z_2|)_{y=K} \int_0^{+\infty} P(x-K) |z_1(x)| dx \\
&\quad + |z_1(0)| \int_K^{+\infty} P'(-y) |z_2(y)| dy - |z_2(K)| \int_0^{+\infty} P'(x-K) |z_1(x)| dx \\
&\quad - \int_{\max\{0,K\}}^{+\infty} |z_1(t,x)||z_2(t,x)| dx.
\end{aligned}$$

Since we assume that the source term are integrable in the quarter plane and of order δ_0^2, then we have that

$$\int_0^T \int_0^{+\infty} \int_K^{+\infty} P(x-y) (|s_1(x)||z_2(y)| + |z_1(x)||s_2(y)|) dx dy \leq \mathcal{O}(1) \delta_0^3.$$

Moreover, by using the assumption on the oscillations of the boundary terms, we obtain

$$\int_0^T \Big(|z_1|_x - \lambda_1 |z_1| \Big)_{x=0} \int_K^{+\infty} P(-y) |z_2(y)| dy \leq \mathcal{O}(1) \delta_0^2,$$
$$\int_0^T \Big(|z_2|_x - \lambda_2 |z_2| \Big)_{y=K} \int_0^{+\infty} P(x-K) |z_1(x)| dx \leq \mathcal{O}(1) \delta_0^2.$$

Since also $Q(0) = \mathcal{O}(1)\delta_0^2$, to conclude that the interaction term $|z_1||z_2|$ is quadratic, we have to estimate the terms which contains the boundary data, not its oscillation:

$$|z_1(0)| \int_K^{+\infty} P'(-y) |z_2(y)| dy, \quad |z_2(K)| \int_0^{+\infty} P'(x-K) |z_1(x)| dx.$$

We have to consider different cases, depending on the relation among the drift λ and the speed of the boundary data.

(1) neither λ_1 nor λ_2 are characteristics. In this case we recall that from the results of Sections 5.2, 5.4 that the boundary data either is 0 or is integrable and of order δ_0. We can thus write

$$\int_0^T |z_1(x_b)| \int_{y_b}^{+\infty} P'(x_b - y)|z_2(y)|dy \leq \mathcal{O}(1)\delta_0^2,$$

and the same computations can be used for the other integral;

(2) if one drift speed (and only one, due to strict hyperbolicity) is boundary characteristic, then we know that its boundary data is bounded by $\mathcal{O}(1)\delta_0$ in L^∞. In this case the other variable can be integrated along vertical curves, and P' is integrable in space: if z_1 is characteristic,

$$\begin{aligned}\int_0^T |z_1(x_b)| \int_K^{+\infty} P'(-y)|z_2(t,y)|dy &\leq \mathcal{O}(1)\delta_0 \int_0^T \int_{\max\{0,K\}}^{+\infty} e^{-cy/2}|z_2(t,y)|dydt \\ &= \mathcal{O}(1)\delta_0 \int_{\max\{0,K\}}^{+\infty} e^{-cy/2} \int_0^T |z_2(t,y)|dtdy \leq \mathcal{O}(1)\delta_0^2.\end{aligned}$$

The other case can be treated similarly.

In all cases, the integrals are bounded by δ_0^2, so that

E:abrrgnao1

$$\int_0^T \int_{\max\{0,K\}}^{+\infty} |z_1(t,x)||z_2(t,x)|dx \leq C\delta_0^2. \tag{6.6}$$

We thus have proved that all the terms (4.38) which do not contain derivatives are bounded by $C\delta_0^2$, if C is a large constant.

The same technique used to estimate case 2) above proves that the terms (4.39) are bounded by $C\delta_0^2$. The idea is that the integral in space is associated to the boundary layer term, while the integral in time is associated to the other characteristic field. Since the latter is not boundary characteristic, its flow along any vertical curve is of the order of δ_0. As an example, consider

$$\int_0^T \int_0^{+\infty} |h_i(t,x)||h_b(t,x)| \leq \int_0^{+\infty} \mathcal{O}(1)\delta_0 e^{-cx} \int_0^T |h_i(t, x_b + k)|dtdk \leq C\delta_0^2.$$

The first inequality follows from (5.29), while last inequality follows from the estimates (5.9) of Section 5.2 and (5.21) of Section 5.4, depending if the i-th characteristic field is leaving or entering the domain. By means of (5.11), (5.22), (5.29), also the terms containing derivatives of h_b, ι_b, h_i, ι_i, $i \neq k$, can be estimated with the same technique.

We are thus left with the estimate of integrals in the quarter plane of the products $z_{1,x}z_2$, $z_1 z_{2,x}$. This will conclude the analysis of all terms in (4.38). We consider only the first case, since for the other one the computations are similar.

By Duhamel formula, we can write for $t \geq \delta t$ (with Tot.Var.$(\lambda_1 - \lambda_{1,0}) \leq C\delta_0$)

E:duhamtrv1

$$\begin{aligned} z_{1,x}(t,x) = &\int_{\mathbb{R}^+} \Gamma_x^{\lambda_{1,0}}(\delta t, x; y) z_1(t - \delta t, y)dy - \int_0^{\delta t}\int_{\mathbb{R}^+} \Gamma_x^{\lambda_{1,0}}(s,x;y)((\lambda_1 - \lambda_{i,0})z_1)_x(t-s,y)dsdy \\ &+ \int_0^{\delta t}\int_{\mathbb{R}^+} \Gamma_x^{\lambda_{1,0}}(s,x;y)s_1(t-s,y)dsdy + \int_0^{\delta t} K_x^{\lambda_{1,0}}(s)\dot{z}_b(t-s)ds + K_x^{\lambda_{1,0}}(\delta t)z_b(t-\delta t).\end{aligned} \tag{6.7}$$

It is clear that we have only to consider the integrals

$$\int_0^T \int_{\mathbb{R}^+} |z_{1,x}z_2|dxdt$$

for $T \geq \delta t$, with $\delta t = \mathcal{O}(1)$, because from the regularity estimates in $[0, \delta t]$ the quadratic estimate $\leq C\delta_0^2$ follows. We now consider the following integrals:

(1) by means of the results of the estimate for transversal terms with shift, it follows

E:trvestder1

$$\begin{aligned}
&\int_{\max\{0,\delta t-\tau\}}^{\min\{T,T-\tau\}} \int_{\max\{0,q\}}^{+\infty} |z_2(t,x)| \int_{\mathbb{R}^+} |G_x^{\lambda_{1,0}}(\delta t, x-y-q)||z_1(t+\tau-\delta t, y)|dydtdx \\
&= \int_{\mathbb{R}} |G_x^{\lambda_{1,0}}(\delta t, y)| \int_{\max\{0,\delta t-\tau\}}^{\min\{T,T-\tau\}} \int_{\max\{0,q,y-q\}}^{+\infty} |z_2(t,x)||z_1(t+\tau-\delta t, x-y-q)|dxdt \\
&\leq \mathcal{O}(1) \int_{\mathbb{R}} |G_x^{\lambda_{1,0}}(\delta t, y)| e^{\mathcal{O}(1)\delta_0 y} \delta_0^2 \leq \mathcal{O}(1) \frac{\delta_0^2}{\sqrt{\delta t}}, \qquad (6.8)
\end{aligned}$$

where we used the estimate (6.6) (time shift has no influence in the computation leading to (6.6)). The function $e^{\mathcal{O}(1)\delta_0 y}$ enters in the computation because the L^1 norm of a boundary characteristic field at a distance y from the boundary is of the order of $e^{\mathcal{O}(1)\delta_0 y}$;

(2) using the integrability of $(y/t)e^{-xy/t}G^{\lambda_{1,0}}(t, x-y)$, we can write

E:trvestder2

$$\begin{aligned}
&\int_{\max\{0,\delta t-\tau\}}^{\min\{T,T-\tau\}} \int_{\max\{0,q\}}^{+\infty} |z_2(t,x)| \\
&\times \int_q^{+\infty} ((y-q)/\delta t) e^{-(x-q)(y-q)/\delta t} G^{\lambda_{1,0}}(\delta t, x-y) |z_1(t+\tau-\delta t, y-q)| dydtdx \\
&= \int_{\mathbb{R}} G^{\lambda_{1,0}}(\delta t, y) \int_{\max\{0,\delta t-\tau\}}^{\min\{T,T-\tau\}} \int_{\max\{0,q,y+q\}}^{+\infty} (x-y-q) e^{-(x-q)(x-y-q)/\delta t}/\delta t \\
&\times |z_2(t,x)||z_1(t+\tau-\delta t, x-y-q) dtdxdy \\
&\leq \mathcal{O}(1)(\delta t)^{-1} \int_{\mathbb{R}} G^{\lambda_{1,0}}(\delta t, y) \int_{\max\{0,\delta t-\tau\}}^{\min\{T,T-\tau\}} \int_{\max\{0,q,y+q\}}^{+\infty} |z_2(t,x)||z_1(t+\tau-\delta t, x-y-q)|dtdxdy \\
&\leq \mathcal{O}(1)(\delta t)^{-1}\delta_0^2; \qquad (6.9)
\end{aligned}$$

(3) from the integrability of K_x, it follows that since at least one field is not characteristic (assume z_1, the other case being similar)

E:trvestder3

$$\begin{aligned}
&\int_{\max\{0,\delta t-\tau\}}^{\min\{T,T-\tau\}} \int_{\max\{0,q\}}^{+\infty} |z_2(t,x)| \int_0^{\delta t} K_x(s, x-q)|z_{1,b}(t+\tau-s)|ds \qquad (6.10) \\
&\qquad \leq \mathcal{O}(1)\delta_0 \int_0^{\delta t} \int_{\max\{0,q\}} K_x(s, x-q) \int_{\max\{0,\delta t-\tau\}}^{\min\{T,T-\tau\}} |z_1(t,x)|dtdxds \leq \mathcal{O}(1)\delta_0^2.
\end{aligned}$$

A similar estimate holds for the term

E:trvestder4

$$\int_{\max\{0,\delta t-\tau\}}^{\min\{T,T-\tau\}} \int_{\max\{0,q\}}^{+\infty} |z_1(t,x)| K_x(\delta t, x-q) |z_{2,b}(t+\tau-\delta t)| ds \leq \mathcal{O}(1)\delta_0^2. \qquad (6.11)$$

Define now

E:KTTRE1

$$\mathbb{K}(T) = \sup \left\{ \int_{\delta t}^{T} \int_y^{+\infty} |z_2(t,x)||z_1(t-s, x-y)|dtdx; 0 \leq s \leq \delta t, y \geq 0 \right\}. \qquad (6.12)$$

Multiplying (6.7) by $z_2(t-\tau, x+K)$, $0 \leq s \leq \delta t$, and integrating in $[\max\{0, \delta t-\tau\}, \min\{T, T+\tau\}] \times [\max\{0,q\}, +\infty)$, we obtain from the previous computations

$$\int_{\max\{0,\delta t-\tau\}}^{\min\{T,T-\tau\}} \int_{\max\{0,q\}}^{+\infty} |z_2(t,x)||z_1(t+\tau, x-q)| \leq \mathcal{O}(1)\delta_0^2 + \mathcal{O}(1)\delta t \delta_0 \mathbb{K}(T),$$

so that it follows for $\delta t = \mathcal{O}(1)$ that

E:final010

$$\mathbb{K}(T) \leq C\delta_0^2. \qquad (6.13)$$

This concludes the estimate of the transversal terms containing a derivative of a component.

The estimate of these terms makes the estimates of the next sections much simpler. In fact, we can now replace any time we need the variable ι_i with $h_{i,x}$ or h_i, depending on the particular case, because the remaining part are terms of different families, hence generating terms which are integrable.

Ss:same01

6.2. **Interactions of waves of the same families.** In this section we estimate the non transversal terms (4.34), (4.36). The main tools are two functionals, defined for a couple of equation with the same drift speed λ. These functionals are the extension of the Area functional and Length functional for the boundary free case, hence we will keep their name. As in the previous section, the estimates on the boundary data and their derivatives will be of fundamental importance.

Sss:area07

6.2.1. *Area functional.* Consider thus two equations,

E:exsyt1

$$\left\{ \begin{array}{rcl} z_{1,t} + (\lambda(t,x) z_1)_x - z_{1,xx} &=& s_1(t,x) \\ z_{2,t} + (\lambda(t,x) z_2)_x - z_{2,xx} &=& s_2(t,x) \end{array} \right. \tag{6.14}$$

with integrable boundary data and integrable initial data. As before, the L^1 norm of s_1, s_2 in $[0,T] \times \mathbb{R}^+$ is of the order of δ_0^2.

Following [4], we define

E:Qarea01

$$Q(t) = \frac{1}{2} \iint_{0 \leq x \leq y} \big| z_1(x) z_2(y) - z_2(x) z_1(y) \big| dx dy. \tag{6.15}$$

Equivalently one can consider the curve

E:gamma02

$$\gamma(t,x) = \int_{+\infty}^{x} \left(\begin{array}{c} z_1(t,y) \\ z_2(t,y) \end{array} \right) dy, \tag{6.16}$$

which satisfies the equation

E:gammaeq1

$$\gamma_t + \lambda(t,x) \gamma_x - \gamma_{xx} = S(t,x), \qquad \gamma_x(t,0) = \left(\begin{array}{c} z_{1,b}(t), \\ z_{2,b}(t) \end{array} \right), \tag{6.17}$$

with Tot.Var.$(S) \leq C \delta_0^2$. Then one has an equivalent representation of Q,

E:Qpot0998

$$Q(t) = \frac{1}{2} \int_0^{+\infty} \int_x^{+\infty} \big| \gamma_x \wedge \gamma_y \big| dx dy. \tag{6.18}$$

We have

$$\begin{aligned} \frac{dQ}{dt} &= \frac{1}{2} \int_0^{+\infty} \int_x^{+\infty} \mathrm{sgn}(\gamma_x \wedge \gamma_y) \big(\gamma_{xx} - \lambda \gamma_x + S \big)_x \wedge \gamma_y dx dy \\ &\quad + \frac{1}{2} \int_0^{+\infty} \int_x^{+\infty} \mathrm{sgn}(\gamma_x \wedge \gamma_y) \gamma_x \wedge \big(\gamma_{yy} - \lambda_1 \gamma_y + S \big)_y dx dy \\ &\leq -\frac{1}{2} \int_0^{+\infty} \Big(\lim_{y \to 0^+} \mathrm{sgn}(\gamma_y \wedge \gamma_x) \Big) \gamma_t(x_b) \wedge \gamma_x dx - \int_0^{+\infty} \big| \gamma_t \wedge \gamma_x \big| dx + \mathcal{O}(1) \delta_0 \int_0^{+\infty} |S_x(t,x)| dx \\ &\leq \mathcal{O}(1) \delta_0 \left(|\gamma_t(x_b)| + \int_0^{+\infty} |S_x(t,x)| dx \right) - \int_{x_b}^{+\infty} \big| \gamma_t \wedge \gamma_x \big| dx. \end{aligned}$$

Since we know that the boundary data and its first derivative are integrable and of order δ_0 in $[0,T]$, we conclude that

E:areleav1

$$\int_0^{+\infty} \int_0^{+\infty} \big| z_{1,x} z_2 - z_1 z_{2,x} \big| dx dt \leq C \delta_0^2. \tag{6.19}$$

In the above estimate we can substitute z_1, z_2 with any couple of variables v_i, w_i, h_i, ι_i: we thus have estimated the $6n$ terms

E:areater1

$$\begin{array}{c} |v_i w_{i,x} - v_{i,x} w_i|, \ |v_i h_{i,x} - v_{i,x} h_i|, \ |v_i \iota_{i,x} - v_{i,x} \iota_i|, \\ |w_i h_{i,x} - w_{i,x} h_i|, \ |w_i \iota_{i,x} - w_{i,x} \iota_i|, \ |h_i \iota_{i,x} - h_{i,x} \iota_i|, \end{array} \tag{6.20}$$

with $i = 1, \dots, n$. Their integral in $\{0 \leq t \leq T, x \geq 0\}$ less than $C \delta_0^2$, C being a large constant.

At this point we can estimate many terms of (4.35), (4.36). First we observe that in the regions where $|w_i/v_i| \geq 2\delta_0$, $|\iota_i/h_i| < \delta_0$ we have (neglecting the waves of different family, which generate transversal terms and thus are estimated by the results of the above section)

$$|h_i v_{i,x}| \le \frac{1}{2}|h_i v_i(v_{i,x}/v_i - h_{i,x}/h_i)| = \frac{1}{2}|h_i v_{i,x} - v_i h_{i,x}|, |h_i w_{i,x}| \le \qquad |h_i w_{i,x} - w_i h_{i,x}| + |w_i h_{i,x}|$$
$$\le |h_i w_{i,x} - w_i h_{i,x}| + \mathcal{O}(1)|w_i h_i| = |h_i w_{i,x} - w_i h_{i,x}| + \mathcal{O}(1)|h_i v_{i,x}|,$$

and similarly for the other terms

$$|v_i h_{i,x}| = C/2|h_i v_i(\sigma_i - \zeta_i)|, \qquad |v_i \iota_{i,x}| \le |v_i \iota_{i,x} - \iota_i v_{i,x}| + \mathcal{O}(1)|v_i h_{i,x}|.$$

Moreover we have that in the region where θ, ϑ are not active,

E:101
$$\begin{aligned} h_i(w_{i,x} + \sigma_i v_{i,x}) &= (h_i w_{i,x} - w_i h_{i,x}) + \sigma_i(h_i v_{i,x} - h_{i,x} v_i), \\ v_i(\iota_{i,x} + \zeta_i h_{i,x}) &= (v_i \iota_{i,x} - \iota_i v_{i,x}) + \zeta_i(v_i h_{i,x} - v_{i,x} h_i), \end{aligned} \tag{6.21}$$

so that the only term to estimate is $v_i h_i(\sigma_i - \zeta_i)$. Since we have, a part from from transversal terms, $w_i = v_{i,x} - \tilde\lambda_i v_i$, $\iota_i = h_{i,x} - \tilde\lambda_i h_i$, it follows that

E:102
$$h_i v_i(\sigma_i - \zeta_i) = h_i v_{i,x} - v_i h_{i,x}. \tag{6.22}$$

Thus all terms of (4.35), (4.36) not in the first row are estimated by (6.20), i.e. they are less than $C\delta_0^2$.

Sss:leghtv1

6.2.2. *Length functional.* Next, we consider the length functional,

E:legtho11
$$L(t) = \int_0^{+\infty} |\gamma_x| dx, \tag{6.23}$$

for which we obtain

$$\begin{aligned} \frac{dL}{dt} &= \int_0^{+\infty} \left\langle \frac{\gamma_x}{|\gamma_x|}, \big(\gamma_{xx} - \lambda\gamma_x + S\big)_x \right\rangle dx \\ &\le -\left\langle \frac{\gamma_x}{|\gamma_x|}, \gamma_t(x_b) \right\rangle - \int_0^{+\infty} \Big(z_1 (z_2/z_1)_x^2 \chi\{|z_2/z_1| \le 3\delta_1\} \Big) dx + \mathcal{O}(1)\delta_0^2. \end{aligned}$$

Note that the integration by parts is valid because we are assuming that $|h_i| + |\iota_i| \neq 0$. By using again the integrability of the boundary data, we conclude that

E:leght11ee
$$\int_0^{+\infty}\int_0^{+\infty} \Big(z_1^2 (z_2/z_1)_x^2 \chi\{|z_2/z_1| \le 3\delta_1\} \Big) dx \le \mathcal{O}(1)\delta_0^2. \tag{6.24}$$

This implies that all the terms of the form $v_i h_i(\zeta_{i,x})^2$, $i = 1, \dots, n$ are estimated by $\mathcal{O}(1)\delta_0^2$.

Sss:energyv1

6.2.3. *Energy functional.* Finally we study the energy for the waves travelling with speed much differently than the eigenvalue λ. These terms correspond to the terms of the first row of (4.35), (4.36), i.e.

E:enetttr1
$$\sum_{i=1}^n (|v_{i,x}|^2 + |w_{i,x}|^2 + |h_{i,x}|^2 + |\iota_{i,x}|^2) \chi\{|w_i/v_i|, |\iota_i/h_i| \ge 2\delta_0\}. \tag{6.25}$$

The estimate which shows that these terms are integrable and of order δ_0^2 is divided into 2 parts: for $h_{i,x}$ ($v_{i,x}$), it follows from the fact that the effective speed $\iota_i/h_i \simeq h_{i,x}/h_i$ ($w_i/v_i \simeq v_{i,x}/v_i$) of h_i (v_i) is much higher than $\tilde\lambda_i$, so that by multiplying by h_i (v_i) and integrating by parts the dissipative term $h_{i,x}^2$ ($v_{i,x}^2$) wins over $h_{i,x}\tilde\lambda_i h_i$ ($v_{i,x}\tilde\lambda_i v_i$). The estimate for $\iota_{i,x}^2$ ($w_{i,x}^2$) follows because one can show by using the estimate on $h_{i,x}^2$ ($v_{i,x}^2$) that ι (w) in the regions where $|\iota_i/h_i| \ge 2\delta_0$ ($|w_i/v_i| \ge 2\delta_0$) satisfies a linear equation plus integrable terms of order δ_0^2.

Note that this type of estimates is possible because of the Neumann boundary condition for the characteristic fields leaving the domain: in fact if we allow the variable z, solution to a scalar equation with drift $\lambda \le c < 0$ to generate a boundary layer, then this stationary solution has the speed of the boundary, which is different from the drift λ, and this boundary layer is certainly not going to decay. We will show these estimates only for h_i, ι_i.

We thus consider the equations for the i-th components h_i, ι_i, $i = 1, \dots, n$

E:hioener1
$$\left\{ \begin{array}{rcl} h_{i,t} + (\tilde\lambda_i h_i)_x - h_{i,xx} &=& \phi_i(t,x) \\ \iota_{i,t} + (\tilde\lambda_i \iota_i)_x - \iota_{i,xx} &=& \psi_i(t,x) \end{array} \right. \tag{6.26}$$

with initial-boundary data small in L^1. We recall that from Remark 4.6, in the regions where $|\iota_i/h_i| \geq 2\delta_0$ we can write

$$|\iota_i| \leq \mathcal{O}(1) \left(|h_{i,x}| + \sum_{j \neq i} (|v_j| + |w_j| + |h_j| + |\iota_j|) \right).$$

Multiplying the scalar equation for $h_i \varrho(\iota_i/h_i) = h_i \varrho_i$, with a suitably chosen cutoff function ϱ,

E:varrho8
$$\varrho(x) = \begin{cases} 0 & |x| \leq \delta_0 \\ \text{smooth connection} & \delta_0 < |x| \leq 2\delta_0 \\ 1 & |x| < 2\delta_0 \end{cases} \tag{6.27}$$

integrating by parts, we obtain

E:rutto1
$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^+} \frac{h_i^2}{2} \varrho_i - \int_{\mathbb{R}^+} \frac{h_i^2}{2} \varrho_{i,t} - \int_{\mathbb{R}^+} h_i h_{i,x} \tilde{\lambda}_i \varrho_i - \int_{\mathbb{R}^+} \tilde{\lambda}_i h_i^2 \varrho_{i,x} + \int_{\mathbb{R}^+} h_{i,x}^2 \varrho_i \\ &+ 2 \int_{\mathbb{R}^+} h_i h_{i,x} \varrho_{i,x} + \int_{\mathbb{R}^+} \frac{h_i^2}{2} \varrho_{i,xx} + h_i \Big(h_{i,x} \varrho_i - h_i \tilde{\lambda}_i \varrho_i + h_i \varrho_{i,x} \Big) \Big|_{x=0} = \int_{\mathbb{R}^+} h_i \varrho_i \phi_i(t,x) dx. \end{aligned} \tag{6.28}$$

We recall that, neglecting from now on the transversal terms, in the regions where $\varrho_i \neq 0$ one can write

$$|h_i|, |\iota_i| \leq \mathcal{O}(1) |h_{i,x}|,$$

and conversely in the regions where the speed is bounded by $2\delta_0$, one has

$$|h_{i,x}|, |\iota_1|, \leq \mathcal{O}(1) |h_i|.$$

We can thus estimate the following integrals by means of (6.19), (6.24) and the fact that when $\varrho' \neq 0$ we have $|\iota_i/h_i| \leq 2\delta_0$, i.e. $|h_{i,x}| \leq \mathcal{O}(1)|h_i|$:

$$\int_0^T \int_{\mathbb{R}^+} \Big(h_i^2 |\varrho_{i,x}| + |h_i h_{i,x} \varrho_{i,x}| \Big) dxdt \leq \mathcal{O}(1) \delta_0^2,$$

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^+} h_i^2 \big(\varrho_{i,t} + \tilde{\lambda}_i \varrho_{i,x} - \varrho_{i,xx} \big) dxdt \\ &\quad = \int_0^T \int_{\mathbb{R}^+} 2 h_i h_{i,x} \varrho' \zeta_{i,x} - h_i^2 \varrho'' \zeta_{i,x}^2 dxdt + \int_0^T \int_{\mathbb{R}^+} |h_i| |\phi_i| + |\iota_i| |\psi_i| dxdt \leq \mathcal{O}(1) \delta_0^2, \end{aligned}$$

so that we obtain finally integrating (6.28) in $[0,T]$

E:energy08
$$\int_0^T \int_{\mathbb{R}^+} |h_{i,x}^2| \chi\{|\iota_i/h_i| \geq \delta_0\} dxdt \leq 2 \int_0^T \int_{\mathbb{R}^+} |h_{i,x}(h_{i,x} - \tilde{\lambda}_i h_i)| \varrho_i dxdt \leq C \delta_0^2, \tag{6.29}$$

where we used the fact that the boundary data are integrable and $2|h_{i,x} - \tilde{\lambda}_i h_i| \varrho_i \geq |h_{i,x}^2| \varrho_i$.

A similar estimate can be done for w_1, where the only difference is in the term

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^+} |\iota_i \iota_{i,x} \varrho_{i,x}| dxdt &= \int_0^T \int_{\mathbb{R}^+} \Big| \varrho_i' \iota_{i,x} (\iota_{i,x} - h_{i,x} \iota_i/h_i) \iota_i/h_i \Big| dxdt \\ &\leq \mathcal{O}(1) \int_0^T \int_{\mathbb{R}^+} \Big| \varrho_i' h_i^2 (\zeta_{i,x})^2 \Big| dxdt + \int_0^T \int_{\mathbb{R}^+} \Big| \varrho_i' h_i^2 \zeta_{i,x} \Big| dxdt \leq \mathcal{O}(1) \delta_0^2, \end{aligned}$$

and in the fact that we cannot estimate $\iota_{i,x} - \tilde{\lambda}_i \iota_i$. We thus write

$$\begin{aligned} &- \int_0^T \int_{\mathbb{R}^+} \iota_i \iota_{i,x} \tilde{\lambda}_i \varrho_i dxdt = \int_0^T \int_{\mathbb{R}^+} \frac{\iota_i^2}{2} (\tilde{\lambda}_{i,x} \varrho_i + \tilde{\lambda}_i \varrho_{i,x}) dxdt + \int_0^T \frac{\iota_i^2}{2} \tilde{\lambda}_i \varrho_i \Big|_{x=0} dt \\ &\quad \leq \mathcal{O}(1) \int_0^T \int_{\mathbb{R}^+} h_{i,x}^2 \varrho_i dxdt + \mathcal{O}(1) \int_0^T \int_{\mathbb{R}^+} h_i^2 \varrho_{i,x} dxdt + \mathcal{O}(1) \int_0^T \frac{h_i^2}{2} \tilde{\lambda}_i \varrho_i \Big|_{x=0} dt \leq \mathcal{O}(1) \delta_0^2, \end{aligned}$$

where we used the previous estimate (6.29). Thus one can also write the estimate

E:energy081
$$\int_0^T \int_{\mathbb{R}^+} |\iota_{i,x}^2| \chi\{|\iota_i/h_i| \geq \delta_0\} dxdt \leq \int_0^T \int_{\mathbb{R}^+} |\iota_{i,x}^2| \varrho_i dxdt \leq C \delta_0^2. \tag{6.30}$$

The above estimates (6.29), (6.30) implies that the terms $h_{i,x}^2$, $\iota_{i,x}^2$ are bounded in the regions $|\iota_i/h_i| \geq 2\delta_0$, for $i = 1, \dots, n$. The same computations can be used for v_i, $i \neq k$, while for the boundary-characteristic field we have to estimate the boundary term $v_{k,x}^2$ in the intervals where $|w_k/v_k(x=0)| \geq \delta_0$. This estimate follows because in this case one has

E:oscill1
$$|v_{k,x} - \hat{\lambda}_k v_k| \varrho(w_k/v_k) \geq \frac{1}{2} |v_{k,x}| \varrho(w_k/v_k), \tag{6.31}$$

i.e. the oscillations of the boundary terms controls the quantity $v_{k,x}$ in the regions where $|w_k| \gg |v_k|$. For the boundary terms of w_k instead, the same discussion as for h_k holds, since its boundary term is in L^1.

Ss:bouter1

6.3. **Boundary source terms.** We finally consider the boundary terms (4.37). These terms can be divided into 2 categories:

(1) oscillations of the boundary variable h_b multiplied by a term whose L^∞ norm if of order δ_0,

E:bdrsuc1
$$(|v_b| + |v_{b,x}| + |v_k| + |v_{k,x}|)(|\iota_b| + |\iota_{b,x}|); \tag{6.32}$$

(2) oscillations of the boundary characteristic field h_k, multiplied by the exponentially decreasing boundary terms v_b, $v_{b,x}$,

E:bdrsuc2
$$(|v_b| + |v_{b,x}|)(|\iota_k| + |\iota_{k,x}|). \tag{6.33}$$

Basically we have to estimate the terms

E:bdrsouc3
$$\mathcal{O}(1)\delta_0 \int_0^T \int_{\mathbb{R}^+} (|\iota_b(t,x)| + |\iota_{b,x}(t,x)|) dxdt, \qquad \mathcal{O}(1)\delta_0 \int_0^T \int_{\mathbb{R}^+} e^{-cx} (|\iota_k(t,x)| + |\iota_{k,x}(t,x)|) dxdt. \tag{6.34}$$

The first integral can be estimated by (5.29) to be $\leq C\delta_0^2$, while from (5.15), (5.17) it follows

$$\int_0^T \int_{\mathbb{R}^+} e^{-cx} (|\iota_k(t,x)| + |\iota_{k,x}(t,x)|) dxdt \leq \int_0^T \int_{\mathbb{R}^+} e^{(d-c)x} (e^{-dx} |\iota_k(t,x)| + e^{-dx} |\iota_{k,x}(t,x)|) dxdt \leq \mathcal{O}(1)\delta_0.$$

Thus also these terms are of order δ_0^2.

S:proconc1

7. Conclusion of the proof of Theorem 1.2

Form the results of Sections 4, 5, it follows that if $C\delta_1 \leq \delta_0 \ll 1$ each component of h has L^1 norm less than

E:final873
$$\|h_b(t)\|_{L^1}, \|h_i(t)\|_{L^1}, \|\iota_b(t)\|_{L^1}, \|\iota_i(t)\|_{L^1} \leq C(\delta_1 + \delta_0^2) < 2\delta_0, \qquad i = 1, \dots, n. \tag{7.1}$$

In particular we can continue the solution for a small interval $[T, T + \delta t]$.

We have proved in Section 6 that if $\|h_i\|_{L^1} \leq 2\delta_0$, and the initial-boundary data have L^1 norm in $[0,T]$ less that δ_0, then the source term is of the order of $C\delta_0^2$, with C depending only on A and its derivative. Let $\bar{T}$ be the maximal time T such that either $\|h_b(\bar{T})\|_{L^1}$, $\|\iota_b(\bar{T})\|_{L^1}$ or some $\|h_i(\bar{T})\|_{L^1}$, $\|\iota_i(\bar{T})\|_{L^1}$, $i = 1, \dots, n$, has L^1 norm equal to $2\delta_0$. It is clear that the existence of such a time $\bar{T}$ is in contradiction with (7.1). This prove Theorem 1.2 with $k_1 = k_2$.

To show that general case, recall the following lemma for Lipschitz continuous semigroup in Banach spaces [6]:

L:apprLipfl1

Lemma 7.1. *Let $S : [0, +\infty) \times D \mapsto D$, D closed subset of a Banach space, be a semigroup satisfying*

E:liptispa1
$$\|S_t u - S_t v\| \leq L(|t - s| + \|u - v\|). \tag{7.2}$$

For any Lipschitz continuous map $w : [0,T] \mapsto D$ it follows that

E:Lipetre1
$$\|w(T) - S_T w(0)\| \leq L \int_0^T \left\{ \liminf_{h \to 0+} \frac{\|w(t+h) - S_h w(t)\|}{h} \right\} dt. \tag{7.3}$$

Let S be the semigroup defined by the parabolic equation (1.11), with the parameter $\kappa_1(t)$,

E:semi665
$$\begin{array}{rcccc} S & : & [0 + \infty) \times ((L^1_{\text{loc}} \cap \mathrm{BV})(\mathbb{R}^+))^3 & \mapsto & ((L^1_{\text{loc}} \cap \mathrm{BV})(\mathbb{R}^+))^3 \\ & & (t, \kappa, u_b, u) & \mapsto & S_t(\kappa, u_b, u) = (\kappa_1(\cdot + t), u_b(\cdot + t), u(t)) \end{array} \tag{7.4}$$

(observe that S substitute κ with κ_1), where $u(t)$ is the solution to the initial boundary problem

$$u_t + A(\kappa_1(t), u) u_x = u_{xx}, \qquad u(0,x) = u_0(x), \; u(t,0) = u_b(t), \quad t, x > 0.$$

It is clear that the difference between the solution u_1 and u_2 of (1.11) with the same initial and boundary data and with parameters $\kappa_1(t)$, $\kappa_2(t)$ remains in L^1, so that we can apply the above Lemma (the map $t \mapsto (\kappa(\cdot+t), u_b(\cdot+t))$ is trivially Lipschitz continuous in L^1 for BV functions). Since one has that

E:Liperty

$$\begin{aligned}\liminf_{h\to 0+} & \frac{\big\|(\kappa_2(t+h), u_b(t+h), u_2(t+h)) - S_h(\kappa_2, u_b(t), u_2(t))\big\|_{L^1}}{h} \\ &= |\kappa_2(t)-\kappa_1(t)| + \Big\|(A(\kappa_2(t), u_2(t)) - A(\kappa_1(t), u_2(t)) u_{2,x}(t)\Big\|_{L^1} \\ &\leq \Big(1 + C\mathrm{Tot.Var.}(u)\Big)|\kappa_1(t)-\kappa_2(t)|, \end{aligned} \tag{7.5}$$

it follows from formula (7.3) the Lipschitz dependence w.r.t. κ in L^1. Noting that for the variable u its dependence has the total variation as a factor, a slight variation of the above argument yields the proof of Theorem 1.2.

S:hyplim02

8. The hyperbolic limit

In this section we show that the solution $u^\epsilon(t)$ to the parabolic system

E:rescal1

$$u_t + A(\kappa(t), u)u_x = \epsilon u_{xx}, \qquad u(0,x) = u_0(x),\ u(t,0) = u_b(t), \qquad t,x>0, \tag{8.1}$$

converges in L^1 to a unique function u, "vanishing viscosity" solution to the hyperbolic quasilinear system with boundary

E:hyppe02

$$u_t + A(\kappa(t), u)u_x = 0, \qquad u(0,x) = u_0(x),\ u(t,0) = u_b(t), \qquad t,x>0. \tag{8.2}$$

We will show below in which sense the solution u is related to the boundary condition u_b, i.e. which is the trace of the BV function $u(t,x)$ at $x=0$. Since in the theorem we are going to prove the map $t \mapsto u(t)$ is Lipschitz continuous in $L^1(\mathbb{R}^+)$ w.r.t. t, then the initial data is assumed in the strong L^1 topology.

First, by the rescaling $(t,x) \mapsto (\epsilon t, \epsilon x)$, we can rewrite Theorem 1.2 as

T:parae2

Theorem 8.1. *Consider the parabolic time dependent system*

E:hooo79

$$u_t + A(\kappa(t), u)u_x = \epsilon u_{xx}, \qquad t,x \geq 0, \tag{8.3}$$

with initial data u_0, u_b satisfying $|u_0-\bar u|, |u_b-\bar u| \leq \delta_0$ and

E:iniassu01

$$\mathrm{Tot.Var.}(u_0), \mathrm{Tot.Var.}(u_b) \leq \delta_1, \quad \|d^k u_0/dx^k\|_{L^1}, \|d^k u_b/dt^k\|_{L^1} \leq C\epsilon^{1-k}\delta_1, \quad k=1,\dots,K. \tag{8.4}$$

Assume moreover that the time dependent parameter $\kappa(t)$ is smooth and satisfies

E:kappart1

$$\|d^k\kappa/dt^k\|_{L^1} \leq C\epsilon^{1-k}\delta_0, \quad k=1,\dots,K. \tag{8.5}$$

The constant $K \in \mathbb{N}$ is chosen large enough. Then, if $\delta_1 \leq \delta_0$ are sufficiently small, the solution $u^\epsilon(t,x)$ exists for all $t \geq 0$ and has total variation uniformly bounded by $2\delta_0$.

Moreover, if $(u_{1,0}, u_{1,b})$, $(u_{2,0}, u_{2,b})$ are two different initial boundary data of (8.3) with parameters κ_1, κ_2, then the respective solution u_1, u_2 to the time dependent parabolic system satisfy for $t \geq s$

E:Lipscht2

$$\|u_1^\epsilon(t) - u_2^\epsilon(s)\|_{L^1(\mathbb{R}^+)} \leq L\Big(|t-s| + \|u_{1,0}(x) - u_{2,0}\|_{L^1(\mathbb{R}^+)} + \|u_{1,b}-u_{2,b}\|_{L^1(0,s)} + 2\delta_0\|\kappa_1-\kappa_2\|_{L^1(0,s)}\Big). \tag{8.6}$$

The constant L does not depend on ϵ.

For any sequence $\epsilon_i \to 0$ and for any fixed κ, u_0, u_b, by Helly's theorem it is possible to extract a converging sequence $u^{\epsilon_{i_j}}(t)$ converging in a countable dense set of functions in $L^1(\mathbb{R}^+)$ with uniformly small BV norm, and by a diagonalization argument we can assume that the sequence is convergent in L^1 on a countable dense set of times $\{t_n\}_{n\in\mathbb{N}}$. By uniform Lipschitz dependence in (8.6), it follows that the sequence $u^{\epsilon_{i_j}}(t)$ is convergent for all $t \in \mathbb{R}^+$, and it generates a Lipschitz continuous semigroup $\mathcal{S}^\kappa$ defined on $\{\|\kappa\|_{BV} \leq C\delta_0\} \cap C^K(\mathbb{R}^+)$, $(u_b, u_0) \in L^1_{\mathrm{loc}} \cap \{\|u\|_{BV} \leq \delta_0\} \cap C^K(\mathbb{R}^+)$ by

E:semitut1

$$\mathcal{S}_t(\kappa, u_b, u_0) \doteq (\kappa(\cdot+t), u_b(\cdot+t), u(t)), \tag{8.7}$$

(as in (7.4), the boundary data is only translated in t) and satisfying for $t \geq s$

E:Lipscht3

$$\begin{aligned}\big\|\mathcal{S}_t(\kappa_1, u_{1,b}, u_{1,0}) - \mathcal{S}_s(\kappa_2, u_{2,b}, u_{2,0})\big\|_{L^1(\mathbb{R}^+)} &\leq L\Big(|t-s| + \|u_{1,0}(x) - u_{2,0}\|_{L^1(\mathbb{R}^+)}\Big) \\ &\quad + L\Big(\|u_{1,b}-u_{2,b}\|_{L^1(\mathbb{R}^+)} + \|\kappa_1-\kappa_2\|_{L^1(\mathbb{R}^+)}\Big). \end{aligned} \tag{8.8}$$

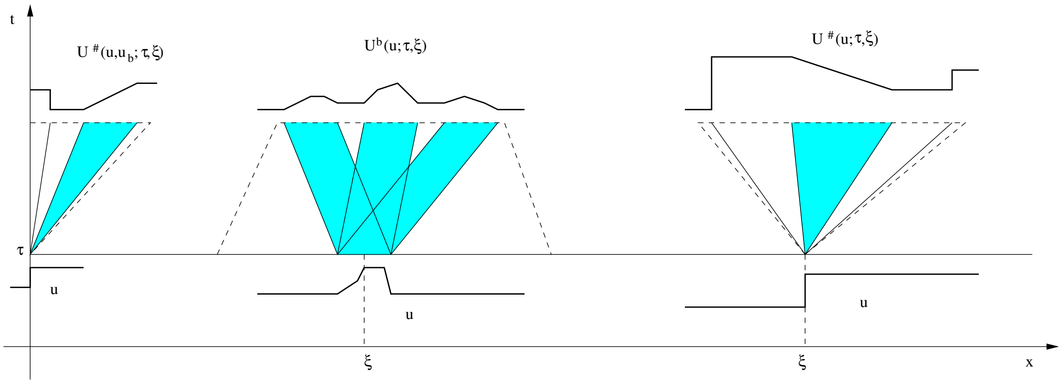

FIGURE 9. The three special functions used to define the viscosity solutions.

By means of the above Lipschitz estimate, using a density argument in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ it is easy to extend the domain of the semigroup $\mathcal{S}$ to the functions (κ, u_b, u_0) with sufficiently small total variation,

E:iniassu07 (8.9)
$$\mathrm{Tot.Var.}(u_0), \mathrm{Tot.Var.}(u_b) \leq \delta_1, \quad \mathrm{Tot.Var.}(\kappa) \leq C\delta_0.$$

With the same procedure as in [3], one can show that the semigroup has finite speed of propagation, in the sense that if $(\kappa(\cdot + t), u_b(\cdot + t), u(t; \kappa, u_0, u_b)) = \mathcal{S}_t(\kappa, u_b, u_0)$, then

E:Lipestsem1 (8.10)
$$\begin{aligned}\int_a^b \Big| u(t,x;\kappa_1,u_{1,b},u_{1,0}) - u(t,x;\kappa_2,u_{2,b},u_{2,0}) \Big| dx &\leq L \left(|t-s| + \int_{\max\{0,a-\Lambda s\}}^{b+\Lambda s} |u_{1,0}(x) - u_{2,0}(x)| dx \right) \\ &+ L \int_0^{s-a/\Lambda} \Big(|k_1(\tau) - k_2(\tau)| + |u_{1,b}(\tau) - u_{2,b}(\tau)| \Big) ds,\end{aligned}$$

for Λ sufficiently large.

The semigroup $\mathcal{S}$, obtained by a diagonalizing process, may not be the unique limit of the vanishing viscosity solution u^ϵ: to different converging subsequences there maybe correspond different semigroups. To prove that there is only one limiting semigroup $\mathcal{S}$, and to characterize it completely, we extend the definition of viscosity solution for the general hyperbolic system with boundary

E:genhyp6 (8.11)
$$u_t + A(\kappa(t), u) u_x = 0, \qquad x \geq 0,$$

with κ is a BV function. Without any loss of generality, we can assume $\kappa(t)$ right continuous.

Let $u(t,x)$ be a BV function w.r.t. x. Given a point (τ, ξ), with $\xi > 0$, denote with $U^\sharp_{(u;\tau,\xi)}$ the solution to the Riemann problem

E:init008899 (8.12)
$$u(\tau, x) = \begin{cases} \lim\limits_{y \to \xi -} u(\tau, y) & x \leq \xi \\ \lim\limits_{y \to \xi +} u(\tau, y) & x > \xi \end{cases}$$

for $0 < t - \tau < \xi/\Lambda$. This solution is obtained by the Riemann solver defined in [1], i.e. it is the unique limit of $u^\epsilon(t)$ with the special initial data (8.12) and parameter $\kappa(\tau) = \lim_{t \to \tau^+} \kappa(t)$ (due to the finite speed of propagation Λ the boundary has no influence).

We denote with $U^\flat_{(u;\tau,\xi)}$ the solution to the linear system

E:linerVS1 (8.13)
$$u_t + A\big(\kappa, u(\tau, \xi)\big) u_x = 0,$$

with initial data $u(\tau, x)$, and $\kappa = \lim_{t \to \tau^+} \kappa(t)$. This solution can be defined also for $\xi = 0$ by a the standard linear analysis of hyperbolic systems.

Finally, for $\xi = 0$, let us denote again with $U^\sharp_{(u,u_b;\tau,0)}$ the solution to the Boundary Riemann problem, i.e. (8.11) for $t > \tau$, $x > 0$, with initial boundary data

E:init008900 (8.14)
$$u_0 = \lim_{x \to 0^+} u(\tau, x), \qquad u_b = \lim_{s \to 0^+} u_b(\tau + s), \qquad \kappa = \lim_{t \to \tau^+} \kappa(t).$$

This solution, which we construct below, has bounded variation and is self similar as the solution $U^\sharp$, but it is different from the function obtained by considering the solution to the Riemann problem

$$u_0(x) = \begin{cases} u_b & x < 0 \\ u_0 & x > 0 \end{cases}$$

for the equation

$$u_t + A\big(\kappa(\tau^+), u\big) u_x = 0$$

on the whole real line and cutting it at $x = 0$.

A *Viscosity Solution* to (8.11) is now a function $u(t,x)$ satisfying the integral estimates:

(i) At every point (τ, ξ), $\xi > 0$, for some $\beta' > 0$

E:sharp008
$$\lim_{h \to 0^+} \frac{1}{h} \int_{\xi - h\beta'}^{\xi + h\beta'} \Big| u(\tau + h, x) - U^\sharp_{(u;\tau,\xi)}(\tau + h, x) \Big| dx = 0; \tag{8.15}$$

(ii) at every point $(\tau, 0)$,

E:sharp007
$$\lim_{h \to 0^+} \frac{1}{h} \int_{0}^{\xi + h\beta'} \Big| u(\tau + h, x) - U^\sharp_{(u,u_b;\tau,0)}(\tau + h, x) \Big| dx = 0; \tag{8.16}$$

(iii) there are constant C, $\beta \leq \beta'$ such that for every $0 \leq a < \xi < b$

E:sharp006
$$\lim_{h \to 0^+} \frac{1}{h} \int_{a + h\beta}^{b - h\beta} \Big| u(\tau + h, x) - U^\flat_{(u;\tau,\xi)}(\tau + h, x) \Big| dx \leq C\big(\text{Tot.Var.}(u;]a, b[)\big)^2. \tag{8.17}$$

For an account of viscosity solution of hyperbolic systems we refer to [5]. We want only to recall the following result, which is an easy consequence of the results on viscosity solutions:

L:unqieviso02 **Lemma 8.2.** *If $\mathcal{S}_t(\kappa, v, u) = (\kappa(\cdot + t), v(\cdot + t), u(t))$ is a Lipschitz semigroup satisfying* (8.10) *and moreover $u(t)$ is a viscosity solution to* (8.11)*, then $\mathcal{S}$ is unique.*

At this point, using the same technique of [3] and adapting it to the boundary Riemann problem case, one can prove the following Lemma:

L:visco00 **Lemma 8.3.** *Let $\mathcal{S} : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$, $\mathcal{S}_t(\kappa, v, u) = (\kappa(\cdot + t), v(\cdot + t), u(t))$, be a semigroup of solutions, constructed as limit of a sequence S^{ϵ_m} of the vanishing viscosity with boundary* (8.3) *and defined on a domain $\mathcal{D} \subset L_1$ of functions (v, u) with small total variation. Let $u : [0, T] \mapsto \mathcal{D}$ be Lipschitz continuous w.r.t. time, i.e.*

E:liptt11
$$\big\| u(t) - u(s) \big\|_{L^1} \leq L|t - s| \tag{8.18}$$

for some constant L and all $s, t \in [0, T]$ (it is clear that since κ, v are BV and the semigroup only translates in t, then they satisfy such a property trivially). Then

E:lipcoins
$$(\kappa(t), v(t), u(t)) = \mathcal{S}_t(\kappa(0), v(0), u(0)) \qquad t \in [0, T] \tag{8.19}$$

if and only if u is a viscosity solution of (8.11)*.*

In particular $(\mathcal{S}_t(\kappa, v, u))_u$ is a viscosity solution to (8.11). It follows by a standard argument that the whole family of viscous approximations converges to a unique limit.

Since $U^\sharp(u, u_b; \tau, 0)$ is BV, then for any (u_0, u_b) there exists the limit

E:trace01
$$\lim_{x \to 0^+} U^\sharp(u, u_b; \tau, 0)(t, x) = \bar{u}(\tau), \qquad t > 0, \tag{8.20}$$

and it is independent on $t > \tau$. By using the finite propagation speed of perturbation and the definition of viscosity solution, it is possible to show that if $u(t)$ is the boundary viscosity solution to (8.11), for a.e. $t > 0$ one has

E:trace02
$$\lim_{x \to 0^+} u(t, x) = \bar{u}(t), \qquad t > 0, \tag{8.21}$$

where $\bar{u}(t)$ is the function defined in (8.20). Due to the fact that $u \in L^\infty$, by Lebesgue's theorem this means that $\bar{u}(t)$ is the strong trace of u at $x = 0$.

To conclude this section, we will construct the Boundary Riemann Solver, which is used to define the function $U^\sharp_{(u,u_b;\tau,0)}$, and hence we will compute $\bar{u}$.

Ss:boundarysol1

8.1. **Solution of the boundary Riemann Problem.** We construct the solution $u(t)$ of the boundary Riemann problem, i.e. of the hyperbolic problem

E:boundaryhy1

$$\left\{ \begin{array}{rcl} u_t + A(\kappa,u)u_x &=& 0 \\ u(0,x) &=& u_0 \\ u(t,0) &=& u_b \end{array} \right. \tag{8.22}$$

which is the limit of the vanishing viscosity approximation: this means that the solution of this problem will be the unique L^1 limit of the solutions to

E:boundarypar1

$$\left\{ \begin{array}{rcl} u_t + A(\kappa,u)u_x &=& \epsilon u_{xx} \\ u(0,x) &=& u_0 \\ u(t,0) &=& u_b \end{array} \right. \tag{8.23}$$

as $\epsilon \to 0$. Observe that since the Boundary Riemann Solver is defined when the parameter κ is constant, we can neglect the dependence of A from κ.

As we will show, the solution to (8.22) is a BV self similar solution of the form

$$u(t,x) = u(x/t), \qquad x,t > 0.$$

Let $\bar{u}$ be the limit point of u as $x \to 0$:

E:realbb41

$$\bar{u} = \lim_{x \to 0^+} u(t,x), \qquad t > 0. \tag{8.24}$$

Since u is a BV function, this limit exists and is constant for $t > 0$. This point is determined in the following way:

(1) u_b is connected to $\bar{u}$ by a characteristic boundary layer, and waves (shocks or contact discontinuities) of the characteristic family with the same speed of the boundary, i.e. 0 in our case;
(2) the Riemann problems $[\bar{u}, u_0]$ is solved with waves of the families $i \geq k$ which have a speed strictly grater than the speed of the boundary.

Using the results on the uniformly stable manifold of Section 3, we can write more explicitly the composition of the Boundary Riemann problem:

(1) the uniformly exponentially stable boundary profile, which is given by the reduced ODE on the uniformly exponentially stable invariant manifold, coupled to a wave of the characteristic field entering the domain. These waves generate the boundary layer, and connect u_b to some point u_1;
(2) waves of the boundary characteristic family k with the same speed of the boundary, but not generating any boundary layer. These waves in the parabolic system do not travel with speed $\sigma_b = 0$, because of the interaction with the boundary, but in the hyperbolic limit this interaction disappears being due to diffusion. We thus arrive to the point $\bar{u}$;
(3) waves of the boundary characteristic field k with speed strictly greater than the speed of the boundary, connecting $\bar{u}$ to some point u_2;
(4) waves of the characteristic fields $i > k$ entering the domain, connecting u_2 to u_0.

The part 1) generates the boundary layer, while the remaining parts can be obtained by means of the standard technique of [1]. The fundamental point is thus to construct the boundary layer for any small Boundary Riemann problem $[u_b, u_0]$.

For completeness, we begin by recalling the results of [1] on the construction of the admissible curve of the i-th characteristic family, and hence how these curves can be used to solve the Riemann problem

E:boundaryhy2

$$\left\{ \begin{array}{rcl} u_t + A(\kappa,u)u_x &=& 0 \\ u(0,x) &=& \begin{cases} u^- & x \leq \xi \\ u^+ & x > \xi \end{cases} \end{array} \right. \tag{8.25}$$

with $\xi > 0$. We thus construct the function $U^\sharp$ solution to (8.12).

Sss:nonbdrch7 8.1.1. *Admissible curve of the i-th characteristic families.* We first reduce $2n+1$ system (3.24), where we neglect the equation for the parameter κ, to the $n+2$ ODE on the center manifold $p = v_i \tilde{r}_i(u, v_i, \sigma_i)$

E:inteiwav1
$$\left\{ \begin{array}{rcl} u_x &=& v_i \tilde{r}_i(u, v_i, \sigma_i) \\ v_{i,x} &=& (\tilde{\lambda}_i(u, v_i, \sigma_i) - \sigma_i) v_i \\ \sigma_{i,x} &=& 0 \end{array} \right. \tag{8.26}$$

where $\tilde{\lambda}_i$ is defined in (3.27). We associate the integral system

E:inteiwav2
$$\left\{ \begin{array}{rcl} u(\tau) &=& u + \int_0^\tau \tilde{r}_i(u(\varsigma), v_i(\varsigma), \sigma_i(\varsigma)) d\varsigma \\ v_i(\tau) &=& \mathrm{conc}_{[0,s]} \tilde{f}_i(\tau) - \tilde{f}_i(\tau) \\ \sigma_i(\tau) &=& d\mathrm{conc}_{[0,s]} \tilde{f}_i / d\tau \end{array} \right. \tag{8.27}$$

where $\tilde{f}_i$ is defined by

E:tildef08
$$\tilde{f}_i(\tau) = \int_0^\tau \tilde{\lambda}(u(\varsigma), v_i(\varsigma), \sigma_i(\varsigma)) d\varsigma, \tag{8.28}$$

and $\mathrm{conc}_{[0,s]} g$ is the concave envelope of g in the interval $[0, s]$. If s is negative, then one has to consider the convex envelope instead of the concave one.

In [1] it is shown that for any s sufficiently small, there exists a unique solution to system (8.27): we denote with $T_s^i u = u(s)$, i.e. the solution $u(\tau)$ computed at $\tau = s$. This curve is called the i-th admissible curve. The curve $u(\tau)$, $\tau \in [0, s]$, satisfies $u(0) = u$, and has contacts with $T_\tau^i u$ whenever $v_i(\tau) = 0$. The solution to the Riemann problem $[T_s^i u, u]$ is thus given by

E:iwavRie1
$$u(x/t) = \begin{cases} T_s^i u & x/t < \sigma_i(s) \\ u(\tau) & \sigma_i(\tau) = x/t \\ u & x/t > \sigma_i(0) \end{cases} \tag{8.29}$$

The i-th admissible curve $T_s^i u$ is Lipschitz continuous, with derivative at $s = 0$ equal to $r_i(u)$, while the curve $Z^i(\tau) = u(\tau)$ is smooth and $\sigma_i(\tau)$ is Lipschitz. The above construction holds for any $i = 1, \dots, n$.

The solution to the Riemann problem $[u^-, u^+]$ is constructed as follows. Consider the Lipschitz map

E:riemnmap1
$$(s_1, \dots, s_n) \mapsto T_{s_1}^1 \circ \cdot \circ T_{s_n}^n u^+, \tag{8.30}$$

whose Jacobian matrix at $s = 0$ is $[r_1(u^+), \dots, r_n(u^+)]$. It follows that the map (8.30) is invertible in a a neighborhood of u^+. Thus, if $|u^- - u^+|$ is sufficiently small, it is possible to find a unique vector $(s_1, \dots, s_n) \in \mathbb{R}^n$ such that
$$u^- = T_{s_1}^1 \circ \cdots \circ T_{s_n}^n u^+.$$
Define $u^\ell = T_{s_\ell}^\ell u^{\ell+1}$, $\ell = 1, \dots, n$, $u^{n+1} = u^+$, $u^1 = u^-$. The solution to (8.25) is thus obtained by piecing together the solutions to the elementary jumps $[u^\ell, u^{\ell+1}]$, constructed by (8.29).

The above construction generates the functions $U^\sharp(u; \tau, \xi)$ for $\xi > 0$, while it is clear that for $U^\sharp(u, \tau, 0)$ we cannot consider waves with speed $\sigma < 0$, i.e. the waves of the characteristic fields leaving the boundary ($i = 1, \dots, k-1$), and the part of the curve $Z_k(\tau)$ for the boundary characteristic fields which has speed $\sigma_k(\tau) < 0$. Aim of the next two part of this section is to prove that it is possible to construct a new curve, the boundary admissible k-th curve $\Upsilon^k u$, and a $k-1$ dimensional manifold Σu so that the map

E:boundarymp9
$$(s_1, \dots, s_n) \mapsto \Sigma_{(s_1, \dots, s_{k-1}), s_k} \circ \Upsilon^k_{(s_1, \dots, s_{k-1}), s_k} \circ T_{s_{k+1}}^{k-1} \circ \cdot \circ T_{s_n}^n u \tag{8.31}$$

is Lipschitz continuous, differentiable at $s = 0$ and invertible in a neighborhood of u.

Fixed thus u_0, let $u_2 \doteq T_{s_{k+1}}^{k+1} \circ \cdots \circ T_{s_n}^n u_0$, and consider the function

E:nonbdrpart1
$$\sigma_k(\tau) = \text{speed of the } k\text{-th wave at } u^k(\tau), \tag{8.32}$$

where $(u_k(\tau), v_k(\tau), \sigma_k(\tau))$ is the solution of the system (8.27) with $i = k$ and starting point u_2. The function $\sigma_k(\tau)$ is Lipschitz continuous and decreasing, and we consider 3 cases:

(1) $\sigma_k(0) < 0$, i.e. all the waves of the k-th family have strictly negative speed. Define then $u_1 = u_2$;
(2) there are two numbers $0 \leq \bar{\tau} \leq \tau_1 < s_k$ such that $\sigma_k(\bar{\tau}) = \sigma(\tau_1) = 0$, and for $0 \leq \tau < \bar{\tau}$ the function $\sigma_k(\tau) > 0$, while for all $\tau_1 < \tau \leq s_k$ $\sigma_k(\tau) < 0$. Define

E:u1der1
$$\bar{u} = T_{\bar{\tau}}^k u_2, \qquad u_1 = T_{\tau_1}^k u_2; \tag{8.33}$$

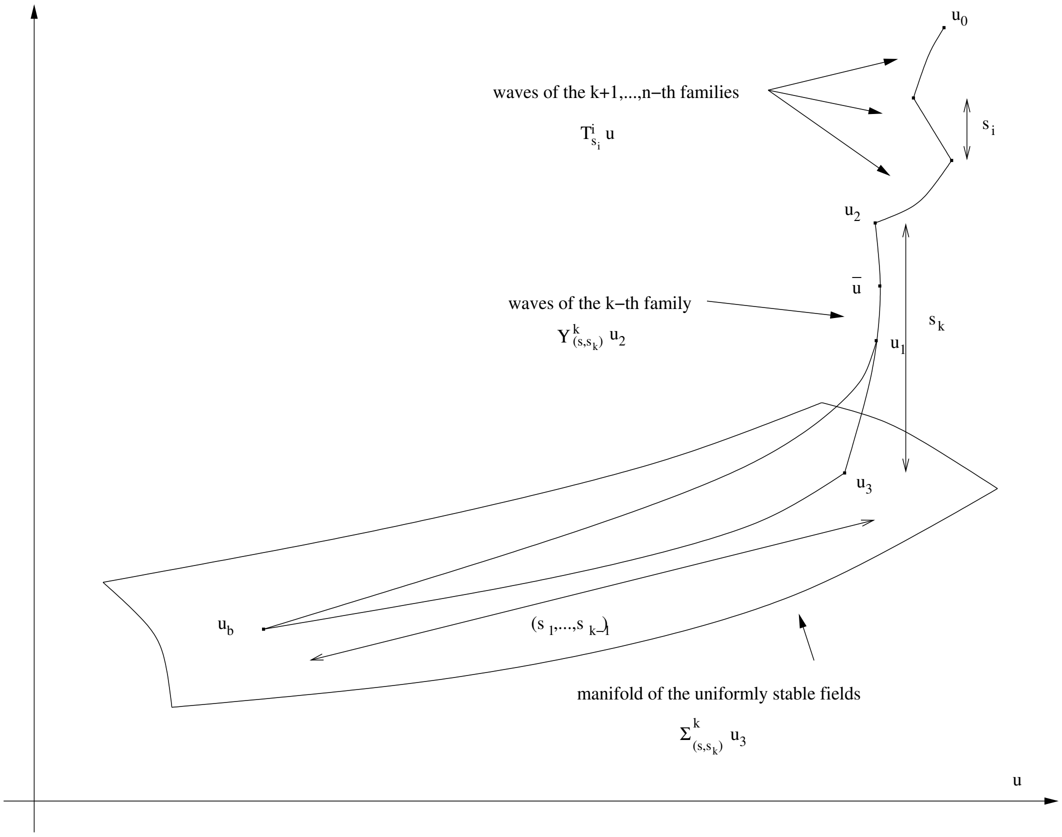

FIGURE 10. The construction of the Boundary Riemann solver. Fi:bounriem2

(3) $\sigma_k(\tau) \geq 0$ for all $0 \leq \tau \leq s_k$. Let $\bar{\tau}$ be the first point such that $\sigma(\bar{\tau}) = 0$, and define

E:u1der2 (8.34)
$$u_1 = T^k_{s_k} u_2, \qquad \bar{u} = T^k_{\bar{\tau}} u_2.$$

It is clear that, following [1], the Riemann problem $[u_1, u_0]$ can be solved with waves of the i-th characteristic families, $i \geq k$, and we do not have to consider any boundary layer: in fact, since the speed of the waves is greater than 0, no interaction with the boundary occurs.

In the remaining part of this section we show that it is possible to connect u_1 to u_b by a boundary layer $u(x)$. Using the diagonalization of the equations on the boundary layer given by (3.23), we can split the boundary layer u in two parts, $u(x) = u_s(x) + u_k(x)$, satisfying two coupled systems of ODE:

- the function $u_s(x)$ is the trajectory of the system

E:syste711 (8.35)
$$\begin{cases} u_{s,x} &= \hat{R}_s(u_s + u_k(x), p_s, p_k(x))p_s \\ p_{s,x} &= \hat{A}_s(u_s + u_k(x), p_s, p_k(x))p_s \end{cases}$$

  starting at a distance $(s_1, \dots, s_{k-1})$ from the origin in the first $k-1$ directions, and converging to 0 as $x \to +\infty$;
- the function $u_k(x)$ is the solution to

E:centerpar11 (8.36)
$$\begin{cases} u_{k,x} &= p_k \hat{r}_k(u_k + u_s, p_s, p_k, 0) \\ p_{k,x} &= \hat{\lambda}_k(u_k + u_s, p_s, p_k, 0)p_k \end{cases}$$

  which converge for $x \to +\infty$ to u_1 and has length $s_k - \tau_1$.

The function u_s is the uniformly stable part of the boundary layer, while the part u_k is the boundary characteristic part. We note that while the existence of u_s follows because the matrix $\hat{A}_s$ is negative definite, the existence of u_k is more subtle, because $\hat{\lambda}_k$ is close to 0. We will prove that if for system (8.36) with $u_s = 0$, $p_s = 0$ there is a solution $(u_{k,0}(x), p_{k,0}(x))$ such that $u_{k,0}(x) \to u_1$, $p_{k,0} \to 0$ as $x \to \infty$ and the length of the orbit $u_{k,0}$ is $s_k - \tau_1$ (i.e. there is a characteristic boundary layer of length

$s_k - \tau_1$ without any uniformly exponentially decaying part), then for any small perturbations u_s, p_s which decays exponentially, there exists a solution u_k, p_k with the same properties.

Sss:unifstcont1

8.1.2. *The uniformly stable part of the boundary layer.* We first prove that for any fixed bounded small functions $u_k(x)$, $p_k(x)$, it is possible to construct a $k-1$ dimensional manifold $S(u_k, p_k)$ of solutions to the system (8.35) which converges to $u = 0$. This manifold can be parameterized by the components of u in the first $k-1$ directions, i.e. $S(u_k, p_k) = S(u_1, \dots, u_{k-1}; u_k, p_k)$.

Fix the functions $p_k(x)$, $u_k(x)$ bounded and small, and consider the perturbed $n+k-1$ system of ODE

E:syste61

$$\left\{ \begin{array}{rcl} u_{s,x} &=& \hat{R}_s(u_s + u_k(x), p_s, p_k(x)) p_s \\ p_{s,x} &=& \hat{A}_s(u_s + u_b(x), p_s, p_k(x)) p_s \end{array} \right. \tag{8.37}$$

By the assumption that $\hat{A}_s$ has uniformly strictly negative eigenvalues $< -c$, we obtain the estimate

E:epotde10

$$p_s(x) = p_s(x; u_k, p_k, u_s(0), p_s(0)) = \mathcal{O}(1) p_s(0) e^{-cx}, \tag{8.38}$$

where $u_s(0)$, $p_s(0)$ are the initial data for u_s, p_s respectively. We can thus integrate the first equation for u_s,

E:uint02

$$u_s(x) = u_s(0) + \int_0^x R_s\Big(u_s(y) + u_k(y), p_s(y), p_k(y)\Big) p_s\big(y; u_k, p_k, u_s(0), p_s(0)\big) dy, \tag{8.39}$$

so that, to have $u_s(+\infty) = 0$, we must impose

E:aburu8

$$u_s(0; u_k, p_k) = -\int_0^{+\infty} R_s\Big(u_s(y) + u_k(y), p_s(y), p_k(y)\Big) p_s\big(y; u_k, p_k, u_s(0), p_s(0)\big) dy. \tag{8.40}$$

We can think of the above relation as a map from $\mathbb{R}^{k-1} \ni p_s(0) \mapsto u_s(0) \in \mathbb{R}^n$, defined by

E:aburubaba8

$$\begin{aligned} u_s^n(0; u_k, p_k) &= \mathcal{M}(u_s^n(0), p_s(0)) \\ &= -\int_0^{+\infty} R_s\Big(u_s^{n-1}(y) + u_k(y), p_s(y), p_k(y)\Big) p_s\big(y; u_k, p_k, u_s^{n-1}(0), p_s(0)\big) dy. \end{aligned} \tag{8.41}$$

A simple computation shows that for $p_s = 0$ the Jacobian of the above map is

$$D_{u_s(0)} \mathcal{M}(u_s(0), 0) = 0, \qquad D_{p_s(0)} \mathcal{M}(u_s(0), 0) = \left[\begin{array}{c} I \\ 0 \end{array} \right] + \mathcal{O}(1) \delta_0 \in \mathbb{R}^{n \times (k-1)}.$$

where the last result follows because, if $|u - \bar{u}| \leq \mathcal{O}(1) \delta_0$,

$$\hat{R}_s(\kappa, u, p_s, p_k, 0) = \left[\begin{array}{c} I \\ 0 \end{array} \right] + \mathcal{O}(1) \delta_0 \in \mathbb{R}^{n \times (k-1)}.$$

Hence for $|p_s|$ sufficiently small there is only one fixed point $u_s(0) = u_s(0)(p_s(0))$, which can be written by the Inverse Function Theorem as a manifold of the first $k-1$ components of $u_s(0)$.

We observe also that the dependence of the above map w.r.t the functions u_k, p_k is smooth w.r.t. the C^n norm: in fact, one may show that

E:differ02

$$\begin{aligned} \Big\| (u_s(0; u_k, p_k) - u_s(0, ; u_k', p_k')) e^{cx/2} \Big\|_{L^\infty} &\leq \mathcal{O}(1) \delta_0 \Big(\|u_k - u_k'\|_{L^\infty} + \|p_k - p_k'\|_{L^\infty} \Big) \\ \Big\| (p_s(0; u_k, p_k) - p_s(0, ; u_k', p_k')) e^{cx/2} \Big\|_{L^\infty} &\leq \mathcal{O}(1) \delta_0 \Big(\|u_k - u_k'\|_{L^\infty} + \|p_k - p_k'\|_{L^\infty} \Big), \end{aligned} \tag{8.42}$$

Sss:cenmaoo

8.1.3. *Characteristic part of the boundary layer.* Consider the system (8.36), and let $u_{k,0}$, $p_{k,0}$ be the orbit such that $p_k(+\infty) = 0$, $u_k(+\infty) = u_1$ and the length of the curve $u_{k,0}(x)$ is $s_k - \tau_1$, when u_s, p_s are equal to 0. The existence of such a orbit follows from the fact that the k-th waves starting from u_2 have strictly negative speed for $\tau_1 < \tau \leq s_k$, see [1]. By using the same change of variable yielding (8.27), i.e. $d\tau = p_k(x) dx$, we can say that the reduced flux function $\tilde{f}^k$ defined as in (8.28) satisfies

E:reduflu0p1

$$\tilde{f}_k(\tau) - \tilde{f}_k(\bar{\tau}) = \int_{\bar{\tau}}^{\tau} \tilde{\lambda}_k(u_k(\varsigma), p_k(\varsigma), 0) d\varsigma = \int_{\bar{\tau}}^{\tau} \hat{\lambda}_k(u_k, 0, p_k, 0) d\varsigma < 0, \qquad \varsigma \in (\bar{\tau}, s_k]. \tag{8.43}$$

Assuming $p_k(x) < 0$, set

E:y-comp1

$$\tau(x) = \tau_1 - \int_x^{+\infty} p_k(x) dx, \qquad x = \int_{s_k}^{\tau} \frac{dy}{p_k(y)}. \tag{8.44}$$

and rewrite the system in integral form (8.27), which here takes the form

E:centerpar2
$$\left\{\begin{array}{rcl} u_k(\tau) & = & u_1 + \int_{\tau_1}^{\tau} \hat r_k(u_k(\varsigma)+u_s(\varsigma), p_s(\varsigma), v_k(\varsigma), 0) d\varsigma \\ p_k(\tau) & = & - \int_{\tau_1}^{\tau} \hat \lambda_k(u_k(\varsigma)+u_s(\varsigma), v_s(\varsigma), v_k(\varsigma)) d\varsigma \end{array}\right. \tag{8.45}$$

with

$$u_s(\tau) = \mathcal{O}(1) u_s(0) \exp\left\{ -c \int_{s_k}^{\tau} \frac{dy}{p_k(y)} \right\}, \qquad p_s(\tau) = \mathcal{O}(1) p_s(0) \exp\left\{ -c \int_{s_k}^{\tau} \frac{dy}{p_k(y)} \right\}.$$

We associate to (8.45) the map

E:centerpar21
$$(u_k(\tau), p_k(\tau)) \mapsto \left\{\begin{array}{rcl} u_k'(\tau) & = & u_1 + \int_{\tau_1}^{\tau} \hat r_k(u_k(\varsigma)+u_s(\varsigma), p_s(\varsigma), v_k(\varsigma), 0) d\varsigma \\ p_k'(\tau) & = & - \int_{\tau_1}^{\tau} \hat \lambda_k(u_k(\varsigma)+u_s(\varsigma), v_s(\varsigma), v_k(\varsigma)) d\varsigma \end{array}\right. \tag{8.46}$$

defined for $\tau_1 \leq \tau \leq s_k$ and $p_k(\tau) \geq p_{k,0}(\tau)/2$.

A sufficient condition for (8.45) to be equivalent to the original system (8.36) is that the map (8.46) yields functions p_k' which remains greater than $p_{k,0}/2$ for $\tau_1 < \tau \leq s_k$, if p_k satisfies the same condition. We will moreover assume that

E:initaad4
$$\int_{\tau_1}^{\tau} \Big(|u_k'(\varsigma) - u_{k,0}(\varsigma)| + |p_k(\varsigma) - p_{k,0}(\varsigma)| \Big) d\varsigma \leq C \delta_0 p_{k,0}(\tau). \tag{8.47}$$

If $u_{k,0}$, $p_{k,0}$ are the original solution, we can write

E:uperrors1
$$\begin{aligned} u_k'(\tau) &= u_{k,0}(\tau) + \int_{\tau_1}^{\tau} (\hat r_k(u_k+u_s, p_s, p_k) - \hat r_k(u_k, 0, p_k)) d\varsigma \\ &= u_{k,0}(\tau) + \mathcal{O}(1) \int_{\tau_1}^{\tau} \Big(|u_k(\varsigma) - u_{k,0}(\varsigma)| + |p_k(\varsigma) - p_{k,0}(\varsigma)| \Big) d\varsigma \\ &\qquad + \mathcal{O}(1) \int_{\tau_1}^{\tau} (|u_s(0)| + |p_s(0)|) \exp\left\{ -c \int_{s_k}^{\varsigma} \frac{dy}{p_k(y)} \right\} d\varsigma, \end{aligned} \tag{8.48}$$

E:uperrors3
$$\begin{aligned} p_k'(\tau) &= p_{k,0}(\tau) + \int_{\tau_1}^{\tau} (\hat \lambda_k(u_k+u_s, p_s, p_k) - \hat \lambda_k(u_k, 0, p_k)) d\varsigma \\ &= v_{k,0}(\tau) + \mathcal{O}(1) \int_{\tau_1}^{\tau} \Big(|u_k(\varsigma) - u_{k,0}(\varsigma)| + |p_k(\varsigma) - p_{k,0}(\varsigma)| \Big) d\varsigma \\ &\qquad + \mathcal{O}(1) \int_{\tau_1}^{\tau} (|u_s(0)| + |p_s(0)|) \exp\left\{ -c \int_{s_k}^{\varsigma} \frac{dy}{p_k(y)} \right\} d\varsigma. \end{aligned} \tag{8.49}$$

We used the fact that $u_{k,0}$, $p_{k,0}$ are fixed points for the unperturbed map. We estimate the integral

E:deriet2
$$\begin{aligned} \int_{\tau_1}^{\tau} \exp\left\{ -c \int_{s_k}^{\varsigma} \frac{dy}{p_k(y)} \right\} d\varsigma = & \int_{\tau_1}^{\tau} \frac{p_k(\varsigma)}{p_k(\varsigma)} \exp\left\{ -c \int_{s_k}^{\varsigma} \frac{dy}{p_k(y)} \right\} d\varsigma \\ = & - \left[\frac{p_k(\varsigma)}{c} \exp\left\{ -c \int_{s_k}^{\varsigma} \frac{dz}{p_k(z)} \right\} \right]_{\tau_1}^{\tau} + \frac{1}{c} \int_{\tau_1}^{\tau} \frac{dp_k(\tau)}{d\tau} \exp\left\{ -c \int_{s_k}^{\varsigma} \frac{dz}{p_k(z)} \right\} \\ = & \frac{p_k(\tau)}{c} + \frac{1}{c} \int_{\tau_1}^{\tau} \hat \lambda_k(u_k+u_s, p_s, p_k, 0) \exp\left\{ - \int_{s_k}^{\varsigma} \frac{dz}{p_k(z)} \right\} d\varsigma. \end{aligned} \tag{8.50}$$

so that we obtain, since $\hat\lambda_k/c$ is of order $\delta_0 \ll 1$,

E:tuozicul1
$$(1 - \mathcal{O}(1)\delta_0) \int_{\tau_1}^{\tau} \exp\left\{ -c \int_{s_k}^{\varsigma} \frac{dz}{p_k(z)} \right\} d\varsigma \leq \int_{\tau_1}^{\tau} (1 - \hat\lambda_k(\varsigma)/c) \exp\left\{ -c \int_{s_k}^{\varsigma} \frac{dz}{p_k(z)} \right\} d\varsigma = p_k(\tau)/c. \tag{8.51}$$

We thus obtain from (8.49) and the fact that $u_s(0)$, $p_s(0)$ are of order δ_0

E:uperrors11
$$p_k(\tau) = (1 + \mathcal{O}(1)\delta_0) p_{k,0}(\tau) + \mathcal{O}(1) \int_{\tau_1}^{\tau} \Big(|u_k(\varsigma) - u_{k,0}(\varsigma)| + |v_k(\varsigma) - v_{k,0}(\varsigma)| \Big) d\varsigma \geq \frac{p_{k,0}(\tau)}{2}. \tag{8.52}$$

To verify that also u'_k, p'_k satisfy (8.47), we integrate (8.48), (8.49) from τ_1 to τ, we obtain that

E:uperrors2

$$\int_{\tau_1}^{\tau} |u'_k(\varsigma) - u_{k,0}(\varsigma)| d\varsigma + \mathcal{O}(1)\delta_0 \int_{\tau_1}^{\tau} \Big(|u_k(\varsigma) - u_{k,0}(\varsigma)| + |p_k(\varsigma) - p_{k,0}(\varsigma)| \Big) d\varsigma \leq \mathcal{O}(1)\delta_0 p_k(\tau),$$
$$\int_{\tau_1}^{\tau} |p'_k(\varsigma) - p_{k,0}(\varsigma)| d\varsigma + \mathcal{O}(1)\delta_0 \int_{\tau_1}^{\tau} \Big(|u_k(\varsigma) - u_{k,0}(\varsigma)| + |p_k(\varsigma) - p_{k,0}(\varsigma)| \Big) d\varsigma \leq \mathcal{O}(1)\delta_0 p_{k,0}(\tau), \tag{8.53}$$

so that we conclude that

E:integretot1

$$\int_{\tau_1}^{\tau} \Big(|u'_k(\varsigma) - u_{k,0}(\varsigma)| + |p'_k(\varsigma) - p_{k,0}(\varsigma)| \Big) d\varsigma \leq \mathcal{O}(1)\delta_0 p_{k,0}(s). \tag{8.54}$$

The same computation used to prove that (8.27) is a contraction can be used to show that there is a unique solution to (8.45), with $p_k > 0$ for $\tau > \tau_1$. Since the curve $u_{k,0}(\tau)$ is Lipschitz continuous, the same holds for $u_k(\tau)$. We observe moreover that by (8.54) it follows that when $p_{k,0}(s_k) \to 0$, the functions u_k, p_k converge to $u_{k,0}$, $p_{k,0}$ in L^∞.

Sss:endofstory

8.1.4. *Construction of* Υ^k *and* Σ. Using the fact that the map (8.41) depends with coefficients of the order of δ_0 w.r.t. u_k, p_k, it follows that there exists a unique solution to the coupled system (8.41), (8.45). Since $p_k > 0$ for $\tau > \tau_1$, this means that this solution is actually a boundary layer, starting from u_1 and parameterized by $(s_1, \dots, s_{k-1})$ and $s_k - \tau_1$.

The point $\Upsilon^k_{\vec{s}, s_k} u_2$ is thus the end point $u_3 = u_k(s_k)$ of the solution to (8.45). To prove that this curve is Lipschitz, we observe that the only critical case is when $p_k(s_k)$ tends to 0. However, since in this case we know that u_k, p_k tends to $u_{k,0}$, $p_{k,0}$, which correspond to a travelling profile with 0 speed, it follows that Υ^k tends to T^k. Thus the curve obtained by piecing together $T^k_{\tau_1} u_2$ with the solution $u_k(s_k)$ of the characteristic part of the boundary layer has no jumps in s_k. Moreover, its derivative for $s_k = 0$ is $r_k(u_2)$.

The manifold $\Sigma_{\vec{s}, s_k} u_3$ is defined by $\Sigma_{\vec{s}, s_k} u = u_3 + S(\vec{s}; u_k, p_k)$, where u_k, p_k are two bounded functions describing the boundary characteristic field. As we noted before, its Jacobian for $s \vec{=} 0$ is given by $[r_1(u_2) \cdots r_{k-}(u_2)]$.

It follows that the map of (8.31) is invertible in a neighborhood of u_0, i.e. we can find a vector $(s_1, \dots, s_n)$ such that

E:boundarymp91

$$u_b = \Sigma_{(s_1, \dots, s_{k-1}), s_k} \circ \Upsilon^k_{(s_1, \dots, s_{k-1}), s_k} \circ T^{k-1}_{s_{k+1}} \circ \cdot \circ T^n_{s_n} u. \tag{8.55}$$

Defining $u^\ell = T^\ell s_\ell u^{\ell+1}$, $\ell = k+1, \dots, n$, $u^{n+1} = u_0$, and $u^k = T^k_{\bar{\tau}} u^{k+1} = \bar{u}$, the solution to the Boundary Riemann Problem $[u_b, u_0]$ is thus given by piecing together the solutions to the elementary jumps $[u^\ell, u^{\ell+1}]$. Note that by construction these jumps has only waves with strictly positive speed.

From $\bar{u}$ to u_b we have the k-th wave $[u_1, \bar{u}]$ with speed 0, and the boundary layer connecting u_b to u_1.

S:gencas1

Appendix A. The case with Tot.Var.(κ) large

In this section we consider the extension of the previous results to the case where κ is not small, but has only bounded total variation (maybe large). More precisely, we consider the system

E:prar77
$$u_t + A(\kappa, u)u_x = u_{xx}, \tag{A.1}$$

with initial boundary data close to a state $\bar{u}$ and with sufficiently small total variation. The parameter κ has large but bounded total variation in $(0 + \infty)$. In particular κ belongs to a compact set, let us say $-K \leq \kappa \leq K$. Other than smoothness of the functions A, κ, we assume a strictly hyperbolicity condition, in the sense that

E:unfirr
$$\lambda_{i+1}(\kappa, u) - \lambda_i(\kappa, v) \geq c > 0 \qquad \forall \kappa, u, v. \tag{A.2}$$

This means that for any fixed κ the system is uniformly strictly hyperbolic in a neighborhood of $\bar{u}$, but for different κ, κ' it may happen that $\lambda_{i+1}(\kappa', u) < \lambda_i(\kappa, u)$.

As a consequence, the boundary $x = 0$ can be boundary characteristic for different families, or no boundary characteristic at all: hence to analyze this more general case, we need to refine our travelling profile-boundary layer analysis.

In Section A.1 we extend the invariant manifold analysis of Section 3. We will first obtain the center manifolds of travelling profiles which depend smoothly on κ in a compact set, not necessarily smooth. Next we show that by (A.2) it is possible to decompose the stable manifold of the boundary layer into invariant submanifold of increasing dimension, contained one into the other. Using this decomposition, we will show that these manifold will depend smoothly on κ, also when the boundary layer looses one dimension, i.e. when for some k the eigenvalue $\lambda_k(\kappa, \bar{u})$ crosses 0 as κ varies. In particular, we will smoothly connect one particular direction in the stable manifold of boundary layers with the characteristic part of the boundary layer.

In Section A.2 we will define the decomposition of the perturbation $h \in L^1$ of (A.1) in terms of the above vectors. The computations to find the source terms and to evaluate them are exactly the same considered during the main part of the paper, with the only variation that there is some redundancy when a stable direction in the boundary layer becomes boundary characteristic: in fact in this case the same object will be described by one direction of the boundary layer and by a family of travelling profiles. We will only show which are the new terms to be evaluated and how to prove that these terms are quadratic w.r.t. the L^1 norm of h.

Finally, in Section A.3 a simple Gronwall inequality concludes the proof of the L^1 estimate. The idea is that there is a bounded and small solution to the ODE

$$\frac{dy}{dt} = y^2 + c(t)y,$$

if $c(t) \in L^1(\mathbb{R}^+)$ and the initial datum $y(0)$ is small, of the order of $e^{-\|c\|_{L^1}}$. The Lipschitz dependence on κ, the convergence to the hyperbolic limit, the uniqueness of the semigroup and the definition of trace at $x = 0$ follow the same line of Section 8.

Ss:gendecODE1

A.1. **Invariant manifolds for travelling profiles and boundary layer.** In the first two parts of this section we will find two families of vectors:

(1) generalized eigenvectors $\tilde{r}_i(\kappa, u, v_i, \sigma_i)$ for travelling profiles of the i-th family, defined in the set $\{|u - \bar{u}|, |v_i|, |\sigma_i - \lambda_i(\kappa, \bar{u})| \leq \delta\}$, δ being a small constant;
(2) vectors $R_i(\kappa, u, p_1, \dots, p_i)$ of the non characteristic part of the boundary layer, defined for when $\lambda_i(\kappa, \bar{u}) \leq -\delta < 0$ and for $|u - \bar{u}| \leq \delta$, $|p_i| \leq \delta$.

Using these vectors $\tilde{r}_i$, R_i, in the last part we will write explicitly the generalized eigenvectors $\hat{r}_i$, $\hat{R}_i$ which we will use in our decomposition.

Sss:center87

A.1.1. *Center manifold for travelling profiles.* Consider the equation defining the center manifold,

E:bdrsys668
$$\begin{cases} u_x = p \\ p_x = (A(\kappa, u) - \sigma I)p \\ \kappa_x = 0 \\ \sigma_x = 0 \end{cases} \tag{A.3}$$

A standard application of the center manifold theorem to the above system of ODE gives that there is a center manifold $p = v_i \tilde{r}_i(\kappa, u, v_i, \sigma)$ of dimension $n+3$ in a neighborhood of radius δ of all equilibria of the form $(\kappa, \bar{u}, 0, \lambda_i(\kappa, \bar{u}))$. Since κ belongs to a compact set of $\mathbb{R}$, the radius of the neighborhood can be chosen uniformly for all equilibria. However, due to lack of uniqueness of the center manifold, one cannot expect that these manifolds join smoothly.

We have the following Lemma:

L:centerext1 **Lemma A.1.** *Consider the set of equilibria*

E:euiset1

$$E_i \doteq \{(\kappa, \bar{u}, 0, \lambda_i(\kappa, \bar{u}); -K \leq \kappa \leq K\}, \tag{A.4}$$

for the system of ODE (A.3). *Then there exists a local invariant manifold of the form*

$$p = v_i \tilde{r}_i(\kappa, u, v_i, \sigma), \tag{A.5}$$

defined on the set

$$F_i = \Big\{(\kappa, u, v_i, \sigma), |\kappa| \leq K, |u - \bar{u}|, |v_i|, |\sigma_i - \lambda_i(\kappa, \bar{u})| \leq \delta\Big\},$$

which contains all the solutions to (A.3) *which remains close to an equilibrium in E_i for all $x \in \mathbb{R}$. Moreover,*

E:p=0

$$|\tilde{r}_i| = 1, \qquad \tilde{r}_i(\kappa, u, 0, \sigma_i) = r_i(\kappa, \bar{u}), \qquad \tilde{r}_{i,\sigma} = \mathcal{O}(1) v_i. \tag{A.6}$$

Proof. Fix an equilibrium $\bar{e} = (\bar{\kappa}, \bar{u}, 0, \lambda_i(\bar{\kappa}, \bar{u})$. The proof of the existence of the center manifold $\mathcal{M} - i$ near $\bar{e}$ relies on the construction of a contracting map for the system

E:bdrsys669

$$\left\{ \begin{array}{lcc} u_x & = & p \\ p_x & = & (A(\bar{\kappa}, \bar{u}) - \lambda_i(\bar{\kappa}, \bar{u}) I) p + \alpha(\kappa, u, \sigma) p \\ \kappa_x & = & 0 \\ \sigma_x & = & 0 \end{array} \right. \tag{A.7}$$

where the non linear function α is given by

E:alpha7

$$\alpha(\kappa, u, \sigma) = \theta(|u - \bar{u}|) \Big(A(\kappa, u) - A(\bar{\kappa}, \bar{u}) - (\sigma_i - \lambda_i(\bar{\kappa}, \bar{u})) I \Big). \tag{A.8}$$

the constants κ, σ are assumed sufficiently close to $\bar{\kappa}$, $\lambda_i(\bar{\kappa}, \bar{u})$, and θ is a cut off function of the form

$$\theta(x) = \begin{cases} 1 & |x| \leq \delta \\ \text{smooth connection} & \delta < |x| < 2\delta \\ 0 & |x| \geq 2\delta \end{cases}$$

We used explicitly the fact that κ, σ remain constant, so that we do not need to consider a cut off function for the non linearity associated to them: it is sufficient to assume that $\kappa - \bar{\kappa}$, $\sigma - \lambda_i(\bar{\kappa}, \bar{u})$ are small.

For any fixed cut off function θ, the center manifold $\mathcal{M}_i(\bar{\kappa}, \bar{u})$ for (A.7) (defined as the invariant manifold such that the trajectories on it do not explode exponentially for $x \to \pm\infty$) is unique. We now observe that the function α defined in (A.8) generates a contracting map for all equilibrium points in E_i, if σ is sufficiently close to $\lambda_i(\kappa, \bar{u})$, i.e. for δ sufficiently small, for any $\kappa \in [-K, K]$ there is a unique center manifold $\mathcal{M}_i(\kappa, \bar{u})$ of (A.7) near the equilibrium $(\kappa, \bar{u}, 0, \lambda_i(\kappa, \bar{u}))$. Note that to state this result we used the essential assumption (A.2) that the separation of the eigenvalues of $A(\kappa, \bar{u})$ is uniform.

The uniqueness of $\mathcal{M}_i(\kappa, \bar{u})$ implies that these manifolds join smoothly: in fact, in the regions where they have the same κ they must coincide.

The remaining parts of the Lemma follow from the same computations of Section 3.1. □

Sss:bouryyyt1

A.1.2. *Decomposition of the boundary layer.* Consider the system defining the boundary layer,

E:bdrsys665

$$\left\{ \begin{array}{lcc} u_x & = & p \\ p_x & = & A(\kappa, u) p \\ \kappa_x & = & 0 \end{array} \right. \tag{A.9}$$

Fixed $\bar{\kappa}$, let $\lambda_i(\bar{\kappa}, \bar{u})$, $i = 1, \dots, \bar{k}$, be the eigenvalues of $A(\kappa, \bar{u})$ which are less that $-\delta$. By the uniform hyperbolicity assumption (A.2), for any $\check{u}$ close to $\bar{u}$ and to any subset $(1, \dots, \alpha) \subset (1, \dots, \bar{k})$ there correspond by the Hadamar Perron Theorem 3.3 a unique manifold

E:firtty78

$$\begin{pmatrix} u \\ p \end{pmatrix} = \mathcal{H}_\alpha(\bar{\kappa}, \check{u}; p_\alpha), \qquad p_\alpha \in \mathbb{R}^\alpha, \tag{A.10}$$

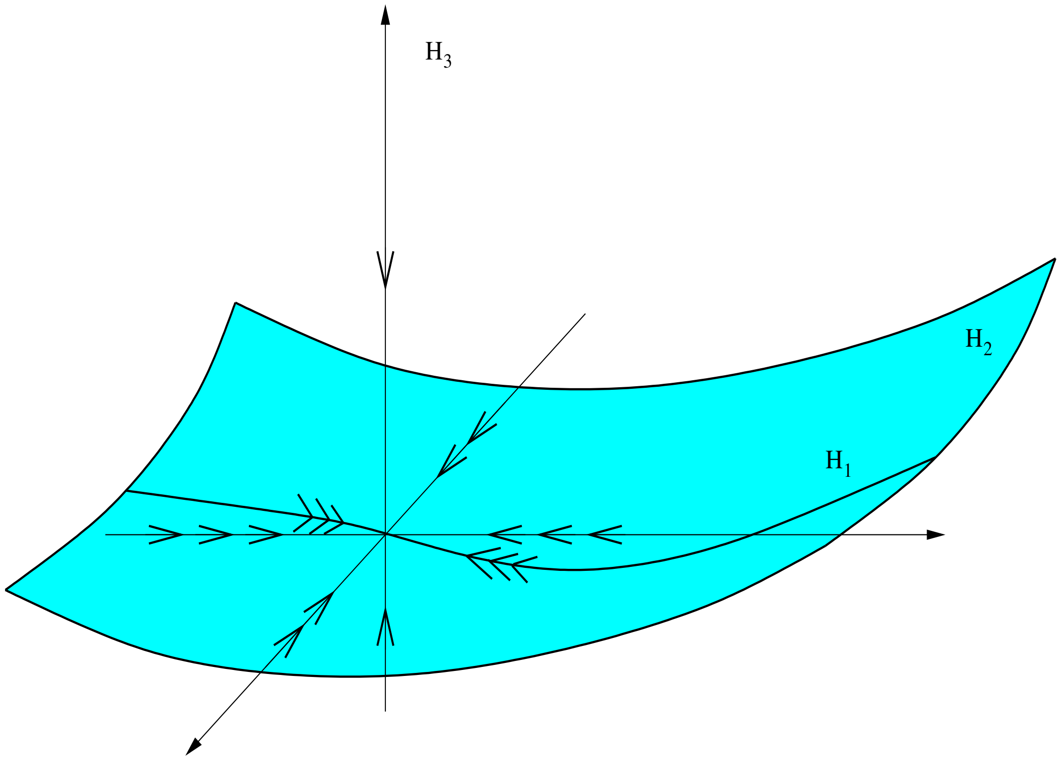

FIGURE 11. The nested manifolds $\mathcal{H}_\alpha$, for $\alpha = 1, 2, 3$. Fi:bounriem5

tangent to

$$\Big\{u = \check u, p = \mathrm{span}\big(r_1(\bar\kappa, \check u), \dots, r_\alpha(\bar\kappa, \check u)\big)\Big\}$$

for $p_\alpha = 0$, which contains all the trajectories of (A.9) converging to $\check u$ at least as $e^{(\lambda_\alpha+\nu)x}$, $0 \le \nu \ll 1$. This manifold depends smoothly on the parameters $\bar\kappa$, $\check u$, and by the same procedure used to prove Theorem 3.1 we can extend it for (κ, u) close to $(\bar\kappa, \check u)$ as the manifold

E:invarint1 (A.11) $$p = R_\alpha(\kappa, u, p_\alpha) p_\alpha, \qquad p_\alpha \in \mathbb{R}^\alpha, \quad \langle R_\alpha, R_\alpha \rangle = I \in \mathbb{R}^{\alpha\times\alpha}.$$

We have thus a sequence of invariant manifolds

(A.12) $$\mathcal{H}_\alpha = \Big\{p = R_\alpha(\kappa, u, p_\alpha)p_\alpha, |\kappa - \bar\kappa|, |u - \bar u|, |p_\alpha| \le \delta\Big\}, \qquad 1 \le \alpha \le \bar\kappa,$$

which satisfy $\mathcal{H}_\alpha \subset \mathcal{H}_{\alpha+1}$. By a smart choice of the unitary vectors $\tilde R_i$ generating $R_{\bar k}$,

$$R_{\bar k}(\kappa, u, p_1, \dots, p_{\bar k}) = \Big[\ \tilde R_1(\kappa, u, p_1, \dots, p_{\bar k}) \quad \cdots \quad \tilde R_{\bar k}(\kappa, u, p_1, \dots, p_{\bar k})\ \Big],$$

we can assume that for all $1 \le \alpha \le \bar k$ the following holds:

E:genderew1 (A.13) $$R_\alpha(\kappa, u, p_1, \dots, p_\alpha) = \Big[\ \tilde R_1(\kappa, u, p_1, \dots, p_\alpha, 0, \dots, 0) \quad \cdots \quad \tilde R_\alpha(\kappa, u, p_1, \dots, p_\alpha, 0 \dots, 0)\ \Big].$$

In particular $R_1(\kappa, u, p_1) = \tilde R_1(\kappa, u, p_1, 0, \dots, 0)$.

Due to uniqueness of the manifolds $\mathcal{H}_\alpha$, with the same arguments we used in Lemma A.1) each $\mathcal{H}_\alpha$ can be extended with continuity to the whole intervals $\{\kappa\colon \lambda_\alpha(\kappa, \bar u) \le -\delta < 0\}$, and it is defined for $|u - \bar u|$, $|q_i|$ less than δ, if $\delta \ll 1$.

We next consider the system (A.9) in the case $|\lambda_{\bar k}(\bar\kappa, \bar u)| \le 2\delta \ll 1$. In this case, contained in the center stable manifold $R_{cs}(\kappa, u, p_{cs})p_{cs}$, $p_{cs} \in \mathbb{R}^{\bar k}$, there exist a smooth invariant (center) manifold $r_{\bar\kappa}(\kappa, u, p_{\bar\kappa})p_{\bar\kappa}$, $p_{\bar\kappa} \in \mathbb{R}$ and uniformly stable manifold $R_s(\kappa, u, p_s)p_s$, $p_s \in \mathbb{R}^{\bar k-1}$. As in Section 3.1, we thus define the unitary matrix-vectors as (3.20)

E:gendeco21
$$\begin{aligned}\hat R_s(\kappa,u,p_s,p_{\bar k}) &= R_{cs}(\kappa,u,R_s p_s + p_{\bar k} r_{\bar k}) R_s(\kappa,u,p_s) \\ \hat r_{\bar k}(\kappa,u,p_s,p_{\bar k}) &= R_{cs}(\kappa,u,R_s p_s + p_{\bar k} r_{\bar k}) r_{\bar k}(\kappa,u,p_{\bar k}).\end{aligned} \tag{A.14}$$

In the region where the manifolds $\mathcal{H}_\alpha$, $\alpha = 1,\dots,\bar k$ coexist with the center manifold $\hat r_{\bar k} p_{\bar k}$, by uniqueness of the invariant manifold $\mathcal{H}_{\bar k - 1}$ it follows that

E:genderew2
$$\hat R_s(\kappa,u,p_1,\dots,p_{\bar k}) = \begin{bmatrix} \tilde R_1(\kappa,u,p_1,\dots,p_{\bar k}) & \cdots & \tilde R_{\bar k - 1}(\kappa,u,p_1,\dots,p_{\bar k}) \end{bmatrix}, \tag{A.15}$$

and by a smart choice of $\tilde R_{\bar k}$ we can assume that

E:stbcer1
$$\tilde R_{\bar k}(\kappa,u,p_1,\dots,p_{\bar k}) = \hat r_{\bar k}(\kappa,u,p_1,\dots,p_{\bar k}). \tag{A.16}$$

R:nonunirt
Remark A.2. We note that the vectors $\tilde R$ of (A.13) are not uniquely defined: we only require that these vectors describe all the nested manifold $\mathcal{H}_\alpha$, $\alpha = 1,\dots,\bar k$, and that $\tilde R_{\bar k}$ is smoothly connected to $\hat r_{\bar k}$ in the regions where $-2\delta \le \lambda_{\bar k}(\kappa,\bar u) \le -\delta$. Moreover when the stable manifold looses the $\bar k$ dimension (which enters into the center manifold), we have that the remaining manifold $\mathcal{H}_{\bar k - 1}$ becomes the uniformly stable manifold $R_s p_s$.

Observe that in general it is impossible to diagonalize the equations on all invariant manifolds $\mathcal{H}_\alpha$ as we did in Section 3.1, because of resonances [7].

Sss:vecure1

A.1.3. *The vectors used in the decomposition.* We now define which unitary vectors $\hat r_i$, $\hat R_i$ are used in our decomposition.

The vectors of the non characteristic part of the boundary layer are the vectors $\tilde R_i$ of Section A.1.2, defined only when $\lambda_i(\kappa,\bar u) \le -\delta < 0$.

For the center manifold vectors $\hat r_i$, $i = 1,\dots,n$, we only add the correction $\hat r_i$ when the field is boundary characteristic, as in (3.31),

E:decovecc2
$$\hat r_k(\kappa,u,p_1,\dots,p_{i-1},v_i,\sigma_i) \doteq \tilde r_k(\kappa,u,v_i,\sigma_i) + \varsigma(\lambda_i(\bar\kappa,\bar u))\big(\hat r_i(\kappa,u,p_1,\dots,p_{i-1},v_i) - \tilde r_i(\kappa,u,v_i,0)\big), \tag{A.17}$$

where $\varsigma(x)$ is a cutoff function equal to 1 when $|x| \le \delta$ and 0 when $|x| \ge 2\delta$.

The regions of existence of the various vectors are shown in figure 12.

Ss:decgerr

A.2. **Decomposition in the general case.** Define the cutoff function ϑ by

$$\vartheta(x) = \begin{cases} 1 & x \le -\delta \\ \text{smooth connection} & -\delta < x < 0 \\ 0 & x \ge 0 \end{cases} \tag{A.18}$$

and let $\bar k(\kappa)$ the maximal index such that $\lambda_{\bar k}(\kappa,\bar u) \le \delta$.

Decompose the perturbation h solution to

$$h_t + (A(\kappa,u)h)_x - h_{xx} = (DAu_x)h - (DAh)u_x,$$

and its effective flux $\iota = h_x - A(u)h$,

$$\begin{aligned}\iota_t + (A(\kappa,u)\iota)_x - \iota_{xx} = &- A_\kappa \dot\kappa h + \Big[DA(u_x \otimes h - h \otimes u_x)\Big]_x \\ &- A(\kappa,u) DA(u_x \otimes h - h \otimes u_x) + DA(u_x \otimes \iota - u_t \otimes h),\end{aligned}$$

as

E:hdeco01
$$\begin{aligned}h &= \sum_{i=1}^{\bar k} h_{i,b} \vartheta(\lambda_i(\kappa,\bar u)) \tilde R_i(\kappa,u,v_{1,b}+v_1,\dots,v_{\bar k,b}+v_{\bar k}) + \sum_{i=1}^n h_i \hat r_i(\kappa,u,v_{b,1},\dots,v_{b,i-1},v_i,\zeta_i) \\ &= \sum_{i=1}^{\bar k} h_{i,b} \vartheta_i \hat R_i + \sum_{i=1}^n h_i \check r_i,\end{aligned} \tag{A.19}$$

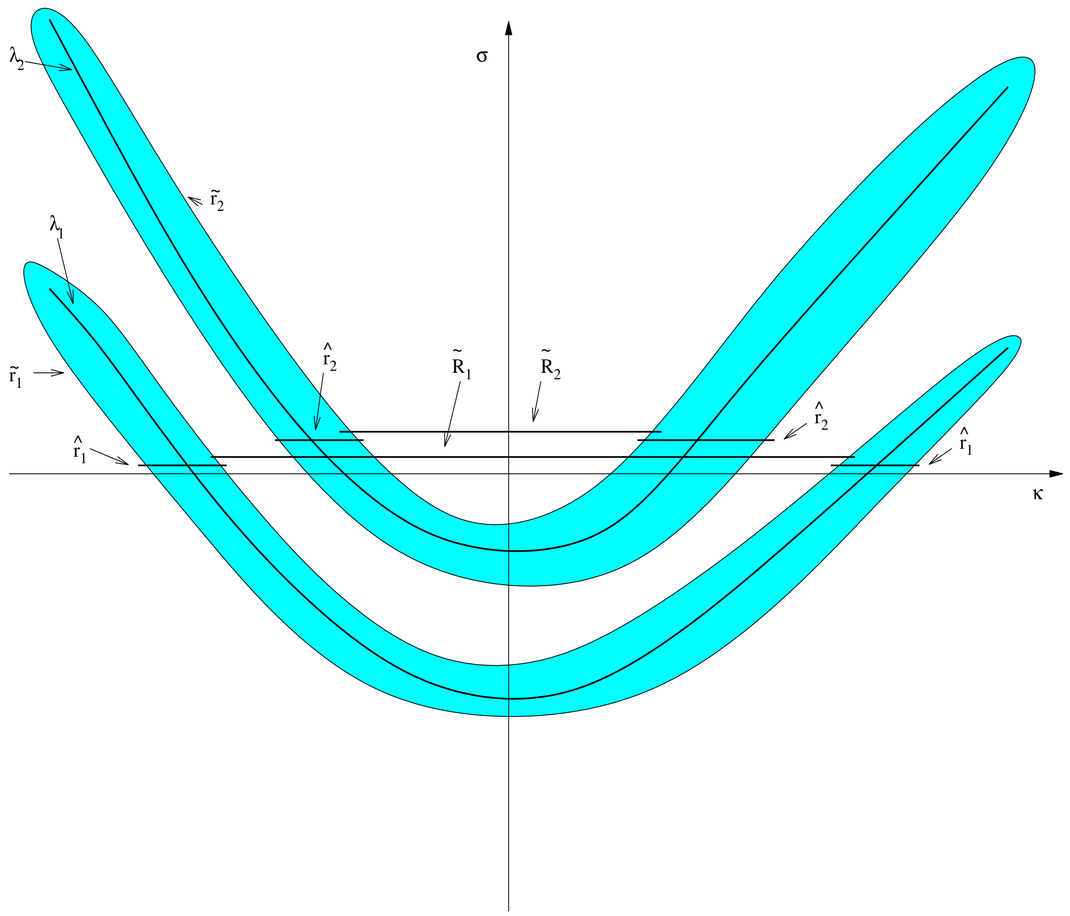

FIGURE 12. The regions of existence of the various vectors considered for a 2×2 system. Fi:bounriem10

$$\iota = \sum_{i=1}^{\bar{k}} \iota_{i,b} \vartheta(\lambda_i(\kappa, \bar{u})) \tilde{R}_i(\kappa, u, v_{1,b}, \dots, v_{\bar{k},b} + v_{\bar{k}}) + \sum_{i=1}^{n} \iota_i \hat{r}_i(\kappa, u, v_{b,1}, \dots, v_{b,i-1}, v_i, \zeta_i)$$

E:iotadeco01 (A.20)
$$= \sum_{i=1}^{\bar{k}} \iota_{i,b} \vartheta_i \hat{R}_i + \sum_{i=1}^{n} \iota_i \check{r}_i,$$

with ζ_i given by

E:zetai8 (A.21)
$$\zeta_i = \lambda_i(\kappa, \bar{u}) - \vartheta(\iota_i/h_i), \qquad \vartheta(x) = \begin{cases} x & |x| \leq \delta_0 \\ \text{smooth connection} & \delta_0 < x \leq 3\delta_0 \\ 0 & |x| > 3\delta_0 \end{cases}$$

The form $\hat{R}_i(v_{\bar{k},b} + v_{\bar{k}})$ just recall that in the regions where $\hat{R}_{\bar{k}} = \hat{r}_{\bar{k}}$, i.e. when $-2\delta \leq \lambda_{\bar{k}}(\kappa 2, \bar{u}) \leq \delta$, we consider the sum of the two scalars $v_{i,x}$, v_i as representing the "almost characteristic" part of the boundary layer. Moreover, the dependence of $\hat{r}_i$ from the boundary layer variables occurs only when the i-th field is boundary characteristic.

We can repeat the computations leading to (4.23), (4.24), and obtain the equations for the components of the form

E:hequ011

$$\begin{aligned}&\sum_{i=1}^{\bar k}\hat R_i(h_{b,i,t}+(\hat A_{b,i}h_b)_x-h_{b,i,xx})+\sum_{i=1}^{n}(\check r_i+h_i\zeta_{i,h}\check r_{i,\sigma})(h_{i,t}+(\tilde\lambda_i h_i)_x-h_{i,xx})\\&\quad+\sum_{i=1}^{n}h_i\zeta_{i,\iota}\check r_{i,\sigma}(\iota_{i,t}+(\hat\lambda_i\iota_i)_x-\iota_{i,xx})+\sum_{i,j}((\hat R_{i,v_{b,j}}\cdot)h_{b,i}+\check r_{i,v_{b,j}}h_i)(v_{b,j,t}+(\hat A_{b,j}v_{b,j})_x-v_{b,j,xx})\\&\quad+\sum_{i=1}^{n}\Big(R_{i,v_i}h_{b,i}+\check r_{i,v}h_i\Big)(v_{i,t}+(\hat\lambda_i v_i)_x-v_{i,xx})\\&=\phi(\kappa,u,v,v_x,w,w_x,h,h_x,\iota,\iota_x)+\sum_{i=1}^{n}\mathcal O(1)(|v_{b,i}|+|v_i|)h_i(\zeta_{i,x})^2+\mathcal O(1)\sum_{i=1}^{n}(|h_{b,i}|+|h_i|)\dot\kappa,\end{aligned}\tag{A.22}$$

E:iotaeq021

$$\begin{aligned}&\sum_{i=1}^{\bar k}\hat R_i(\iota_{b,i,t}+(\hat A_{b,i}h_b)_x-h_{b,i,xx})+\sum_{i=1}^{n}(\check r_i+\iota_i\zeta_{i,h}\check r_{i,\sigma})(h_{i,t}+(\tilde\lambda_i h_i)_x-h_{i,xx})\\&\quad+\sum_{i=1}^{n}\iota_i\zeta_{i,\iota}\check r_{i,\sigma}(\iota_{i,t}+(\hat\lambda_i\iota_i)_x-\iota_{i,xx})+\sum_{i,j}((\hat R_{i,v_{b,j}}\cdot)\iota_{b,i}+\check r_{i,v_{b,j}}\iota_i)(v_{b,j,t}+(\hat A_{b,j}v_{b,j})_x-v_{b,j,xx})\\&\quad+\sum_{i=1}^{n}\Big(R_{i,v_i}\iota_{b,i}+\check r_{i,v}\iota_i\Big)(v_{i,t}+(\hat\lambda_i v_i)_x-v_{i,xx})\\&=\psi(\kappa,u,v,v_x,w,w_x,h,h_x,\iota,\iota_x)+\sum_{i=1}^{n}\mathcal O(1)(|v_{b,i}|+|v_i|)\iota_i(\zeta_{i,x})^2+\mathcal O(1)\sum_{i=1}^{n}(|\iota_{b,i}|+|\iota_i|)\dot\kappa,\end{aligned}\tag{A.23}$$

where $\hat A_{b,i}$ is the flux of the i-th component $v_{i,b}$ in the stable manifold, and ϕ, ψ are quadratic functions of the form

E:formphi21

$$\begin{aligned}\phi(t,x),\psi(t,x)=&\sum_{i,j}a_{ij}(t,x)h_iv_j+\sum_{i,j}b_{ij}(t,x)h_iv_{j,x}+\sum_{i,j}c_{ij}(t,x)h_{i,x}v_j+\sum_{i,j}d_{ij}(t,x)\iota_iv_j\\&+\sum_{i,j}e_{ij}(t,x)\iota_iv_{j,x}+\sum_{i,j}f_{ij}(t,x)\iota_{i,x}v_j+\sum_i g_i(t,x)h_{i,x}v_{i,x}+\sum_i m_i(t,x)\iota_{i,x}v_{i,x}\\&+\sum_{i,j}n_{ij}(t,x)h_iw_j+\sum_{i,j}o_{ij}(t,x)\iota_iw_j+\sum_{i,j}p_{ij}(t,x)h_{i,x}v_{b,j,x}+\sum_{i,j}q_{ij}(t,x)h_{b,i,x}v_{k,j,x}\\&+\sum_i r_{ij}(t,x)w_ih_{i,x}+\sum_i s_{ij}(t,x)w_{i,x}h_i+\sum_i r'_{ij}(t,x)w_i\iota_{i,x}+\sum_i s'_{ij}(t,x)w_{i,x}\iota_i\\&+\sum_{i,j}t_{ij}(t,x)\iota_{i,x}v_{b,j,x}+\sum_{i,j}t'_{ij}(t,x)\iota_{b,i,x}v_{j,x},\end{aligned}\tag{A.24}$$

As before, we suppose that $v_{b,i}$, v_i, $w_{i,b}$, w_i satisfy scalar equations of the form

E:vbcond41

$$(v_{b,i}\vartheta_i)_t+(\hat A_{b,i}v_{b,i}\vartheta_i)_x-(v_{b,i}\vartheta_i)_{xx}=0,\qquad(w_{b,i}\vartheta_i)_t+(\hat A_{b,i}w_{b,i}\vartheta_i)_x-(w_{b,i}\vartheta_i)_{xx}=0,\tag{A.25}$$

E:vbcond31

$$v_{i,t}+(\tilde\lambda_iv_i)_x-v_{i,xx}=\omega_i(t,x),\qquad w_{i,t}+(\tilde\lambda_iw_i)_x-w_{i,xx}=\varpi_i(t,x),\tag{A.26}$$

with ω_i, ϖ_i integrable for $0\le t\le T$, $x\ge0$ and of order δ_0^2.

When computing the exact form of ϕ, ψ, one may use the same computations of Sections 4.2.1, 4.2.2, 4.2.3, to obtain the same structure of (4.40) outside the regions where ϑ_i is active.

In these regions, the $\bar k$ boundary layer field has two representations, namely as a part of travelling wave $h_{\bar k}\check r_{\bar k}$ with speed 0, and the "almost characteristic" part of the boundary layer $h_{b,\bar k}\hat R_{\bar k}$, with $\hat R_{\bar k}=\check r_{\bar k}(\zeta_{\bar k}=0)$. As in Section 4.2.3, one can check that the terms involving $h_{b,\bar k}$ $h_{\bar k}$ have the form

E:bdrsouc12

$$\begin{aligned}&\mathcal O(1)(|v_{b,\bar k}|+|v_{b,\bar k,x}|+|v_{\bar k}|+|v_{\bar k,x}|)(|\iota_{b,\bar k}|+|\iota_{b,\bar k,x}|)+\mathcal O(1)(|v_{b,\bar k}|+|v_{b,\bar k,x}|)(|\iota_{\bar k}|+|\iota_{\bar k,x}|)\\&+\mathcal O(1)(|h_{b,\bar k}|+|h_{b,\bar k,x}|+|h_{\bar k}|+|h_{\bar k,x}|)(|w_{b,\bar k}|+|w_{b,\bar k,x}|)+\mathcal O(1)(|h_{b,\bar k}|+|h_{b,\bar k,x}|)(|w_{\bar k}|+|w_{\bar k,x}|).\end{aligned}\tag{A.27}$$

Since $\lambda_{\bar{k}}$ in these regions is less that $-\delta$, the scalar variable $v_{b,\bar{k}}$ and its derivative are exponentially decreasing as $e^{-\delta x/2}$, while $w_{b,\bar{k}}$, $h_{b,\bar{k}}$, $\iota_{b,\bar{k}}$ (and their derivatives) are integrable in time for any fixed x, other than exponentially decreasing in space.

We are thus left with the estimate of the integrability in time of $h_{\bar{k}}$, $h_{\bar{k},x}$ (and similarly we can estimate $w_{\bar{k}}$, $w_{\bar{k},x}$), with the difference that even if the characteristic field is leaving the domain its boundary data is not 0, as we will specify below. Repeating the computations of Section 5.3 it follows that

E:bouderc321

$$\int_0^T e^{-C\delta y}|h_{\bar{k},x}(t,y)|dt \leq C\delta_0^2. \tag{A.28}$$

so that the result follows by means of the same computations of Section 5.2, with boundary term which is not 0 but integrable. Note that as in formulas (5.21), (5.22) we will not obtain any exponential growth of its $L^1([0,T])$ norm.

To split the initial boundary data in the various components, we use the same ideas of Section 4.3, with the novelty that for the boundary data we need to change the number of characteristic fields leaving the domain. This is done according to the variable $\bar{k}$, i.e. we set

E:bdrdec8

$$h_b = \sum_{i=1}^{\bar{k}} h_{i,b}\vartheta_i \hat{R}_i + \sum_{i=1}^{n} h_i(1-\vartheta_i)\check{r}_i, \tag{A.29}$$

and similarly for ι_b. Note that when $0 < \vartheta_i < 1$ the $\bar{k}$ component of h_b is decomposed into two scalar quantities, namely $h_{b,\bar{k}}$ and $h_{\bar{k}}$.

The source terms in the left hand sides of (A.22), (A.23) are as before decomposed only along the generalized eigenvectors of the travelling profiles $\hat{r}_i$, hence no source terms are in the equations for $(h_{b,i}\vartheta_i)$, $(\iota_{b,i}\vartheta_i)$.

Ss:finnr

A.3. **The stability estimate.** Using similar computations to the ones in Section 6 (with the study of the redundancy case $h_{\bar{k}}$, $h_{b,\bar{k}}$), one can prove that if the L^1 norm of h, ι is less than $\delta_0 \ll 1$, then the interaction terms in ϕ, ψ and the quadratic terms containing $\zeta_{i,x}^2$ of (A.22), (A.23) are of order δ_0^2.

Define thus

E:uppereqq1

$$M(t) = \sup_{0\leq s\leq T}\Big\{\|h(s)\|_{L^1}, \|\iota(s)\|_{L^1}\Big\}. \tag{A.30}$$

By integrating from 0 to T the equations for the components we have that

E:free1

$$M(t) \leq M(0) + CM(t)^2 + C\int_0^T M(s)|\dot{\kappa}(s)|ds. \tag{A.31}$$

where C is a large constant. By Gronwall estimate, if

$$M(0) < \left(4C\exp\left\{2C\int_0^T |\dot{\kappa}(s)|ds\right\}\right)^{-1},$$

it follows that

E:firtyy1

$$M(t) \leq 2M(0)\exp\left\{2C\int_0^T |\dot{\kappa}(s)|ds\right\}. \tag{A.32}$$

Fixed thus the total variation of κ in $\mathbb{R}^+$, if the initial data is sufficiently small in BV, there is a unique solution with total variation uniformly small and depending Lipschitz continuously on the initial boundary data and the parameter κ in L^1.

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