GLIMM INTERACTION FUNCTIONAL FOR BGK SCHEMES

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1. Introduction

Consider the $n \times n$ hyperbolic system of conservation laws

$$u_t + \mathcal{F}(u)_x = 0, \qquad u \in \mathbb{R}^n. \tag{1.1}$$

Under the assumptions that the matrix $A(u) = D\mathcal{F}(u)$ is strictly hyperbolic and its eigenvalues are genuinely non linear or linearly degenerate, the global existence of a solution $u : [0, +\infty) \mapsto \mathbb{R}^n$ for small BV initial data has first been proved in [18].

The uniqueness of solutions and their Lipschitz dependence in L^1 w.r.t. the initial data has been established in a series of papers [9, 11, 12, 13, 14, 15]. See [10, 17] for a general introduction to the theory of hyperbolic systems in one space dimension.

In the two papers [6, 4], a different approach has been used. Instead of constructing approximate solutions of the hyperbolic system (1.1) and studying their properties, the authors consider two approximations:

- vanishing viscosity approximation,

$$u_t + \mathcal{F}(u) = \epsilon u_{xx}; \tag{1.2}$$

- semidiscrete upwind approximation,

$$u_t(t,x) + \frac{1}{\epsilon}\Big(\mathcal{F}(u(t,x)) - \mathcal{F}(u(t,x-\epsilon))\Big) = 0. \tag{1.3}$$

Aim of the two papers [6, 4] is to prove that the solutions of the two schemes (1.2), (1.3) are well defined, globally in times, and satisfy some estimates which are independent on ϵ. More precisely:

1. the solution has uniformly bounded total variation for all $t \geq 0$, and its BV norm depends only on the BV norm of the initial data;
2. the solution depends Lipschitz continuously on the initial data in L^1, and on time.

It is easy to verify that both properties (1), (2) are invariant for the *hyperbolic rescaling* $(t, x) \mapsto (t/\epsilon, x/\epsilon)$. Because of this rescaling, it suffices to consider the case $\epsilon = 1$ in (1.2), (1.3). As an example of some non scaling invariant property, note that the L^1 norm of a solution tends to 0 as $\epsilon \to 0$: thus to obtain non trivial hyperbolic limits one has to assume $u \in L^\infty$, for example.

One advantage of the approach in [6, 4] is that strict hyperbolicity is the only assumption needed on $A(u)$: as an example, the analysis of stability for a wave front tracking or Glimm scheme solution becomes quite difficult without the usual assumption on genuine non linearity or linear degeneracy of the eigenvalues. It is an open problem whether the results on hyperbolic systems obtained in [6, 4] can be proved directly at the hyperbolic level (1.1). We note also that in [1, 2] it is shown how a similar approach cannot easily been extended to fully discrete schemes, e.g. Lax-Friedrichs or upwind Godunov scheme.

In the literature, there are other schemes used to approximate (1.1): the *relaxation schemes*. The easiest example is the scheme

$$\left\{\begin{array}{rcl} u_t + v_x &=& 0 \\ v_t + \Lambda^2 u_x &=& \frac{1}{\epsilon}(\mathcal{F}(u) - v) \end{array}\right. \tag{1.4}$$

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where Λ is a positive constant. The above system is a special case of a class of relaxation systems introduced in [20], but we will refer to it as the *Jin-Xin relaxation system* from the name of the authors. For a general introduction and survey to relaxation schemes see [23].

At a formal level, one sees that, as $\epsilon \to 0$,

$$v \to \mathcal{F}(u), \qquad u_t + \mathcal{F}(u)_x \to 0.$$

As in the previous cases, the linearized version of (1.4) around $\bar{u}$ defines a continuous semigroup if $A(\bar{u})$ is hyperbolic and its eigenvalues $\lambda_i(\bar{u})$ satisfy

$$-\Lambda + c \le \lambda_i(\bar{u}) \le \Lambda - c, \tag{1.5}$$

for some $c > 0$. The transformation $x \mapsto x/\Lambda$, $v \mapsto \Lambda v$ allows us to set $\Lambda = 1$ in (1.4), and with the hyperbolic rescaling $(t,x) \mapsto (t/\epsilon, x/\epsilon)$ we can take $\epsilon = 1$ in (1.4).

Troughout the following, for notational convenience we denote by

$$u_{0,t}, u_{0,tx}, \dots,$$

the derivatives of u at time $t = 0$.

We should look for properties of the solution (u,v) which are invariant under the hyperbolic rescaling.

By differentiating the second equation in (1.4) w.r.t. x and using the first, one obtains the nonlinear wave equation

$$u_t + A(u)u_x = u_{xx} - u_{tt}, \tag{1.6}$$

with $A(u) = D\mathcal{F}(u)$. The above equation is meaningful also in the case $A(u)$ is not a Jacobian matrix, so that one cannot write it in the conservative form (1.4).

We consider the wave equation (1.6), with $A(u)$ strictly hyperbolic but not necessarily a Jacobian of some vector function $\mathcal{F}(u)$. The first main result presented in the meeting is the following theorem:

Theorem 1.1. *Consider the nonlinear wave equation* (1.6)*, with $A(u)$ strictly hyperbolic and satisfying the stability conditions* (1.5) *with $\Lambda = 1$*

$$-1 + c \le \lambda_i(\bar{u}) \le 1 - c. \tag{1.7}$$

Assume that the initial data $(u_0, u_{0,t})$ are sufficiently smooth and with total variation less than δ_1:

$$\|u_0\|_{L^\infty}, \|u_{0,t}\|_{L^\infty} \le \delta_1, \ \|u_{0,x}\|_{L^1}, \|u_{0,tx}\|_{L^1} \le \delta_1, \ \|\partial_x^k u_0\|_{L^1}, \|\partial_x^k u_{0,t}\|_{L^1} \le C'\delta_1, \tag{1.8}$$

for some constant C' and $k = 2,3$.

If $\delta_1 \le C^{-1}\delta_0$, with C sufficiently large, then there exists a global solution (u, u_t), defined for all $t \ge 0$, and with the L^1 norm of u_x, u_{tx} less than $4\delta_0$:

$$\|u(t)\|_{L^1} \le 4\delta_0, \quad \|u_t(t)\|_{L^1} \le 4\delta_0. \tag{1.9}$$

Moreover this solution depends continuously w.r.t. time and the initial data: there exists a constant L such that, for any two solutions u, $\hat{u}$ one has

$$\begin{aligned} &\big\|u(t) + e^{-t}u_{0,t} - (\hat{u}(s) + e^{-s}\hat{u}_{0,t})\big\|_{L^1} + \big\|(u_t(t) - e^{-t}u_{0,t}) - (\hat{u}_t(s) - e^{-s}\hat{u}_{0,t})\big\|_{L^1} \\ &\quad \le L\Big(|t-s| + \big\|(u_0 + u_{0,t}) - (\hat{u}_0 + \hat{u}_{0,t})\big\|_{L^1} + \big\|u_{0,tx} - \hat{u}_{0,tx}\big\|_{L^1} + \big\|u_{0,txx} - \hat{u}_{0,txx}\big\|_{L^1}\Big). \end{aligned} \tag{1.10}$$

We note here that by means of the techniques used in this paper, one can avoid the assumption of smooth initial data. Moreover, if $\bar{u} \in \mathbb{R}^n$ is any constant state, by the shift $u \mapsto u - \bar{u}$ we can replace the first inequality of (1.8) by

$$\|u_0 - \bar{u}\|_{L^\infty}, \|u_{0,t}\|_{L^\infty} \le \delta_1,$$

and assume $A(u)$ strictly hyperbolic in a neighborhood of $\bar{u}$.

It is important to observe that the initial data are not assumed to satisfy $u_{0,t} \in L^1$, which on the other hand is a natural condition for the initial data of (1.4), since $v_x = -u_t$. As we will show in the analysis, apart from exponentially decaying terms, u_t becomes immediately in L^1. More precisely we will show that $u_t - e^{-t}u_{0,t}$ is integrable and has L^1 norm of the order of $4\delta_0$ for all $t > 0$.

Finally, notice that the Lipschitz dependence is w.r.t. the sum of $u_0 + u_{0,t}$. This is clearly more precise that the dependence w.r.t. u_0 and $u_{0,t}$ separately. Moreover this dependence becomes particularly relevant in the hyperbolic limit $\epsilon \to 0$, see (1.14) below.

The second result presented is the analysis of the limit as $\epsilon \to 0$. Denote by $u^\epsilon(t)$ the solution of the rescaled system

$$u_t + A(u)u_x = \epsilon(u_{xx} - u_{tt}), \quad u(0,x) = u_0(x),\ u_t(0,x) = u_{0,t}(x), \tag{1.11}$$

and assume that the initial data are given by $(u_0, u_{0,t}/\epsilon)$, with u_0, $u_{0,t}$ fixed. This assumption on the form of $u_t(t=0)$ is needed in order to make the sum $u + \epsilon u_t$ to converge at $t = 0$. We then prove the following theorem:

Theorem 1.2. *Consider the nonlinear wave equation* (1.11)*, with $A(u)$ strictly hyperbolic and satisfying the stability conditions* (1.7)*. Assume that the initial data $(u_0, \epsilon u_{0,t})$ are sufficiently smooth and with total variation less that δ_1:*

$$\|u_0\|_{L^\infty}, \|u_{0,t}\|_{L^\infty} \leq \delta_1,\ \|u_{0,x}\|_{L^1}, \|\epsilon u_{0,tx}\|_{L^1} \leq \delta_1,\ \epsilon^k \|\partial_x^k u_0\|_{L^1}, \epsilon^{k+1}\|\partial_x^k u_{0,t}\|_{L^1} \leq C'\delta_1,\ k = 2,3. \tag{1.12}$$

for some constant C'.

Then the solution $u^\epsilon(t)$ to (1.11) *converges in L^1_{loc} as $\epsilon \to 0$ to a unique limit $u(t)$.*

The BV functions $u(t)$, $t > 0$, generate a Lipschitz continuous semigroup $u(t) = \mathcal{S}_t u(0)$ in L^1_{loc} w.r.t. time and data: for $t, s \geq \tau > 0$

$$\|u(t) - \hat{u}(s)\|_{L^1} \leq L\Big(|t - s| + \|u(\tau) - \hat{u}(\tau)\|_{L^1}\Big). \tag{1.13}$$

Moreover, we have the estimate

$$\|u(t) - (u_0 + u_{0,t})\|_{L^1} \leq Lt, \tag{1.14}$$

so that the correct initial data for $u(t)$ is given by $u_0 + u_{t,0}$.

This semigroup is defined on a domain $\mathcal{D}$ containing all the function with sufficiently small total variation, and can be uniquely identified by a relaxation limiting Riemann Solver, i.e. the unique Riemann solver compatible with (1.6).

We can thus say that the semigroup $\mathcal{S}$ represent a family of *relaxation limiting solutions* to the quasi-linear hyperbolic system

$$u_t + A(u)u_x = 0. \tag{1.15}$$

We repeat again that as a consequence of the above theorem, the appropriate initial data for the limiting solution $u(t)$ is the sum of $u_0 + u_{0,t}$. Thus $u(t)$ may have a jump at $t = 0$, while, for $t > 0$, $u(t)$ is Lipschitz continuous w.r.t. t. Consider the following easy example.

Example 1.3. Consider the simple model

$$u_t = \epsilon(u_{xx} - u_{tt}).$$

with initial data $u(0) = 0$, $u_t(0) = \epsilon^{-1}$. Then the solution is clearly $1 - e^{-t/\epsilon}$, which converges to $u(t) \equiv 1$, $t > 0$. Thus the hyperbolic limit should have the initial data $u(0) = 1$.

When proving uniform BV estimates (and also stability estimates), the fundamental points are

- to understand the non linear wave structure of the solution of the kinetic scheme,
- to write a Glimm type functional which measure the interaction of non linear waves.

These two steps are strictly related: in fact the knowledge of the wave decomposition yields the form of the interacting terms, hence suggests the form of the functional. Conversely, the form of the functional describes how the solution can be decomposed as a sum of non linear waves.

In this note we want to extend the construction of a Glimm interaction functional to the general case of BGK models, i.e. kinetic models of the form

$$F^\alpha_t + \alpha F^\alpha_x = M^\alpha\bigg(\sum_\beta F^\beta\bigg) - F^\alpha, \quad F^\alpha \in \mathbb{R}, \tag{1.16}$$

with the assumption that for all u

$$\sum_\alpha M^\alpha(u) = u, \quad M^\alpha(u), \frac{dM^\alpha(u)}{du} > 0. \tag{1.17}$$

For simplicity, we will only construct the functional for the linearized version of (1.16), i.e.

$$F^\alpha_t + \alpha F^\alpha_x = c^\alpha \left(\sum_\beta F^\beta \right) - F^\alpha, \quad c^\alpha > 0, \ \sum_\alpha c^\alpha = 1. \tag{1.18}$$

We assume moreover that the speeds α are bounded.

2. Flux on a Dirichlet boundary and Glimm functional

Before constructing the Glimm type functional for relaxation, we recall its construction for the scalar nonlinear parabolic equation

$$u_t + \lambda(u) u_x = u_{xx}.$$

One can in fact introduce the vector valued function $P = (u, f(u) - u_x)$, which satisfies

$$P_t + \lambda(u) P_x = P_{xx}.$$

For the above equation one can show that the functional

$$Q(t) = \frac{1}{2} \iint_{x<y} |P_x(t,x) \wedge P_x(t,y)| dxdy \tag{2.1}$$

is decreasing: precisely

$$\frac{dQ}{dt} \leq - \int_{\mathbb{R}} |P_t(t,x) \wedge P_x(t,x)| dx = - \int_{\mathbb{R}} |P_{xx}(t,x) \wedge P_x(t,x)| dx. \tag{2.2}$$

The last equation can be thought as the instantaneous area swept by the curve $P(t,x)$.

It is possible to give another interpretation of the previous functional. In fact one can consider the variable

$$\tilde{p}(t,x,y) = P_x(t,x) \wedge P_x(t,y) = u_x(t,x) u_t(t,y) - u_x(t,y) u_t(t,x),$$

which satisfies

$$\tilde{p}_t + \mathrm{div}\Big((\lambda(u(t,x)), \lambda(u(t,y)) \tilde{p} \Big) = \triangle \tilde{p}. \tag{2.3}$$

Due to the symmetry $\tilde{p}(t,x,y) + p(t,y,x) = 0$, the above scalar 2-d equation can be considered in the half plane $\{x > y\}$ with boundary data $\tilde{p}(t,x,x) = 0$. The functional Q now has half of the L^1 norm of $\tilde{p}$, and its time derivative is the flux of $\tilde{p}$ across the boundary $\{x = y\}$.

While the first interpretation as a shortening curve is difficult to extend to BGK kinetic schemes, the last interpretation is more suitable: in the next section we will associate to system (1.16) in one space variable a BGK model in the half plane $\{x > y\}$, and we will estimate the flux of the solution through the boundary $\{x = y\}$. To understand better the construction and the final estimate, we consider here the following 1-d example.

Example 2.1. We consider the simple model

$$\begin{cases} z^-_t - z^-_x &= \frac{z^+ - z^-}{2} \\ z^+_t + z^+_x &= \frac{z^- - z^+}{2} \end{cases} \tag{2.4}$$

in $x \geq 0$ with boundary data $f^-(t,0) + f^+(t,0) = 0$. Our goal is to estimate

$$\int_0^{+\infty} |z^-(t,0)| dt. \tag{2.5}$$

To have a better control of the solution, we first notice that if the boundary data is $z^+(t,0) = 0$, then clearly by L^1 contraction

$$\frac{d}{dt} \int_{\mathbb{R}^+} |z^-(t,x)| + |z^+(t,x)| dx \leq -|z^-(t,0)|,$$

so that the integral (2.5) is bounded by the initial L^1 norm of z. The above estimate just tells that the number of particle which cross the boundary (and disappears) is bounded by the total number of particles in $x > 0$.

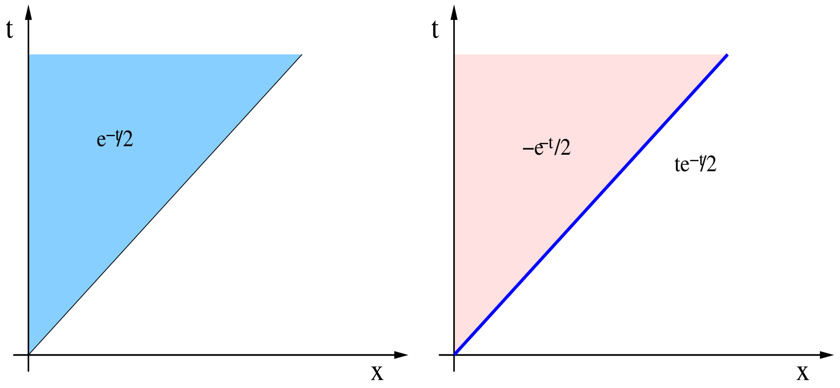

FIGURE 1. The functions $z^{-,1}$ (left) and $z^{+,1}$ (right).

We thus have only to consider the case of an initial Dirac mass δ in f^+ located at the origin. We now decompose the solution to (2.4) with initial data $z^-(0,x) = 0$, $z^+(0,x) = \delta(x)$ as the sum of functions $z^{\pm,i}$, $i = 0, 1, \dots$, each one satisfying

$$\left\{ \begin{array}{lcl} z_t^{-,i+1} - z_x^{-,i+1} & = & \frac{z^{-,i}+z^{+,i}}{2} - z^{-,i+1} \\ z_t^{+,i+1} + z_x^{+,i+1} & = & \frac{z^{-,i}+z^{+,i}}{2} - z^{+,i+1} \end{array} \right. \tag{2.6}$$

with $z^{\pm,-1} = 0$, $z^{-,0} = 0$, $z^{+,0} = \delta(x)$, and null initial data for $i = 1, 2, \dots$. Roughly speaking, we can imagine that each function $z^{\pm,i}$ describes a generation of particles, moving with speed ± 1, and with average decay time of 1. When 2 particles of the i generation decay, then 2 particles of the $i+1$-th generation are created, with speed -1 and $+1$.

It is simple to construct the first solutions: in fact,

$$z^{-,0}(t,x) = 0, \qquad z^{+,0}(t,x) = e^{-t}\delta(x-t),$$

$$z^{-,1}(t,x) = \frac{e^{-t}}{2}\chi\{0 \le x \le t\}, \qquad z^{+,1}(t,x) = -\frac{e^{-t}}{2}\chi\{0 \le x \le t\} + \frac{t}{2}e^{-t}\delta(x-t).$$

At this point we can observe that at the next step the total source $z^{-,1} + z^{+,1}$ has become $1/2$, while the total flux of these solutions across the boundary is $1/2$. We thus have proved the following: after $1 + 1/2$ crossing (1 is due to the initial absorbing boundary), we have that $1/2$ of the initial L^1 norm disappear. It is thus clear that the total amount of crossing is bounded by

$$\frac{1 + 1/2}{1 - 1/2} = 3.$$

We thus conclude with the estimate

$$\int_0^{+\infty} |z^-(t,0)|dt \le 3 \int_{\mathbb{R}^+} |z^-(0,x)| + |z^+(0,x)|dx. \tag{2.7}$$

3. BGK SCHEMES

Consider the linear scalar BGK scheme

$$F_t^\alpha + \alpha F_x^\alpha = c^\alpha \sum_\beta F^\beta - F^\alpha, \quad F^\alpha \in \mathbb{R}. \tag{3.1}$$

where c^α are positive coefficients such that

$$\sum_\alpha c^\alpha = 1,$$

and the speeds α are bounded by K.

Consider the initial data

$$F^\alpha(t,x) = \delta_{\alpha,\bar{\alpha}}\delta(x). \tag{3.2}$$

The solution to (3.1) together with (3.2) is the Green function $\Gamma^\alpha(\bar\alpha, x)$: we suppose that Γ exists and is smooth in (t,x) for $t > 0$. An application of Duhamel formula gives that the solution to (3.1) with general initial data is

$$F^\alpha(t,x) = \sum_\beta \int_{\mathbb{R}} \Gamma^\alpha(\beta, x-y) F^\beta(0,y) dy. \tag{3.3}$$

An important quantity is the viscosity associated to (3.1). Define the state u by

$$u(t,x) = \sum_\alpha F^\alpha(t,x). \tag{3.4}$$

By the Chapman-Enskog expansion (or considering the parabolic part of the Green kernel Γ), we obtain that

$$u_t + \bigg(\sum_\alpha \alpha c^\alpha \bigg) u_x - \bigg(\frac{1}{2} \sum_{\alpha\beta} (\alpha-\beta)^2 c^\alpha c^\beta \bigg) u_{xx} \simeq 0. \tag{3.5}$$

Let

$$\lambda = \sum_\alpha \alpha, \quad \sigma^2 = \frac{1}{2} \sum_{\alpha\beta} (\alpha-\beta)^2 c^\alpha c^\beta > 0. \tag{3.6}$$

Without any loss of generality we can assume $\lambda = 0$.

The last quantity is the asymptotic viscosity of the solution F^α, and is related to the long time diffusive behavior of F^α. We will show that it is also related to the existence of a Glimm functional for (3.1).

Observe that $\sigma > 0$ and the fact that the average speed is 0 imply that

$$\sum_{\alpha > m} c^\alpha \geq \frac{1}{K(K-m)} \sum_\alpha (\alpha - m)(\alpha + K) c^\alpha = \frac{\sigma^2 - mK}{K(K-m)}, \tag{3.7}$$

and similarly

$$\sum_{\alpha < m} c^\alpha \geq \frac{\sigma^2 - mK}{K(K-m)}, \tag{3.8}$$

We introduce the two quantities related to the wave interaction: the Amount of Interaction is defined by

$$\begin{aligned} \mathcal{I} &= \sum_{\alpha\beta} |\alpha - \beta| \int_0^{+\infty} \int_{\mathbb{R}} \big| F^\alpha_t(t,x) F^\beta_x(t,x) - F^\alpha_x(t,x) F^\beta_t(t,x) \big| dxdt \\ &= \sum_{\alpha\beta} |\alpha - \beta| \int_0^{+\infty} \int_{\mathbb{R}} \big| F^\alpha_x(t,x) F^\beta_x(t,x) \big| \bigg| - \frac{F^\beta_t(t,x)}{F^\beta_x(t,x)} - \bigg(- \frac{F^\alpha_t(t,x)}{F^\alpha_x(t,x)} \bigg) \bigg| dxdt. \end{aligned} \tag{3.9}$$

The Glimm Functional is

$$\begin{aligned} \mathcal{Q}(t) &= \sum_{\alpha\beta} \iint_{\mathbb{R}^2} \big| F^\alpha_t(t,x) F^\beta_x(t,x) - F^\alpha_x(t,x) F^\beta_t(t,x) \big| dxdt \\ &= \sum_{\alpha\beta} \iint_{\mathbb{R}^2} \big| F^\alpha_x(t,x) F^\beta_x(t,x) \big| \bigg| - \frac{F^\beta_t(t,x)}{F^\beta_x(t,x)} - \bigg(- \frac{F^\alpha_t(t,y)}{F^\alpha_x(t,y)} \bigg) \bigg| dxdt. \end{aligned} \tag{3.10}$$

The interpretation of the above formulas is the following.

If we define the strength of the wave in the family F^α located at (t,x) as $|F^\alpha_x(t,x)|$, and its speed as the level set speed

$$\sigma^\alpha(t,x) = - \frac{F^\alpha_t(t,x)}{F^\alpha_x(t,x)},$$

then the amount of interaction is the integral over the half plane $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and w.r.t. $\mu \times \mu$ of the elementary interactions

$$\begin{aligned} &|\alpha - \beta| |F^\alpha_x| |F^\beta_x| \big| \sigma^\alpha - \sigma^\beta \big| \\ &\qquad = |\alpha - \beta| \big(\text{strength of } \alpha \text{ wave} \big) \times \big(\text{strength of } \beta \text{ wave} \big) \times \Big| \text{difference in speed between } \alpha,\ \beta \Big|. \end{aligned}$$

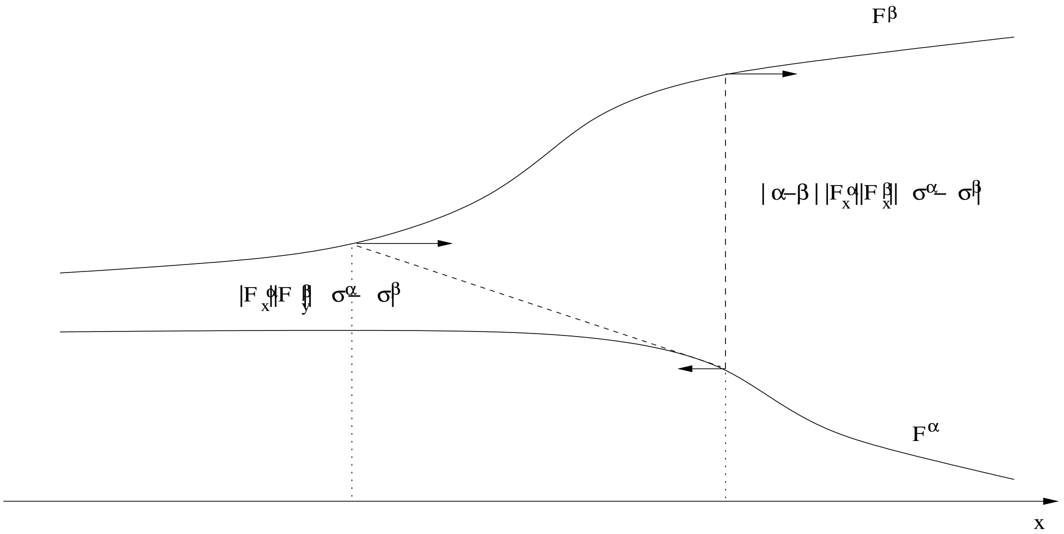

FIGURE 2. Amount of interaction and Glimm functional.

This is, a part from the index α, β the amount of interaction among non linear waves of hyperbolic systems. The coefficient $|\alpha - \beta|$ is related to the time of interaction: if two families have speed α, β, then the time of interaction is of order $|\alpha - \beta|^{-1}$.

Similarly, the Glimm functional is given by the sum of the elementary Glimm interaction potentials

$$\begin{aligned} &|F_x^\alpha(t,x)||F_x^\beta(t,y)|\big|\sigma^\alpha(t,x) - \sigma^\beta(t,y)\big| \\ &\quad = \big(\text{strength of } \alpha \text{ wave in } x\big) \times \big(\text{strength of } \beta \text{ wave in } y\big) \times \Big|\text{difference in speed between } \alpha,\, \beta\Big|. \end{aligned} \tag{3.11}$$

Define now the variables

$$g^\alpha = \frac{\partial F^\alpha}{\partial t}, \quad f^\alpha = \frac{\partial F^\alpha}{\partial x},$$

which satisfy the same linear BGK scheme

$$g_t^\alpha + \alpha g_x^\alpha = c^\alpha \sum_\beta g^\beta - g^\alpha, \quad f_t^\alpha + \alpha f_x^\alpha = c^\alpha \sum_\beta f^\beta - f^\alpha, \tag{3.12}$$

and introduce the functions

$$P^{\alpha\beta}(t,x,y) = f^\alpha(t,x) g^\beta(t,y) - f^\beta(t,y) g^\alpha(t,x). \tag{3.13}$$

A simple computation shows that

$$\begin{aligned} P_t^{\alpha\beta} &= -\alpha P_x^{\alpha\beta} - \beta P_y^{\alpha\beta} + c^\alpha \sum_\gamma P^{\gamma\beta} + c^\beta \sum_\gamma P^{\alpha\gamma} - 2P^{\alpha\beta} \\ &= -(\alpha,\beta) \cdot \nabla P^{\alpha\beta} + \sum_\gamma \big(c^\beta P^{\alpha\gamma} + c^\alpha P^{\gamma\beta}\big) - 2P^{\alpha\beta}. \end{aligned} \tag{3.14}$$

Because of the symmetry of (3.13) one has

$$P^{\alpha\beta}(x,y) = -P^{\beta\alpha}(y,x), \tag{3.15}$$

Thus the above BGK scheme can be considered either in the plane with initial data satisfying (3.15), or in the half plane $\{x > y\}$ with boundary conditions

$$P^{\alpha\beta}(x,x) + P^{\beta\alpha}(x,x) = 0. \tag{3.16}$$

The meaning of the above boundary condition is that when a particle $p^{\alpha\beta}$ travelling with speeds (α,β), $\alpha < \beta$ hits the boundary $\{x = y\}$, then it changes into $-p^{\beta\alpha}$, i.e. it is reflexed $(\alpha,\beta) \mapsto (\beta,\alpha)$ and from positive it becomes negative (or viceversa).

It is thus clear that the total number of particle is preserved, i.e.

$$\frac{d}{dt}\sum_{\alpha\beta}\int_{\mathbb{R}^2}|P^{\alpha\beta}(t,x,y)|dxdy = 2\frac{d}{dt}\sum_{\alpha\beta}\int_{x>y}|P^{\alpha\beta}(t,x,y)|dxdy \leq 0.$$

We can also expect that, because of diffusion, if $P^{\alpha\beta} > 0$ in some points, then it is positive in a small neighborhood. Hence after the boundary reflection some particles become negative and annihilates with the positive ones: observe in fact that the source term is the sum w.r.t. the velocity α of all particles at a fixed location.

The quantities (3.9), (3.10) turn out to be the flux of $P^{\alpha\beta}$ on the boundary $x = y$,

$$\mathcal{I} = \sum_{\alpha\beta}|\alpha-\beta|\int_0^{+\infty}\int_{\mathbb{R}}|P^{\alpha\beta}(t,x,x)|dxdt, \tag{3.17}$$

and the initial L^1 norm of $P^{\alpha\beta}$ in the plane,

$$\mathcal{Q} = \sum_{\alpha\beta}\iint_{\mathbb{R}^2}|P^{\alpha\beta}(0,x,y)|dxdy. \tag{3.18}$$

Aim of the next section is to prove that if $\sigma > 0$, then $\mathcal{I}$ is bounded by $\mathcal{Q}$.

3.1. **Construction of the Glimm functional for BGK kinetic schemes.** We consider the linear BGK model (1.18) in the plane

$$P^{\alpha\beta}_t + (\alpha,\beta)\cdot\nabla P^{\alpha\beta} = \frac{1}{2}\sum_\gamma\big(c^\beta P^{\alpha\gamma} + c^\alpha P^{\gamma\beta}\big) - P^{\alpha\beta}, \tag{3.19}$$

with initial data satisfying

$$P^{\alpha\beta}(0,x,y) + P^{\beta\alpha}(0,y,x) = 0. \tag{3.20}$$

The coefficients have been renormalized by the rescaling $(t,x)\mapsto(t/2,x/2)$.

Since the system is linear, it suffices to consider the special initial data

$$P^{\alpha\beta}(0,x,y) = \delta_{\alpha,\bar\alpha}\delta_{\beta,\bar\beta}\delta(x,y) - \delta_{\alpha,\bar\beta}\delta_{\beta,\bar\alpha}\delta(x,y), \quad \bar\alpha > \bar\beta, \tag{3.21}$$

i.e. two Dirac deltas located at the origin with speed $(\bar\alpha,\bar\beta)$ and $(\bar\beta,\bar\alpha)$, by the symmetry (3.20).

In fact, we can first solve the equation (3.19) with absorbing boundary data,

$$P^{\alpha\beta}(t,x,x) = 0 \quad \text{for } \alpha \geq \beta.$$

Denote this solution with $P^{\alpha\beta,-1}$. As in Example 2.1, from conservation it follows that the boundary flux of this solution is

$$\sum_{\alpha\beta}|\alpha-\beta|\int_0^{+\infty}\int_{\mathbb{R}}|P^{\alpha\beta,-1}(t,x,x)|dxdt \leq \sum_{\alpha\beta}\iint_{\mathbb{R}^2}|P^{\alpha\beta,-1}(0,x,y)|dxdy. \tag{3.22}$$

Then we solve (3.19) with the source term $|\alpha-\beta|P^{\alpha\beta,-1}(t,x,x)$. If we have that the boundary flux of the solution with the initial data (3.21) is bounded by C, independent of $(\bar\alpha,\bar\beta)$, then it follows that

$$\begin{aligned}
\sum_{\alpha\beta}|\alpha-\beta|\int_0^{+\infty}\int_{\mathbb{R}}|P^{\alpha\beta}(t,x,x)|dxdt &\leq C\sum_{\alpha\beta}|\alpha-\beta|\int_0^{+\infty}\int_{\mathbb{R}}|P^{\alpha\beta,-1}(t,x,x)|dxdt \\
&\leq C\sum_{\alpha\beta}\iint_{\mathbb{R}^2}|P^{\alpha\beta}(0,x,y)|dxdy \\
&\leq (1+C)\bigg(\sum_\alpha \text{Tot.Var.}(F^\alpha)\bigg)^2.
\end{aligned} \tag{3.23}$$

We follow the same approach used in the one dimensional example.

The solution to the BGK scheme (3.19) with initial data (3.20) can be written as

$$P^{\alpha\beta}(t,x,y) = \sum_{n=0}^{+\infty}P^{\alpha\beta,n}(t,x,y), \tag{3.24}$$

where each function $P^{\alpha\beta,n}$ satisfies

$$P_t^{\alpha\beta,n} + (\alpha,\beta)\cdot\nabla P^{\alpha\beta,n} = \frac{1}{2}\sum_\gamma \big(c^\beta P^{\alpha\gamma,n-1} + c^\alpha P^{\gamma\beta,n-1}\big) - P^{\alpha\beta,n}. \tag{3.25}$$

We will say that $P^{\alpha\beta,n}$ is the n-th generation of particle.

We will write only the functions $P^{\alpha\beta,n}$ with $n = 0,1,2$, and then compute the source term for $n = 3$. The idea of the proof is that we will show that the source term for $n = 3$ decreases of a positive quantity, and the flux through the boundary of the first 3 generations is bounded.

It is a matter of computation to verify that $P^{\alpha\beta,n}$, $n + 0,1,2$, are given by

$$P^{\alpha\beta,0}(t,x,y) = e^{-t}\delta(x-\bar\alpha t, y-\bar\beta t)\delta_{\alpha,\bar\alpha}\delta_{\beta,\bar\beta} - e^{-t}\delta(x-\bar\beta t, y-\bar\alpha t)\delta_{\alpha,\bar\beta}\delta_{\beta,\bar\alpha}, \tag{3.26}$$

$$\begin{aligned}
P^{\alpha\beta,1}(t,x,y) = {} & \frac{1}{2}te^{-t}(c^{\bar\alpha}+c^{\bar\beta})\Big(\delta(x-\bar\alpha t, y-\bar\beta t)\delta_{\alpha,\bar\alpha}\delta_{\beta,\bar\beta} - \delta(x-\bar\beta t, y-\bar\alpha t)\delta_{\alpha,\bar\beta}\delta_{\beta,\bar\alpha}\Big) \\
& + \frac{1}{2}e^{-t}\bigg(c^\alpha\delta_{\alpha\neq\bar\alpha}\chi[\alpha,\bar\alpha](x/t)\delta(y-\bar\beta t)\delta_{\beta,\bar\beta} + c^\beta\delta_{\beta\neq\bar\beta}\chi[\beta,\bar\beta](y/t)\delta(x-\bar\alpha t)\delta_{\alpha,\bar\alpha}\bigg) \\
& - \frac{1}{2}e^{-t}\bigg(c^\beta\delta_{\beta\neq\bar\beta}\chi[\beta,\bar\beta](x/t)\delta(y-\bar\alpha t)\delta_{\alpha,\bar\alpha} + c^\alpha\delta_{\alpha\neq\bar\alpha}\chi[\alpha,\bar\alpha](y/t)\delta(x-\bar\beta t)\delta_{\beta,\bar\beta}\bigg),
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
P^{\alpha\beta,2}(t,x,y) = {} & \frac{1}{4}t^2e^{-t}\big(c^{\bar\alpha}+c^{\bar\beta}\big)^2\Big(\delta(x-\bar\alpha t, y-\bar\beta t)\delta_{\alpha,\bar\alpha}\delta_{\beta,\bar\beta} - \delta(x-\bar\beta t, y-\bar\beta t)\delta_{\alpha,\bar\beta}\delta_{\beta,\bar\alpha}\Big) \\
& + \frac{1}{4}te^{-t}\delta_{\alpha,\bar\alpha}\delta_{\beta,\bar\beta}\bigg(\sum_{\gamma\neq\bar\alpha} c^\gamma c^{\bar\alpha}\mathrm{tr}[\gamma,\bar\alpha,\bar\alpha](x/t)\delta(y-\bar\beta t) + \sum_{\gamma\neq\bar\beta} c^\gamma c^{\bar\beta}\mathrm{tr}[\gamma,\bar\beta,\bar\beta](y/t)\delta(x-\bar\alpha t)\bigg) \\
& - \frac{1}{4}te^{-t}\delta_{\alpha,\bar\beta}\delta_{\beta,\bar\alpha}\bigg(\sum_{\gamma\neq\bar\beta} c^\gamma c^{\bar\beta}\mathrm{tr}[\gamma,\bar\beta,\bar\beta](x/t)\delta(y-\bar\alpha t) + \sum_{\gamma\neq\bar\alpha} c^\gamma c^{\bar\alpha}\mathrm{tr}[\gamma,\bar\alpha,\bar\alpha](y/t)\delta(x-\bar\beta t)\bigg) \\
& + \frac{1}{4}te^{-t}\delta_{\alpha\neq\bar\alpha}\delta_{\beta,\bar\beta}\bigg(c^\alpha(c^{\bar\alpha}+c^{\bar\beta})\mathrm{tr}[\alpha,\bar\alpha,\bar\alpha](x/t)\delta(y-\bar\beta t) + (c^\alpha)^2\mathrm{tr}[\alpha,\alpha,\bar\alpha](x/t)\delta(y-\bar\beta t) \\
& \qquad + c^\alpha\sum_{\gamma\neq\alpha,\bar\alpha} c^\gamma\mathrm{tr}[\alpha,\gamma,\bar\alpha](x/t)\delta(y-\bar\beta t)\bigg) \\
& + \frac{1}{4}te^{-t}\delta_{\alpha=\bar\alpha}\delta_{\beta\neq\bar\beta}\bigg(c^\beta(c^{\bar\alpha}+c^{\bar\beta})\mathrm{tr}[\beta,\bar\beta,\bar\beta](y/t)\delta(x-\bar\alpha t) + (c^\beta)^2\mathrm{tr}[\beta,\beta,\bar\beta](y/t)\delta(x-\bar\alpha t) \\
& \qquad + c^\beta\sum_{\gamma\neq\beta,\bar\beta} c^\gamma\mathrm{tr}[\beta,\gamma,\bar\beta](y/t)\delta(x-\bar\alpha t)\bigg) \\
& + \frac{1}{4}e^{-t}\bigg(c^\alpha c^{\bar\beta}\delta_{\alpha\neq\bar\alpha}\chi[\alpha,\bar\alpha](x/t)\delta(y-\bar\beta t)\delta_{\beta,\bar\beta} + c^\beta c^{\bar\alpha}\delta_{\beta\neq\bar\beta}\chi[\beta,\bar\beta](y/t)\delta(x-\bar\alpha t)\delta_{\alpha,\bar\alpha}\bigg) \\
& - \frac{1}{4}te^{-t}\delta_{\beta\neq\bar\beta}\delta_{\alpha,\bar\alpha}\bigg(c^\beta(c^{\bar\alpha}+c^{\bar\beta})\mathrm{tr}[\beta,\bar\beta,\bar\beta](x/t)\delta(y-\bar\alpha t) + (c^\beta)^2\mathrm{tr}[\beta,\beta,\bar\beta](x/t)\delta(y-\bar\alpha t) \\
& \qquad + c^\beta\sum_{\gamma\neq\beta,\bar\beta} c^\gamma\mathrm{tr}[\beta,\gamma,\bar\beta](x/t)\delta(y-\bar\alpha t)\bigg) \\
& - \frac{1}{4}te^{-t}\delta_{\beta=\bar\beta}\delta_{\alpha\neq\bar\alpha}\bigg(c^\alpha(c^{\bar\alpha}+c^{\bar\beta})\mathrm{tr}[\alpha,\bar\alpha,\bar\alpha](y/t)\delta(x-\bar\beta t) + (c^\alpha)^2\mathrm{tr}[\alpha,\alpha,\bar\alpha](y/t)\delta(x-\bar\beta t) \\
& \qquad + c^\alpha\sum_{\gamma\neq\alpha,\bar\alpha} c^\gamma\mathrm{tr}[\alpha,\gamma,\bar\alpha](y/t)\delta(x-\bar\beta t)\bigg) \\
& - \frac{1}{4}e^{-t}\bigg(c^\beta c^{\bar\alpha}\delta_{\beta\neq\bar\beta}\chi[\beta,\bar\beta](x/t)\delta(y-\bar\alpha t)\delta_{\alpha,\bar\alpha} + c^\alpha c^{\bar\beta}\delta_{\alpha\neq\bar\alpha}\chi[\alpha,\bar\alpha](y/t)\delta(x-\bar\beta t)\delta_{\beta,\bar\beta}\bigg) \\
& + \frac{1}{4}c^\alpha c^\beta\delta_{\alpha\neq\bar\alpha}\delta_{\beta\neq\bar\beta}e^{-t}\Big(\chi[\alpha,\bar\alpha](x/t)\chi[\beta,\bar\beta](y/t) - \chi[\beta,\bar\beta](x/t)\chi[\alpha,\bar\alpha](y/t)\Big),
\end{aligned} \tag{3.28}$$

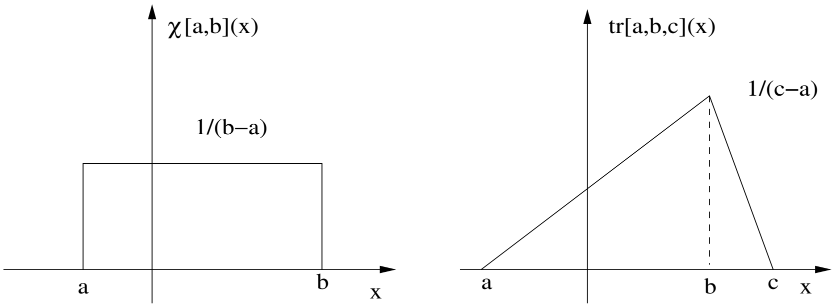

FIGURE 3. The functions $\chi[a,b](x)$ and $\mathrm{tr}[a,,c](x)$.

where the functions $\chi[a,b](x)$ and $\mathrm{tr}[a,b,c](x)$ are defined as

$$\chi[a,b](x) = \begin{cases} |a-b|^{-1} & \min\{a,b\} \le x \le \max\{a,b\} \\ 0 & \text{otherwise} \end{cases} \tag{3.29}$$

for $a \neq b$ and

$$\mathrm{tr}[a,b,c](x) = \begin{cases} |x-a'|/(|b'-a'||c'-a'|) & a' \le x \le b' \\ |c'-x|/(|c'-b'||c'-a'|) & b' \le x \le c' \\ 0 & \text{otherwise} \end{cases} \tag{3.30}$$

$$a' = \min\{a,b,c\},\ c' = \max\{a,b,c\},\ b' = \{a,b,c\} \setminus \{a',b'\}, \quad a' < c'.$$

Note that as a consequence of conservation, we have that the integral over $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^2$ of $P^{\alpha\beta,1}$, $P^{\alpha\beta,2}$ is constant:

$$\sum_{\alpha\beta} \int_0^{+\infty} \iint_{\mathbb{R}^2} |P^{\alpha\beta,i}(t,x,y)| dxdydt = 2, \qquad i = 1,2. \tag{3.31}$$

We can use a similar computation to estimate the flux on the boundary: in fact, integrating only on $\{x > y\}$ and observing that on $\{x = y\}$ the function $P^{\alpha\beta} > 0$ if $\alpha < \beta$, we obtain or $i = 1,2$

$$\begin{aligned} &\sum_{\alpha<\beta} (\beta - \alpha) \int_0^{+\infty} \int_{\mathbb{R}} P^{\alpha\beta,i}(t,x,x) dxdt \\ &\qquad \le \sum_{\alpha<\beta} \int_0^{+\infty} \iint_{\mathbb{R}^2} \sum_\gamma \frac{1}{2} \Big(c^\alpha P^{\gamma\beta,i-1}(t,x,y) + c^\beta P^{\alpha\gamma,i-1}(t,x,y) \Big) dxdydt \le 1. \end{aligned}$$

It thus follows that the total flux, counting also the initial data, is

$$\sum_{i=1}^3 \sum_{\alpha<\beta} (\beta - \alpha) \int_0^{+\infty} \int_{\mathbb{R}} P^{\alpha\beta,i}(t,x,x) dxdt \le 3. \tag{3.32}$$

The cancellation is due to the fact that for $n = 3$ some of the squares

$$Q^+_{\alpha\beta} = \Big\{ (x,y) \in \big[\min\{\alpha,\bar\alpha\}t, \max\{\alpha,\bar\alpha\}t\big] \times \big[\min\{\beta,\bar\beta\}t, \max\{\beta,\bar\beta\}t\big] \Big\},$$

$$Q^-_{\alpha\beta} = \Big\{ \big[\min\{\beta,\bar\beta\}t, \max\{\beta,\bar\beta\}t\big] \times \big[\min\{\alpha,\bar\alpha\}t, \max\{\alpha,\bar\alpha\}t\big] \Big\}$$

overlap. A simple analysis shows that

$$Q^+_{\alpha\beta} \cap Q^-_{\alpha\beta} = \begin{cases} 0 & \alpha \ge \beta \\ (\alpha - \beta)^2 & \bar\beta \le \alpha \le \beta \le \bar\alpha \\ (\alpha - \bar\beta)^2 & \bar\beta \le \alpha \le \beta, \beta > \bar\alpha \\ (\bar\alpha - \beta)^2 & \alpha \le \beta \le \bar\alpha, \alpha < \bar\beta \\ (\bar\alpha - \bar\beta)^2 & \alpha < \bar\beta, \beta < \bar\alpha \end{cases} \tag{3.33}$$

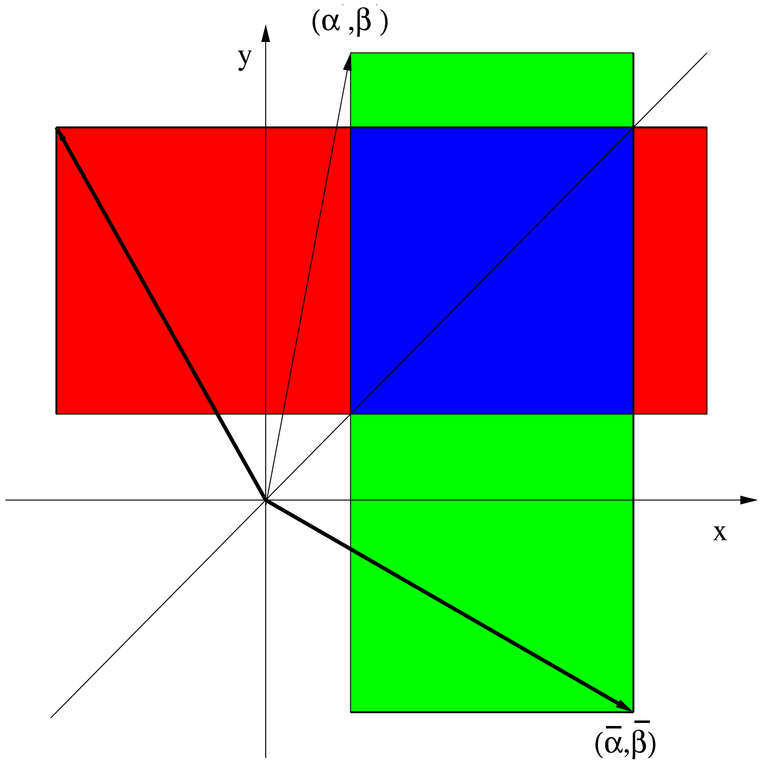

FIGURE 4. The cancellation occurring on $P^{\alpha,\beta,2}$.

We have thus that the cancellation when computing the source term of the step $n = 3$ is greater than

$$\begin{aligned} C = & \sum_{\alpha<\bar\beta,\beta>\bar\alpha} c^\alpha c^\beta \frac{(\bar\alpha-\bar\beta)^2}{4|\alpha-\bar\alpha||\beta-\bar\beta|} + \sum_{\beta>\bar\alpha,\bar\beta\le\alpha\le\beta} c^\alpha c^\beta \frac{|\alpha-\bar\alpha|}{4|\beta-\bar\beta|} \\ & + \sum_{\alpha<\bar\beta,\alpha\le\beta\le\bar\alpha} c^\alpha c^\beta \frac{|\beta-\bar\beta|}{4|\alpha-\bar\alpha|} + \sum_{\bar\beta\le\alpha\le\beta\le\bar\alpha} c^\alpha c^\beta \frac{(\alpha-\beta)^2}{4|\alpha-\bar\alpha||\beta-\bar\beta|}. \end{aligned} \tag{3.34}$$

We now use the assumption that the dissipation is strictly positive to estimate C, and more precisely we consider (3.7) with $m = \sigma^2/(2K)$,

$$\sum_{\alpha>\sigma^2/(2K)} c^\alpha \ge \frac{\sigma^2}{2K^2-\sigma^2}, \qquad \sum_{\alpha<-\sigma^2/(2K)} c^\alpha \ge \frac{\sigma^2}{2K^2-\sigma^2}. \tag{3.35}$$

We consider 3 cases:

(1) If $\bar\alpha \ge \bar\beta > -\sigma^2/(4K)$, then we have

$$\begin{aligned} C \ge & \sum_{\alpha<\bar\beta,\alpha\le\beta\le\bar\alpha} c^\alpha c^\beta \frac{|\beta-\bar\beta|}{4|\alpha-\bar\alpha|} \ge \sum_{\alpha\le\beta\le-\sigma^2/(2K)} c^\alpha c^\beta \frac{|\beta-\bar\beta|}{4|\alpha-\bar\alpha|} \\ & \ge \frac{\sigma^2}{16K^2}\frac{1}{2}\sum_{\alpha,\beta\le-\sigma^2/(2K)} c^\alpha c^\beta \ge \frac{\sigma^6}{32K^2(2K^2-\sigma^2)^2}. \end{aligned} \tag{3.36}$$

(2) Similarly for $\bar\beta \le \bar\alpha \le \sigma^2/(4K)$,

$$\begin{aligned} C \ge & \sum_{\beta>\bar\alpha,\bar\beta\le\alpha\le\beta} c^\alpha c^\beta \frac{|\alpha-\bar\alpha|}{4|\beta-\bar\beta|} \ge \sum_{\beta\ge\alpha\ge\sigma^2/(2K)} c^\alpha c^\beta \frac{|\beta-\bar\beta|}{4|\alpha-\bar\alpha|} \\ & \ge \frac{\sigma^2}{16K^2}\frac{1}{2}\sum_{\alpha,\beta\ge\sigma^2/(2K)} c^\alpha c^\beta \ge \frac{\sigma^6}{32K^2(2K^2-\sigma^2)^2}. \end{aligned} \tag{3.37}$$

(3) Finally, if $\bar\alpha \geq \sigma^2/(4K)$, $\bar\beta \leq -\sigma^2/(4K)$, is follows

$$\begin{aligned} C &\geq \sum_{\alpha<\bar\beta,\beta>\bar\alpha} c^\alpha c^\beta \frac{(\bar\alpha-\bar\beta)^2}{4|\alpha-\bar\alpha||\beta-\bar\beta|} \geq \sum_{\alpha\leq-\sigma^2/(2K),\beta\geq\sigma^2/(2K)} c^\alpha c^\beta \frac{(\bar\alpha-\bar\beta)^2}{4|\alpha-\bar\alpha||\beta-\bar\beta|} \\ &\geq \frac{\sigma^4}{16K^4}\frac{1}{2} \sum_{\alpha\leq-\sigma^2/(2K),\beta\geq\sigma^2/(2K)} c^\alpha c^\beta \geq \frac{\sigma^8}{32K^4(2K^2-\sigma^2)^2}. \end{aligned} \tag{3.38}$$

In all cases, the cancellation is strictly positive. We thus conclude that there exists a constant C such that $\mathcal{I} \leq C\mathcal{Q}$, and the constant can be estimated by

$$C \leq 3\cdot\frac{32K^4(2K^2-\sigma^2)^2}{\sigma^8}.$$

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