THE VECTOR MEASURES WHOSE RANGE IS STRICTLY CONVEX

Stefano Bianchini – Carlo Mariconda

S.I.S.S.A. – Università di Padova

September 1997

Abstract. Let μ be a measure on a measure space (X, Λ) with values in $\mathbb{R}^n$ and f be the density of μ with respect to its total variation. We show that the range $\mathcal{R}(\mu) = \{\mu(E) : E \in \Lambda\}$ of μ is strictly convex if and only if the determinant $\det[f(x_1), \dots, f(x_n)]$ is non zero a.e. on X^n. We apply the result to a class of measures containing those that are generated by Chebyshev systems.

1991 *Mathematics Subject Classification.* Primary: 46G10, 52A20, 28B05. Secondary: 28A35, 41A50.

Key words and phrases. Chebyshev measure, Chebyshev system, exposed point, strictly convex, Lyapunov, range of a vector measure.

We warmly thank R. Cerf for having carefully read the manuscript and for his useful remarks; C.M. wishes to thank G. Colombo for his helpful advices on measurable multifunctions.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Introduction

Let $\mu : (X, \Lambda) \to \mathbb{R}^n$ be a non–atomic vector measure. A Theorem of Lyapunov [15] states that its range $\mathcal{R}(\mu) = \{\mu(E) : E \in \Lambda\}$ is closed and convex. In [5,7] the authors, motivated from the study of some bang–bang control problems, were led to introduce a broad class of measures, Chebyshev measures, whose range is strictly convex. Their definition involves a signed measure $\det \mu$ defined on the product space $(X^n, \Lambda^{\otimes n})$ by the relation

$$\forall A_1, \dots, A_n \in \Lambda \qquad \det \mu(A_1 \times \cdots \times A_n) = \det[\mu(A_1), \dots, \mu(A_n)]$$

(where $\det[u_1, \dots, u_n]$ denotes the determinant of $u_1, \dots, u_n$.)

In the simpler case when $X = I = [0, 1]$ and Λ coincides with the set $\mathcal{L}$ of its Lebesgue measurable subsets the measure μ is said to be Chebyshev with respect to the Lebesgue measure λ in $[0, 1]$ if the measure $\det \mu$ is strictly positive on the non $\lambda^{\otimes n}$–negligible subsets of $\Gamma = \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \le x_1 \le \cdots \le x_n \le 1\}$, $\lambda^{\otimes n}$ denoting the n–product measure of λ. In the case where μ is absolutely continuous with density g with respect to λ the above condition is equivalent to the fact that the determinant $\det[g(x_1), \dots, g(x_n)]$ is strictly positive $\lambda^{\otimes n}$– a.e. in Γ i.e. that g is a Chebyshev system (or T–system, following the terminology of [11]).

As it is shown in [7] the range $\mathcal{R}(\mu)$ of such a measure is strictly convex and contains the origin in its boundary. A peculiar property of a Chebyshev measure is that its range can be described through the values that the measure assumes on the finite union of intervals. It is well known that a compact, convex, centrally symmetric subset of $\mathbb{R}^2$ containing the origin is the range of a two dimensional measure, i.e. a bidimensional zonoid (see [3]). In [2] the authors show that every strictly convex, compact, centrally symmetric subset of $\mathbb{R}^2$ (with O in its boundary) is the range of a Chebyshev measure. It is then natural to ask whether this result can be in some way extended to greater dimensions and, more generally, to try to characterize the measures whose ranges are strictly convex. The latter question was asked during a workshop to R. Schneider who answered with the following result.

Theorem. [16] *$\mathcal{R}(\mu)$ is strictly convex if and only if for every A such that $\mu(A) \neq O$ there exist $A_1, \ldots, A_n$ in A such that $\mu(A_1), \ldots, \mu(A_n)$ are linearly independent.*

It seems difficult however to check whether or not a measure does satisfy these conditions. One of the purposes of this paper is to show that the range of a measure μ is strictly convex if and only if the density f of μ with respect to its total variation $|\mu|$ is such that $\det[f(x_1), \ldots, f(x_n)]$ is non zero a.e. on X^n. The latter determinant being the density of $\det \mu$ with respect to the product measure $|\mu|^{\otimes n}$ it turns out that $\mathcal{R}(\mu)$ is strictly convex if and only if the total variation of $\det \mu$ is equivalent to $|\mu|^{\otimes n}$. The main result is obtained via the study of the exposed faces of $\mathcal{R}(\mu)$; this allows also to give an alternative simple proof of Schneider's Theorem.

In §4 we study some applications of this characterization to Chebyshev measures. First we show that (again considering for simplicity the case where $X = I$ and $\Lambda = \mathcal{L}$) μ is a Chebyshev measure with respect to λ if and only if the measure $\det \mu$ is positive and equivalent to $|\mu|^{\otimes n}$ on Γ. We improve the main result of [2] showing that if the range of a bidimensional measure μ is strictly convex and contains the origin in its boundary then not only is $\mathcal{R}(\mu)$ the range of a suitable Chebyshev measure but μ is itself a Chebyshev measure. Finally we answer the initial question: when $n > 2$ there exist strictly convex zonoids (with the origin in the boundary) that are not the range of a Chebyshev measure. Actually the latter have a non regular boundary.

2. Extreme points and exposed points of the range of a measure

Notation. By "$\cdot$" we denote the usual scalar product, $\|\cdot\|$ is the euclidian norm in $\mathbb{R}^n$ and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^n$; O is the zero vector in $\mathbb{R}^n$.

In what follows X is a set and Λ is a $\sigma-$ algebra of subsets of X. If $\nu, \nu_1, \ldots, \nu_m$ are

measures on (X, Λ) we denote by $\nu_1 \otimes \cdots \otimes \nu_m$ (resp. $\nu^{\otimes m}$) the m–product measure of $\nu_1, \ldots, \nu_m$ (resp. of ν) on $(X^m, \Lambda^{\otimes m})$, where $X^m = X \times \cdots \times X$ (m times) and $\Lambda^{\otimes m}$ is the m–product σ–algebra of Λ. We set $L^1_\nu(X, \mathbb{R}^n)$ to be the space of the ν–integrable functions on X with values in $\mathbb{R}^n$.
In §2, §3 we assume that μ is a non–atomic vector measure on (X, Λ) with values in $\mathbb{R}^n$; we will denote by $|\mu|$ its total variation and by f the density of μ with respect to $|\mu|$; we recall that f belongs to $L^1_{|\mu|}(X, \mathbb{R}^n)$ and that $\|f\| = 1$ almost everywhere (a.e.) in X. The range $\mathcal{R}(\mu)$ of μ is the subset of $\mathbb{R}^n$ defined by $\mathcal{R}(\mu) = \{\mu(E) : E \in \Lambda\}$.
Unless the contrary is expressely stated, for A, B in X by $A \subseteq B$ we mean that $B \setminus A$ is $|\mu|$–negligible and by $A = B$ that $A \subseteq B$ and $B \subseteq A$ i.e. that $|\mu|(A \Delta B) = 0$.

For K being a compact convex subset of $\mathbb{R}^n$ and p in S^{n-1} let $h(K, p) = \max\{p \cdot x : x \in K\}$; the *supporting hyperplane* $H(K, p)$ with *outer normal vector* p is defined by

$$H(K, p) = \{x \in \mathbb{R}^n : p \cdot x = h(K, p)\}$$

and $F(K, p) = H(K, p) \cap K$ is the *exposed face with outer normal vector* p.
We recall that a point x in K is said to be *exposed* if it coincides with an exposed face, i.e. if there exists p in S^{n-1} such that $F(K, p) = \{x\}$; obviously each exposed point of K is an extreme point of K (but the converse is not true, see for instance [14]).
For p in S^{n-1} we introduce the following measurable subsets of X:

$$D^+(p) = \{x \in X : p \cdot f(x) > 0\}, \quad D^-(p) = \{x \in X : p \cdot f(x) < 0\},$$
$$D^0(p) = \{x \in X : p \cdot f(x) = 0\}.$$

Lyapunov's Theorem (see for instance [15]) states that $\mathcal{R}(\mu)$ is closed and convex. We describe here the exposed faces of $\mathcal{R}(\mu)$.

Proposition 2.1. *Assume that p belongs to S^{n-1}; then $h(\mathcal{R}(\mu), p) = p \cdot \mu(D^+(p))$ and*

$$F(\mathcal{R}(\mu), p) = \{\mu(E) : E \in \Lambda,\ D^+(p) \subseteq E \subseteq D^+(p) \cup D^0(p)\}.$$

Proof. For E in Λ we have

$$\begin{aligned}
p\cdot\mu(E) &= \int_E p\cdot f(x)\,d|\mu| = \\
&= \int_{E\cap D^-(p)} p\cdot f(x)\,d|\mu| + \int_{E\cap D^+(p)} p\cdot f(x)\,d|\mu| \le \\
&\le \int_{E\cap D^+(p)} p\cdot f(x)\,d|\mu| \le \int_{D^+(p)} p\cdot f(x)\,d|\mu| = p\cdot\mu(D^+(p)),
\end{aligned}$$

proving the first part of the claim. Moreover the above inequalities show that, for E in Λ, the equality $p\cdot\mu(E) = p\cdot\mu(D^+(p))$ holds if and only if $|\mu|(E\cap D^-(p)) = 0$ and $E\cap D^+(p) = D^+(p)$ or, equivalently, $D^+(p)\subseteq E\subseteq D^+(p)\cup D^0(p)$. $\square$

Corollary 2.2. *For p in S^{n-1} the exposed face $F(\mathcal{R}(\mu),p)$ of $\mathcal{R}(\mu)$ with outer normal vector p is reduced to a point if and only if $|\mu|(D^0(p)) = 0$.*

We recall that a compact convex subset of $\mathbb{R}^n$ is strictly convex if and only if each of its exposed faces is reduced to a point. The above result yields then directly a first characterization of the strict convexity of $\mathcal{R}(\mu)$.

Proposition 2.3. *$\mathcal{R}(\mu)$ is strictly convex if and only if $|\mu|(D^0(p))=0$ for each p in S^{n-1}.*

As an application we give a short alternative proof to Schneider's characterization of the measures whose range are strictly convex.

If $\{u_\iota\}_{\iota\in I}$ is a set of vectors in $\mathbb{R}^n$ we denote by $<u_\iota>_{\iota\in I}$ the vector space spanned by the vectors u_ι. The orthogonal space of a vector space L is denoted by $L^\perp$.

Theorem 2.4. [16] *$\mathcal{R}(\mu)$ is strictly convex if and only if for every A such that $\mu(A)\neq O$ there exist $A_1,\dots,A_n$ in A such that $\mu(A_1),\dots,\mu(A_n)$ are linearly independent.*

Proof. Let A be such that $\mu(A)\neq O$ and assume that the vector space

$$L = <\mu(B) : B\in\Lambda, B\subseteq A>$$

is at most $(n-1)$ dimensional. Then if p belongs to $S^{n-1} \cap L^{\perp}$ we have

$$\forall B \in \Lambda, \quad B \subseteq A \Longrightarrow \int_B p \cdot f(x)\, d|\mu| = p \cdot \int_B f(x)\, d|\mu| = p \cdot \mu(B) = 0$$

so that $A \subseteq D^0(p)$ and thus $|\mu|(D^0(p)) > 0$; Proposition 2.3 implies that $\mathcal{R}(\mu)$ is not strictly convex. Conversely if $\mathcal{R}(\mu)$ is not strictly convex by Proposition 2.3 there exists p in S^{n-1} satisfying $|\mu|(D^0(p)) > 0$: let $A \subseteq D^0(p)$ be such that $\mu(A) \neq O$. Then

$$\forall B \in \Lambda, \quad B \subseteq A \Longrightarrow p \cdot \mu(B) = \int_B p \cdot f(x)\, d|\mu| = 0$$

and thus $<\mu(B) : B \in \Lambda, B \subseteq A> \subseteq <p>^{\perp} \neq \mathbb{R}^n$. $\square$

The next result is traditionally obtained from a celebrated Theorem of Olech [12]; we prove it here in an elementary way.

Proposition 2.5. *Let E, F in Λ be such that $\mu(E) = \mu(F)$ is an extreme point of $\mathcal{R}(\mu)$. Then $|\mu|(E \Delta F) = 0$.*

Proof. Assume that $|\mu|(E \setminus F) > 0$ and let $A \subseteq E \setminus F$ be such that $\mu(A) \neq O$. Set $E_1 = E \setminus A$ and $E_2 = F \cup A$. Clearly we have $\mu(E_1) = \mu(E) - \mu(A) \neq \mu(E)$, $\mu(E_2) = \mu(F) + \mu(A) = \mu(E) + \mu(A) \neq \mu(E)$ and $\mu(E) = \frac{1}{2}\mu(E_1) + \frac{1}{2}\mu(E_2)$, contradicting the extremality of $\mu(E)$. $\square$

Corollary 2.6. *Assume that the origin O is an extreme point of $\mathcal{R}(\mu)$ and let A in Λ be such that $\mu(A) = O$. Then $|\mu|(A) = 0$.*

Proof. Since $\mu(A) = O = \mu(\emptyset)$ is an extreme point of $\mathcal{R}(\mu)$ then Proposition 2.5 implies that $|\mu|(A) = |\mu|(A \Delta \emptyset) = 0$. $\square$

As a consequence we obtain the following characterization of the exposed points of $\mathcal{R}(\mu)$.

Proposition 2.7. *For E in Λ the point $\mu(E)$ is exposed in $\mathcal{R}(\mu)$ if and only if there exists p in S^{n-1} such that $E = D^+(p)$ and $|\mu|(D^0(p)) = 0$.*

Proof. Assume that $|\mu|(D^0(p)) = 0$; then by Proposition 2.1 the exposed face $F(\mathcal{R}(\mu), p)$ coincides with $\{\mu(D^+(p))\}$ so that the latter is an exposed point of $\mathcal{R}(\mu)$.
Conversely, let E in Λ be such that $F(\mathcal{R}(\mu), p) = \{\mu(E)\}$ for some p in S^{n-1}. By Corollary 2.2 necessarily we have $|\mu|(D^0(p)) = 0$ and therefore $F(\mathcal{R}(\mu), p) = \{\mu(D^+(p))\}$ so that $\mu(E) = \mu(D^+(p))$. Since $\mu(E)$ is an exposed (and thus extreme) point of $\mathcal{R}(\mu)$ then Proposition 2.5 yields $E = D^+(p)$. $\square$

Corollary 2.8. *The origin is an exposed point of $\mathcal{R}(\mu)$ if and only if there exists p in S^{n-1} such that $p \cdot f(x) < 0$ a.e. on X.*

Proof. Proposition 2.7 implies that $O = \mu(\emptyset)$ is an exposed point of $\mathcal{R}(\mu)$ if and only if there exists p in S^{n-1} such that $D^+(p) = \emptyset$ and $|\mu|(D^0(p)) = 0$. $\square$

3. The measures whose range is strictly convex

The main result of this section stems from Corollary 2.3: it states that the range of μ is strictly convex if and only if the vectors $f(x_1), \dots, f(x_n)$ are linearly independent for a.e. $(x_1, \dots, x_n)$ in X^n. We introduce the subset Δ of X^n defined by

$$\Delta = \{(x_1, \dots, x_n) \in X^n : \det[f(x_1), \dots, f(x_n)] = 0\}.$$

Theorem 3.1. *$\mathcal{R}(\mu)$ is strictly convex if and only if Δ is $|\mu|^{\otimes n}$–negligible.*

Proof. If $\mathcal{R}(\mu)$ is not strictly convex by Proposition 2.3 there exists p in S^{n-1} such that $|\mu|(D^0(p)) > 0$; since $(D^0(p))^n \subseteq \Delta$ then we obtain $|\mu|^{\otimes n}(\Delta) \geq \left[|\mu|(D^0(p))\right]^n > 0$.

We give two proofs of the opposite implication. For each subset S of X^n and $(x_2, \dots, x_n)$ in X^{n-1} let $S(x_2, \dots, x_n) = \{x_1 \in X : (x_1, \dots, x_n) \in S\}$ be the $(x_2, \dots, x_n)$–section of S.

First proof. We first show that the set

$$B = \{(x_1, \dots, x_n) \in X^n : f(x_1) \in < f(x_2), \dots, f(x_n) > \}$$

is measurable. For $u_1, \ldots, u_m$ in $\mathbb{R}^n$ $(m \le n)$ we denote by $|u_1 \wedge \cdots \wedge u_m|$ their Gramian i.e. the sum of the squares of the minors of order m of the matrix $(e_i \cdot u_j)_{i,j}$ (where $(e_i)_i$ is the standard basis in $\mathbb{R}^n$); clearly $u_1, \ldots, u_m$ are linearly dependent if and only if their Gramian vanishes. For every non empty subset $I = \{i_1, \ldots, i_k\}$ of $\{2, \ldots, n\}$ let B_I be the measurable subsets of B defined by

$$B_I = \{(x_1, \ldots, x_n) \in X^n : |f(x_1) \wedge f(x_{i_1}) \wedge \cdots \wedge f(x_{i_k})| = 0, |f(x_{i_1}) \wedge \cdots \wedge f(x_{i_k})| \neq 0\}$$

and set $Z = \{(x_1, \ldots, x_n) \in X^n : f(x_1) = \cdots = f(x_n) = 0\}$. Let $x = (x_1, \ldots, x_n) \in B$: then either $x \in Z$ or there exists a subset I of $\{2, \ldots, n\}$ such that $\{f(x_i) : i \in I\}$ is a maximal subset of linearly independent vectors among $\{f(x_i) : i \in \{2, \ldots, n\}\}$ and $f(x_1) \in < f(x_i) : i \in I >$ or, equivalently, $x \in B_I$. Thus $B = Z \bigcup \big(\bigcup_{I \subseteq \{2,\ldots,n\}} B_I\big)$, proving the claim.

Fubini's theorem gives

$$|\mu|^{\otimes n}(\Delta) = \int_{X^{n-1}} \left\{ \int_{\Delta(x_2,\ldots,x_n)} d|\mu|(x_1) \right\} d(|\mu|(x_2) \otimes \cdots \otimes |\mu|(x_n)).$$

Assume that $\mathcal{R}(\mu)$ is strictly convex; then Proposition 2.3 yields

$$\forall (x_2, \ldots, x_n) \in X^{n-1} \qquad |\mu|(\{x_1 \in X : f(x_1) \in < f(x_2), \ldots, f(x_n) >\}) = 0$$

so that if Δ_1 is the (measurable) subset of Δ defined by

$$\Delta_1 = \{(x_1, \ldots, x_n) \in \Delta : f(x_1) \notin < f(x_2), \ldots, f(x_n) >\}$$

from the above formula we obtain

$$|\mu|^{\otimes n}(\Delta) = \int_{X^{n-1}} \left\{ \int_{\Delta_1(x_2,\ldots,x_n)} d|\mu|(x_1) \right\} d(|\mu|(x_2) \otimes \cdots \otimes |\mu|(x_n))$$

and thus Tonelli's Theorem yields $\Delta = \Delta_1 \quad |\mu|^{\otimes n}-$ a.e.. Similarly if for i in $\{2, \ldots, n\}$ we put

$$\Delta_i = \{(x_1, \ldots, x_n) \in \Delta : f(x_i) \notin < f(x_1), \ldots, f(x_{i-1}), f(x_{i+1}) \ldots, f(x_n) >\}$$

the same arguments give $\Delta = \Delta_i \quad |\mu|^{\otimes n}$– a.e.. As a consequence

$$\Delta = \bigcap_{i=1}^{n} \Delta_i \quad |\mu|^{\otimes n}\text{– a.e..}$$

Obviously the set $\bigcap_{i=1}^{n} \Delta_i$ is empty; the conclusion follows.

Second proof. Let $g : X^{n-1} \times S^{n-1} \longrightarrow \mathbb{R}$ be the map defined by

$$\forall (y_1, \dots, y_{n-1}) \in X^{n-1} \quad \forall z \in S^{n-1} \qquad g((y_1, \dots, y_{n-1}), z) = \sum_{i=2}^{n} \Big(z \cdot f(y_i) \Big)^2 .$$

The function g is measurable in $(y_1, \dots, y_{n-1})$ and continuous in z: Corollary 6.3 in [10] then implies that the set–valued map $G : X^{n-1} \to \mathcal{P}(S^{n-1})$ defined by

$$G(y_1, \dots, y_{n-1}) = \{z \in S^{n-1} : g((y_1, \dots, y_{n-1}), z) = 0\} = < f(y_1), \dots, f(y_{n-1}) >^{\perp} \cap S^{n-1}$$

has a measurable graph: Theorem 5.7 in [10] then yields the existence of a measurable selection $p : X^{n-1} \to S^{n-1}$ of G, i.e. p is measurable and $p(y_1, \dots, y_{n-1}) \in G(y_1, \dots, y_{n-1})$ a.e. in X^{n-1}. For i in $\{1, \dots, n\}$ let A_i be the measurable subset of X^n defined by

$$A_i = \{(x_1, \dots, x_n) \in X^n : f(x_i) \cdot p(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = 0\}.$$

We claim that $\Delta = \cup_i A_i$ (modulo $|\mu|^{\otimes n}$).
In fact let $x = (x_1, \dots, x_n) \in \Delta$: then $\det[f(x_1) \dots, f(x_n)] = 0$ so that there exists i such that $f(x_i) \in < f(x_1), \dots, f(x_{i-1}), f(x_{i+1}), \dots, f(x_n) >$; modulo a negligible set the latter vector space is contained in $< p(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) >^{\perp}$ and thus x belongs to A_i. Conversely let (for instance) $(x_1, \dots, x_n) \in A_1$. Either $f(x_2), \dots, f(x_n)$ are linearly independent so that $f(x_1) \in < p(x_2, \dots, x_n) >^{\perp} = < f(x_2), \dots, f(x_n) >$ or $f(x_2), \dots, f(x_n)$ are linearly dependent: in both cases we obtain $\det[f(x_1) \dots, f(x_n)] = 0$, proving the claim.

Assume that $|\mu|^{\otimes n}(\Delta) > 0$: then there exists i such that $|\mu|^{\otimes n}(A_i) > 0$; again it is not restrictive to suppose that $i = 1$. Fubini's Theorem gives

$$|\mu|^{\otimes n}(A_1) = \int_{X^{n-1}} \left\{ \int_{A_1(x_2,\ldots,x_n)} d|\mu|(x_1) \right\} d(|\mu|(x_2) \otimes \cdots \otimes |\mu|(x_n))$$

so that there exists $(x_2, \ldots, x_n)$ in X^{n-1} such that $|\mu|(A_1(x_2, \ldots, x_n)) > 0$. Now we have $A_1(x_2, \ldots, x_n) = D^0(p(x_2, \ldots, x_n))$: Proposition 2.3 implies that $\mathcal{R}(\mu)$ is not strictly convex. $\square$

The *determinant measure* $\det \mu$ associated to $\mu = (\mu_1, \ldots, \mu_n)$ was introduced in [7]. It seems natural to use it here.
We shall denote by $\mathcal{S}_n$ the symmetric group of order n and, for σ in $\mathcal{S}_n$, by $\epsilon(\sigma)$ its sign.

Definition 3.2. The determinant measure of μ, denoted by $\det \mu$, is the signed measure defined on $(X^n, \Lambda^{\otimes n})$ by

$$\det \mu \ = \ \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma)\, \mu_{\sigma(1)} \otimes \cdots \otimes \mu_{\sigma(n)}.$$

This is the only measure whose restriction to the product sets $A_1 \times \cdots \times A_n$ satisfy $\det \mu(A_1 \times \cdots \times A_n) = \det[\mu(A_1), \cdots, \mu(A_n)]$.

The next result appears in the proof of [7, Th. 3.4] but is not explicitely stated.

Proposition 3.3. *The function* $\det f$ *defined on* X^n *by*

$$\det f(x_1, \ldots, x_n) = \det[f(x_1), \ldots, f(x_n)]$$

is the density function of $\det \mu$ *with respect to* $|\mu|^{\otimes n}$*.*

Proof. Set $f = (f_1, \ldots, f_n)$. For any measurable subset A of X^n the application of Fubini–

Tonelli's Theorem yields

$$\begin{aligned}
\det\mu(A) &= \sum_{\sigma\in\mathcal{S}_n} \epsilon(\sigma)\big(\mu_{\sigma(1)}\otimes\cdots\otimes\mu_{\sigma(n)}\big)(A) \\
&= \int_A \sum_{\sigma\in\mathcal{S}_n} \epsilon(\sigma) f_{\sigma(1)}(x_1)\cdots f_{\sigma(n)}(x_n)\, d(|\mu|(x_1)\otimes\cdots\otimes|\mu|(x_n)) \\
&= \int_A \det[f(x_1),\dots,f(x_n)]\, d|\mu|^{\otimes n}(x_1,\dots,x_n). \quad \square
\end{aligned}$$

The measure $\det\mu$ allows to reformulate Theorem 3.1 in terms of the behaviour of $|\mu|^{\otimes n}$ with respect to $\det\mu$. We recall that a vector measure τ is said to be absolutely continuous with respect to some other signed measure ξ (both defined in (X,Λ)), in symbols $\tau \ll \xi$, whenever for A in Λ the condition $|\xi|(A) = 0$ implies $\tau(A) = O$. Two positive measures τ,ξ on X are said to be equivalent if each of them is absolutely continuous with respect to the other. We will use the fact that if τ,ξ are finite positive measures and $\tau \ll \xi$ then τ is equivalent to ξ if and only if the density of τ with respect to ξ is strictly positive a.e. on X.

Theorem 3.4. *The range of μ is strictly convex if and only if $|\mu|^{\otimes n}$ is equivalent to $|\det\mu|$ (the total variation of $\det\mu$).*

Proof. Proposition 3.3 together with [1, Ex. 26.10] imply that $|\det\mu|$ is absolutely continuous with density $|\det f|$ with respect to $|\mu|^{\otimes n}$. Thus the condition that $|\det f|$ does not vanish in X^n is equivalent to the absolute continuity of $|\mu|^{\otimes n}$ with respect to $|\det\mu|$. Theorem 3.1 yields the conclusion. $\square$

4. Some applications to Chebyshev measures

As usual X is a set and Λ is a σ–algebra of subsets of X. We consider the following assumption.

Assumption (A)**.**

(A_1): $\mathcal{M} = (M_i)_{i\in[0,1]}$ is an increasing family of measurable sets (i.e. $M_i \subseteq M_j$ if $i < j$)

such that $M_0 = \emptyset, M_1 = X$ and ν is a positive non–trivial bounded measure on (X, Λ) such that the function $i \mapsto \nu(M_i)$ is continuous and strictly increasing.
(A_2): μ is a vector measure on (X, Λ) with values in $\mathbb{R}^n$ and the function $i \mapsto |\mu|(M_i)$ is continuous.

Remark. The existence of such a family implies clearly that both ν and μ are non–atomic; conversely if these measures are non–atomic then Lyapunov's Theorem applied to the vector measure $(\nu, |\mu|)$ yields the existence of a family $(M_i)_{i\in[0,1]}$ such that $\nu(M_i) = i\nu(X)$ and $|\mu|(M_i) = i|\mu|(X)$ for every i (see [8]) and thus satisfying the assumption (A).

The family $\mathcal{M}$ induces an order relation $\underset{\mathcal{M}}{\prec}$ (or simply $\prec$ when no ambiguity may occur) defined by $x \underset{\mathcal{M}}{\prec} y$ if there exists i in $[0, 1]$ such that $x \in M_i$ and $y \notin M_i$. By $P_{\mathcal{M}}$ (or P) we will denote the subset of X^n defined by $P_{\mathcal{M}} = \{(x_1, \ldots, x_n) \in X^n : x_1 \underset{\mathcal{M}}{\prec} \cdots \underset{\mathcal{M}}{\prec} x_n\}$. We will assume for simplicity that $P_{\mathcal{M}}$ is measurable (in the general case one should replace $P_{\mathcal{M}}$ with any of its measurable coverings).

Example. If $\mathcal{M} = ([0, i])_{i\in[0,1]}$ then $P_{\mathcal{M}} = \{(x_1, \ldots, x_n) \in [0, 1]^n : 0 \le x_1 < \cdots < x_n \le 1\}$.

Chebyshev measures with respect to ν and $\mathcal{M}$ have been defined in [7]: they are vector measures whose associated determinant measure is strictly positive on the non ν–negligible subsets of $P_{\mathcal{M}}$.

Definition 4.1. The vector measure μ is a Chebyshev measure with respect to ν and the family $\mathcal{M}$ (or simply a T_ν–measure or T–measure when $\nu = |\mu|$) if $\mu, \nu, \mathcal{M}$ satisfy assumption (A) and the measure $\det \mu$ verifies

$$\forall A \in \Lambda^{\otimes n} \cap P_{\mathcal{M}}, \qquad \nu^{\otimes n}(A) > 0 \quad \Longrightarrow \quad \det \mu(A) > 0.$$

When no ambiguity may occur we shall often omit to mention the dependence with respect to $\mathcal{M}$.

Remark. The measure $\det \mu$ is absolutely continuous with respect to $|\mu|^{\otimes n}$; it follows then directly from the definition that μ is a $T_{|\mu|}$–measure if and only if $\det \mu$ is positive and equivalent to $|\mu|^{\otimes n}$ on $P_{\mathcal{M}}$.

Remark. When $n = 1$ then $P_{\mathcal{M}} = X$ for every family of subsets $\mathcal{M}$; moreover if μ is a signed measure then $\det \mu = \mu$. Therefore μ is a T_ν–measure whenever it assumes strictly positive values on the non ν–negligible subsets of X. In particular a positive measure μ is a T_ν–measure if and only if $\nu \ll \mu$. It may happen however that μ is not absolutely continuous with respect to ν. Let, for instance, μ be the Lebesgue measure on $[0, 1]$, E a measurable set such that $0 < \mu(E \cap I) < \mu(I)$ for every non trivial interval I, ν be the measure defined by $\nu(A) = \mu(A \cap E)$ for every measurable set A and set $\mathcal{M} = ([0, i])_{i\in[0,1]}$. Clearly $\nu, \mu, \mathcal{M}$ verify assumption (A); moreover ν is absolutely continuous with respect to μ so that μ is a T_ν–measure; however μ is not absolutely continuous with respect to ν ($\nu([0, 1] \setminus E) = 0$ whereas $\mu([0, 1] \setminus E) = 1 - \mu(E) > 0$).

We will use the following result [7, Funny corollary 4.5].

Proposition 4.2. *Let μ be a T_ν–measure and A in Λ be such that $\mu(A) = O$ (the origin in $\mathbb{R}^n$); then $\nu(A) = 0$. In particular ν is absolutely continuous with respect to $|\mu|$.*

Sketch of the proof. Assume that $\nu(A) > 0$. If μ is a T_ν–measure with respect to a family $\mathcal{M} = (M_i)_i$ of subsets of X then the continuity of the map $i \mapsto \nu(M_i)$ allows to decompose A as a disjoint union of some non ν–negligible sets $A_1, \dots, A_n$ such that their product $A_1 \times \cdots \times A_n$ is contained in $P_{\mathcal{M}}$. It follows that $\det[\mu(A_1), \dots, \mu(A_n)] = \det \mu(A_1 \times \cdots \times A_n) > 0$ so that the vectors $\mu(A_1), \dots, \mu(A_n)$ are linearly independent. However $O = \mu(A) = \mu(A_1) + \cdots + \mu(A_n)$, a contradiction. $\square$

It follows that if μ is a T_ν–measure then the map $i \mapsto |\mu|(M_i)$ is strictly increasing. The term "Chebyshev" arises from the well–known concept of T–system (where "T" stands for Tchebycheff), applied in approximation theory and to moment problems in statistics, involving continuous functions defined on intervals of $\mathbb{R}$ (see [11]). We recall here a slightly more general definition.

Definition 4.3. [7] Let $\nu, \mathcal{M}$ satisfy (A_1). A function g in $L^1_\nu(X, \mathbb{R}^n)$ is said to be a Chebyshev system with respect to ν and $\mathcal{M}$ (or simply a T_ν–system) if the determinant $\det[g(x_1), \ldots, g(x_n)]$ is positive $\nu^{\otimes n}$– a.e. in $P_{\mathcal{M}}$.

Let $g \in L^1_\nu(X, \mathbb{R}^n)$ and μ be the measure with density function g with respect to ν. The arguments involved in the proof of Proposition 3.3 show that $\det[g(x_1), \ldots, g(x_n)]$ is the density function of $\det \mu$ with respect to $\nu^{\otimes n}$. As a consequence Chebyshev systems generate absolutely continuous Chebyshev measures.

Theorem 4.4. [7, Th. 3.4] *Let $\mu, \nu, \mathcal{M}$ satisfy (A) and μ be absolutely continuous with density g with respect to ν. Then μ is a T_ν–measure if and only if g is a T_ν–system.*

Chebyshev systems arise naturally from linear differential equations; some of their applications to control theory and the calculus of variations where given in [5].

Example. Let $h \in \mathcal{C}^\infty(\mathbb{R})$ satisfy $h^{(i)}(0) = 0$ for $0 \leq i \leq n-2$ and $h^{(n-1)}(0) = 1$. There exists $\delta > 0$ such that the function $f = (h^{(n-1)}, h^{(n-2)}, \ldots, h', h)$ is a Chebyshev system on $[-\delta, \delta]$ with respect to the Lebesgue measure and the family of intervals $M_i = [-\delta, -\delta + 2i\delta]$ $(i \in [0, 1])$.

We give now a remarkable example: a function with values in a half plane of $\mathbb{R}^2$ and whose inverse images of lines are negligible sets generates a Chebyshev measure.
For θ in $\mathbb{R}$ and u in $\mathbb{R}^2 \setminus \{O\}$ we denote by $\arg_\theta u$ the argument of u in $(\theta - \pi, \theta + \pi]$.

Proposition 4.5. *Let (X, Λ, ν) be a measure space (ν being non trivial); g in $L^1_\nu(X, \mathbb{R}^2)$ be such that the set $\{x \in X : p \cdot g(x) = 0\}$ is ν– negligible for every p in S^1 and $q \cdot g(x) > 0$ ν– a.e. for some q in S^1. Then the two dimensional measure μ defined by $\mu(A) = \int_A g \, d\nu$ for every A in Λ is a Chebyshev measure (i.e. there exists a family $\mathcal{M} = (M_i)_{i \in [0,1]}$ of subsets of X with respect to which μ is a T_ν–measure).*

Proof. Let θ in $\mathbb{R}$ be such that $q = e^{i\theta}$: then if we set $a = \theta - \frac{\pi}{2}, b = \theta + \frac{\pi}{2}$ the argument $\arg_\theta g(x)$ of $g(x)$ in $(\theta - \pi, \theta + \pi]$ belongs to $[a, b]$ for x in $X \setminus Z$, for some negligible set Z. For t in $[a, b]$ let

$$N_t = \{x \in X : \arg_\theta g(x) \leq t\}$$

and $\phi : [a, b] \to \mathbb{R}^+$ be the increasing map defined by $\phi(t) = \nu(N_t)$ for every t in $[a, b]$. Our assumption implies that for every t in $\mathbb{R}$ the sets $\{x \in X : \arg_\theta g(x) = t\}$ are negligible. Clearly $\phi(a) = 0$ and $\phi(b) = \nu(X)$. Moreover the family of sets $(N_t)_{t \in [a,b]}$ being increasing it follows that ϕ is continuous and therefore $\phi([a, b]) = [0, \nu(X)]$. Let $\psi : [0, \nu(X)] \longrightarrow [a, b]$ be a right inverse of ϕ and set, for every i in $[0, 1]$, $M_i = N_{\psi(i\nu(X))}$. The definition of ψ then implies that $\nu(M_i) = \phi(\psi(i\nu(X))) = i\nu(X)$ so that the map $i \mapsto \nu(M_i)$ is continuous and strictly increasing. The absolute continuity of μ with respect to ν yields the continuity of $i \mapsto |\mu|(M_i)$ and thus $\nu, \mu, \mathcal{M}$ fulfil assumption (A). For x_1, x_2 in X the relation $x_1 \underset{\mathcal{M}}{\prec} x_2$ here implies that there exists t in $[a, b]$ such that $\arg_\theta g(x_1) \leq t$ and $\arg_\theta g(x_2) > t$. Since $b - a = \pi$ it follows that if x_1, x_2 are not in Z then $\det[g(x_1), g(x_2)] > 0$. Hence this latter condition is fulfilled for $\nu^{\otimes 2}$– almost every couple (x_1, x_2) belonging to the set $P_\mathcal{M}$ associated to $\mathcal{M}$ and therefore g is a T_ν–system. Theorem 4.4 yields the conclusion. □

The main result of [7] states that given a positive measure ν and a prescribed increasing family of sets $\mathcal{M} = (M_i)_i$ the range $\{\mu(E), E \in \Lambda\}$ of a n–dimensional T_ν–measure μ with respect to $\mathcal{M}$ can be described through the values that it assumes on the finite unions of (at most n) sets of the form $M_j \setminus M_i$. Let Γ be the subset of $\mathbb{R}^n$ defined by

$$\Gamma = \{(\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n : 0 \leq \gamma_1 \leq \dots \leq \gamma_n \leq 1\}.$$

Representation theorem 4.6. [7] *Suppose μ is a T_ν–measure with respect to the family $\mathcal{M} = (M_i)_{i \in [0,1]}$ and let ρ be a measurable function such that $0 \leq \rho \leq 1$ a.e.. There exists $\alpha = (\alpha_1, \dots, \alpha_n)$ in Γ satisfying*

$$\mu(E_\alpha) = \int_X \rho \, d\mu \quad \textit{where} \quad E_\alpha = \bigcup_{\substack{1 \leq i \leq n \\ i \text{ odd}}} (M_{\alpha_{i+1}} \setminus M_{\alpha_i}) \quad (\alpha_{n+1} = 1).$$

If $0 < \rho < 1$ on a ν–non negligible set then α is unique, it belongs to the interior of Γ and $\mu(E_\alpha)$ lies in the interior of $\mathcal{R}(\mu)$.

Remark. We recall that Lyapunov's Theorem [15] states that there exists a set E in Λ such that $\int_X \rho\, d\mu = \mu(E)$; the improvement here is that the set E can be chosen among a family of "nice" sets. We refer to [6] for some comments about this fact and to [5] for an application of this result to the bang–bang principle in control theory.

Chebyshev measures are considered here in connection with §2, §3 because they represent a broad class of measures whose range is strictly convex.

Theorem 4.7. [7, Th. 5.3] *The range $\mathcal{R}(\mu)$ of a T_ν–measure μ is strictly convex. The boundary points of $\mathcal{R}(\mu)$ admit a unique representation modulo $|\mu|$; moreover a point $\mu(E)$ belongs to the boundary of $\mathcal{R}(\mu)$ if and only if there exists γ in the boundary of Γ such that $|\mu|(E\Delta E_\gamma) = 0$. In particular $O = \mu(\emptyset) = \mu(E_{(1,\dots,1)})$ belongs to the boundary of $\mathcal{R}(\mu)$.*

Proof of strict convexity. Let E, F in Γ be such that $\mu(E) \neq \mu(F)$: then $|\mu|(E\Delta F) \neq 0$; let for instance $|\mu|(E \setminus F) > 0$. Then for λ in $(0, 1)$ the function $\rho = \lambda\chi_E + (1-\lambda)\chi_F$ is such that $0 < \rho < 1$ a.e. on $E \setminus F$. Theorem 4.6 implies that $\int \rho\, d\mu = \lambda\mu(E) + (1-\lambda)\mu(F)$ belongs to the interior of $\mathcal{R}(\mu)$. $\square$

In what follows we shall denote by μ a n–dimensional vector measure and by f the density of μ with respect to its total variation $|\mu|$.

Assume that μ is a T_ν–measure with respect to a family $\mathcal{M}$ of subsets of X. If μ is absolutely continuous with respect to ν then, trivially, μ is a $T_{|\mu|}$–measure; it follows from Theorem 4.4 that $\det[f(x_1), \dots, f(x_n)] > 0 \quad |\mu|^{\otimes n}$–a.e. on $P_{\mathcal{M}}$. Otherwise, if μ is not absolutely continuous with respect to ν, it does not seem clear at all from the definition 4.1 whether the above conclusion still holds. Certainly, by Proposition 4.2, ν is absolutely continuous with respect to $|\mu|$; however one might think that there exists a $\nu^{\otimes n}$– negligible (but not $|\mu|^{\otimes n}$–negligible) subset A in $P_{\mathcal{M}}$ such that $\det \mu(A) \leq 0$. The next result shows that this pathology does not occur; its proof is based on the fact that a T_ν–measure has a strictly convex range and on our characterization of the measures having this property.

Theorem 4.8. *Let μ be a T_ν–measure (with respect to a family $\mathcal{M}$). Then μ is a $T_{|\mu|}$–measure (with respect to $\mathcal{M}$).*

Proof. By Theorem 4.7 the range of μ is strictly convex; Theorem 3.1 then implies that

$$\det[f(x_1), \dots, f(x_n)] \neq 0 \qquad |\mu|^{\otimes n} - \text{ a.e.}$$

Let

$$P^-_{\mathcal{M}} = \{(x_1, \dots, x_n) \in P_{\mathcal{M}} : \det[f(x_1), \dots, f(x_n)] < 0\}$$

and assume that $|\mu|^{\otimes n}(P^-_{\mathcal{M}}) > 0$. Since, by definition

$$\det \mu(P^-_{\mathcal{M}}) = \int_{P^-_{\mathcal{M}}} \det[f(x_1), \dots, f(x_n)] d|\mu|^{\otimes n},$$

then by the continuity of $|\mu|$ with respect to the sets M_i, there exists a $(2n)$–uple

$$(\alpha_1^1, \alpha_1^2, \cdots, \alpha_n^1, \alpha_n^2) \in \mathbb{R}^{2n}$$

such that

$$0 < \alpha_1^1 < \alpha_1^2 < \cdots < \alpha_n^1 < \alpha_n^2 < 1$$

and

$$\det \mu\big(P^-_{\mathcal{M}} \cup \big(M_{\alpha_1^2} \setminus M_{\alpha_1^1}\big) \times \cdots \times \big(M_{\alpha_n^2} \setminus M_{\alpha_n^1}\big)\big) < 0.$$

However, ν being a positive measure, we have

$$\nu^{\otimes n}\big(P^-_{\mathcal{M}} \cup \big(M_{\alpha_1^2} \setminus M_{\alpha_1^1}\big) \times \cdots \times \big(M_{\alpha_n^2} \setminus M_{\alpha_n^1}\big)\big) \geq \nu^{\otimes n}\big(\big(M_{\alpha_1^2} \setminus M_{\alpha_1^1}\big) \times \cdots \times \big(M_{\alpha_n^2} \setminus M_{\alpha_n^1}\big)\big) > 0,$$

a contradiction. Therefore f is a T–system; Theorem 4.4 yields the conclusion. $\square$

Remark. The above result shows that, in Definition 4.1, the auxiliary positive measure ν can be omitted: in dealing only with T–measures, as we do in the rest of the paper, there is undoubtly a gain of clarity. However in the applications ([5]) it happens that ν and $\mathcal{M}$ are given a priori and that μ is defined through a density function g in $L^1_\nu(X, \mathbb{R}^n)$: the easiest way to see if μ is a Chebyshev measure is then to check whether g is a T_ν–system.

Theorem 4.7 states that the range of a Chebyshev measure is strictly convex and that the origin O belongs to the boundary of its range. It is well known that each compact, convex, centrally symmetric subset of $\mathbb{R}^2$ is the range of a two dimensional measure [2, 3] i.e. a zonoid. Conversely, in [2, Theorem 2] the authors show that, in $\mathbb{R}^2$, each strictly convex zonoid (with O in its boundary) is the range of a Chebyshev measure. The results obtained in §2, §3 allow us to give a much stronger result: the bidimensional Chebyshev measures are exactly those measures whose range satisfies the above geometrical properties. Let $\mu : (X, \Lambda) \to \mathbb{R}^2$ be a non–atomic bidimensional measure and, as usual, let f be the density of μ with respect to $|\mu|$.

Theorem 4.9. *Assume that μ is a bidimensional measure whose range $\mathcal{R}(\mu)$ is strictly convex and contains the origin in its boundary. Then there exists a family of sets with respect to which μ is a Chebyshev measure. Moreover there exists θ in $\mathbb{R}$ such that for every measurable function ρ with values in $[0, 1]$ there exist α, β in $\mathbb{R}$ satisfying*

$$\int_X \rho(x)\, d\mu = \mu\big(\{x \in X : \alpha \le \arg_\theta f(x) \le \beta\}\big).$$

Proof. By Corollary 2.8 there exists q in S^1 such that $q \cdot f(x) > 0$ a.e. on X and, by Proposition 2.3, the sets $\{x \in X : p \cdot f(x) = 0\}$ are negligible for every p in S^1. The application of Proposition 4.5 with $(|\mu|, f)$ instead of (ν, g) yields the existence of a family $\mathcal{M} = (M_i)_i$ of subsets of X with respect to which μ is a Chebyshev measure. Furthermore the proof of Proposition 4.5 shows that there exists θ in $\mathbb{R}$ such that for every i we have $M_i = \{x \in X : \arg_\theta f(x) \le \xi_i\}$ for some ξ_i in $(\theta - \pi, \theta + \pi]$: the conclusion follows from the representation theorem 4.6. $\square$

Let μ be a vector measure on (X, Λ), E be in Λ and let μ_E be the vector measure defined by $\mu_E(B) = \mu(E \setminus B) - \mu(E \cap B)$ for every B in Λ. It is easy to verify (see [3, Lemma 1.3]) that the range $\mathcal{R}(\mu_E)$ of μ_E is a translate of the range of μ; more precisely we have that $\mathcal{R}(\mu_E) = \mathcal{R}(\mu) - \mu(E) = \{x - \mu(E) : x \in \mathcal{R}(\mu)\}$. The next characterization of the bidimensional strictly convex zonoids in $\mathbb{R}^2$ follows then directly.

Corollary 4.10. *Let μ be a measure on (X, Λ) with values on $\mathbb{R}^2$. The range of μ is strictly convex if and only if there exists a subset E in Λ such μ_E is a Chebyshev measure.*

Proof. If there exists E in Λ such that μ_E is a Chebyshev measure then by Theorem 4.7 the range of μ_E is strictly convex. Thus each of its translates, in particular the range of μ, is strictly convex too. Conversely assume that the range of μ is strictly convex. Let E be such that $\mu(E)$ belongs to the boundary of $\mathcal{R}(\mu)$. Then the origin lies in the boundary of the translate $\mathcal{R}(\mu) - \mu(E)$. The latter set is the range of μ_E and is strictly convex: it follows from Theorem 4.9 that μ_E is a T–measure. $\square$

Our next result shows that, when $n > 2$, the boundary of the range of a n–dimensional Chebyshev measures is not regular. In particular Theorem 4.9 cannot be extended to greater dimensions: a measure whose range is a ball (it exists, see for instance [13]) is certainly not the range of some Chebyshev measure.

Proposition 4.11. *Let $\mu = (\mu_1, \ldots, \mu_n)$ be a T–measure and $n \geq 3$. Then the boundary of $\mathcal{R}(\mu)$ is not a $(n-1)$–dimensional C^1–manifold.*

We need the following Lemma, whose proof is postponed at the end of the paper.

Lemma 4.12. *Let μ be a T–measure with respect to an increasing family $(M_\alpha)_{\alpha \in [0,1]}$ of subsets of X. Then there exists a vector measure $\tilde{\mu}$ on the Lebesgue σ–algebra of $[0,1]$ such that $\tilde{\mu}$ is a T–measure with respect to the intervals $([0,\alpha])_{\alpha \in [0,1]}$ and $\mathcal{R}(\mu) = \mathcal{R}(\tilde{\mu})$.*

Proof of Proposition 4.11. By Lemma 4.12 it is not restrictive to suppose that μ is a Chebyshev measure on $[0,1]$ with respect to the intervals $([0,x])_{x \in [0,1]}$. Fix x in $[0,1)$ and let $\lambda_x : [0, 1-x] \to \mathbb{R}^n$ be the curve defined by

$$\forall t \in [0, 1-x] : \qquad \lambda_x(t) = \mu([x, x+t]).$$

Since $n \geq 3$ and $\chi_{[x,x+t]}$ has at most two discontinuity points Theorem 4.7 implies that the set $\Gamma_x = \lambda_x([0,1-x])$ is entirely contained in the boundary $\partial\mathcal{R}(\mu)$ of $\mathcal{R}(\mu)$. Remark further that the origin $O = \lambda_x(0)$ belongs to Γ_x. Assume that $\partial\mathcal{R}(\mu)$ is a manifold of class $\mathcal{C}^1$ and let p be a unit normal vector to the tangent plane Π of $\partial\mathcal{R}(\mu)$ at the origin. For every point q of $\mathcal{R}(\mu)$, the distance from q to the plane Π is the absolute value of $q \cdot p$; therefore $\lim\limits_{\substack{q\to 0\\ q\in\partial\mathcal{R}(\mu)}} \dfrac{q\cdot p}{\|q\|} = 0$. Since, by Proposition 4.2, $\mu(I) \neq O$ for every non trivial interval I it follows that $\lim\limits_{t\to 0} \dfrac{\mu([x,x+t])\cdot p}{\|\mu([x,x+t])\|} = 0$ and therefore, recalling that $\|\mu(A)\| \leq |\mu|(A)$ for every measurable set A, $\lim\limits_{t\to 0} \dfrac{\mu([x,x+t])}{|\mu|([x,x+t])} \cdot p = 0$. Since, by [17], $\lim\limits_{t\to 0} \dfrac{\mu([x,x+t])}{|\mu|([x,x+t])} = f(x)$ a.e. in $[0,1]$ we obtain that $D^0(p) = \{x \in [0,1] : f(x)\cdot p = 0\} = [0,1]$ ($|\mu|-$ a.e.). However the set $\mathcal{R}(\mu)$ is strictly convex: Proposition 2.3 then implies that $|\mu|(D^0(p)) = 0$, a contradiction. $\square$

Proof of Lemma 4.12. Let $\mu = (\mu_1, \ldots, \mu_n)$ and, for each i in $\{1,\ldots,n\}$, let $\mu_i = \mu_i^+ - \mu_i^-$ be the Jordan decomposition of μ_i. Let $g_i : [0,1] \to \mathbb{R}$ be the bounded variation function defined by

$$g_i(\alpha) = \mu_i^+(M_\alpha) - \mu_i^-(M_\alpha)$$

and let $\tilde{\mu}_i$ be the Lebesgue-Stieltjes measure generated by g_i. Clearly $\tilde{\mu}$ is regular and if we set $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_n)$ the continuity of the functions g_i yields

$$\forall \alpha, \beta \in [0,1],\, \alpha \leq \beta \qquad \tilde{\mu}([\alpha,\beta]) = \tilde{\mu}((\alpha,\beta]) = \tilde{\mu}([\alpha,\beta)) = \tilde{\mu}((\alpha,\beta)) = \mu(M_\beta \setminus M_\alpha).$$

We show now that $\tilde{\mu}$ is a T–measure with respect to $\tilde{\mathcal{M}} = ([0,i])_{i\in[0,1]}$.
Notice first that μ being a $T_{|\mu|}$–measure then by Proposition 4.2 for every $\alpha < \beta$ in $[0,1]$ we have $\mu(M_\beta \setminus M_\alpha) \neq O$. Therefore $|\tilde{\mu}|([\alpha,\beta]) \geq |\tilde{\mu}([\alpha,\beta])| = |\mu(M_\beta \setminus M_\alpha)| > 0$. Moreover it is clear that $|\tilde{\mu}| \leq |\mu|$. It follows that $\tilde{\mu}, |\tilde{\mu}|, \tilde{\mathcal{M}}$ satisfy assumption (A). We recall that in this case the set $P_{\tilde{\mathcal{M}}}$ associated to $\tilde{\mathcal{M}}$ is given by $P_{\tilde{\mathcal{M}}} = \{(x_1,\ldots,x_n) \in [0,1]^n : 0 \leq x_1 < \cdots < x_n \leq 1\}$.

Let $A \subseteq P_{\tilde{\mathcal{M}}}$ be such that $|\tilde{\mu}|^{\otimes n}(A) > 0$. The measure $|\tilde{\mu}|^{\otimes n}$ being regular and $P_{\tilde{\mathcal{M}}}$ being open in $[0,1]^n$ there exists a $\mathcal{G}_\delta$ subset E of $P_{\tilde{\mathcal{M}}}$ such that $A \subseteq E$ and $|\tilde{\mu}|^{\otimes n}(E \setminus A) = 0$. We may assume that

$$E = \bigcap_{m=1}^{\infty} V_m$$

where $(V_m)_{m\in\mathbb{N}}$ are open subsets of $P_{\tilde{\mathcal{M}}}$ and $V_1 \supseteq \cdots \supseteq V_m \supseteq V_{m+1} \supseteq \cdots$. Moreover we can choose the sets V_m such that for each m in $\mathbb{N}$

$$V_m = \bigcup_{k=1}^{\infty} I_{m,1}^k \times \cdots \times I_{m,n}^k$$

where the $I_{m,i}^k$ are subintervals of $[0,1]$ satisfying $\sup I_{m,i}^k \leq \inf I_{m,i+1}^k$ and

$$\big(I_{m,1}^k \times \cdots \times I_{m,n}^k\big) \bigcap \big(I_{m,1}^l \times \cdots \times I_{m,n}^l\big) = \emptyset \quad \text{if} \quad k \neq l.$$

If $\alpha_{m,j}^k = \inf I_{m,j}^k$, $\beta_{m,j}^k = \sup I_{m,j}^k$, we define $J_{m,j}^k = M_{\beta_{m,j}^k} \setminus M_{\alpha_{m,j}^k}$ and we set

$$G = \bigcap_{m=1}^{\infty} \Big(\bigcup_{k=1}^{\infty} J_{m,1}^k \times \cdots \times J_{m,n}^k \Big).$$

Clearly G is a subset of P. Moreover, by definition of $J_{m,i}^k$,

$$\begin{aligned}
|\mu|^{\otimes n}(G) &= \lim_{m\to\infty} \Big(\sum_{k=1}^{\infty} |\mu|\big(J_{m,1}^k\big) \cdots |\mu|\big(J_{m,n}^k\big) \Big) \\
&= \lim_{m\to\infty} \Big(\sum_{k=1}^{\infty} |\tilde{\mu}|\big(I_{m,1}^k\big) \cdots |\tilde{\mu}|\big(I_{m,n}^k\big) \Big) \\
&= |\tilde{\mu}|^{\otimes n} \Big(\bigcap_{m=1}^{\infty} \big(\bigcup_{k=1}^{\infty} I_{m,1}^k \times \cdots \times I_{m,n}^k \big) \Big) \\
&= |\tilde{\mu}|^{\otimes n}(E) = |\tilde{\mu}|^{\otimes n}(A)
\end{aligned}$$

so that $|\mu|^{\otimes n}(G) > 0$: the measure μ being $T_{|\mu|}$ we deduce that $\det \mu(G) > 0$.

Let $\sigma = \big(\sigma_1, \dots, \sigma_n\big)$ be a permutation of $1, \dots, n$. We have

$$\begin{aligned}
\big(\mu_{\sigma_1} \otimes \cdots \otimes \mu_{\sigma_n}\big)(G) &= \lim_{m\to\infty}\left(\sum_{k=1}^{\infty} \mu_{\sigma_1}\big(J^k_{m,1}\big)\cdots\mu_{\sigma_n}\big(J^k_{m,n}\big)\right)\\
&= \lim_{m\to\infty}\left(\sum_{k=1}^{\infty} \tilde\mu_{\sigma_1}\big(I^k_{m,1}\big)\cdots\tilde\mu_{\sigma_n}\big(I^k_{m,n}\big)\right)\\
&= \big(\tilde\mu_{\sigma_1} \otimes \cdots \otimes \tilde\mu_{\sigma_n}\big)(E);
\end{aligned}$$

moreover $|\tilde\mu_{\sigma_1} \otimes \cdots \otimes \tilde\mu_{\sigma_n}| \le |\tilde\mu|^{\otimes n}$ and thus

$$\big(\tilde\mu_{\sigma_1} \otimes \cdots \otimes \tilde\mu_{\sigma_n}\big)(E) = \big(\tilde\mu_{\sigma_1} \otimes \cdots \otimes \tilde\mu_{\sigma_n}\big)(A).$$

As a consequence

$$\big(\mu_{\sigma_1} \otimes \cdots \otimes \mu_{\sigma_n}\big)(G) = \big(\tilde\mu_{\sigma_1} \otimes \cdots \otimes \tilde\mu_{\sigma_n}\big)(A)$$

so that

$$\det \tilde\mu(A) = \det \mu(G) > 0$$

and thus $\tilde\mu$ is a Chebyshev measure.

Finally Theorem 4.6 applied to the T–measures μ and $\tilde\mu$ gives

$$\begin{aligned}
\mathcal{R}(\mu) &= \Big\{\mu\Big(\bigcup_{\substack{1\le i\le n\\ i \text{ odd}}} \big(M_{\alpha_{i+1}} \setminus M_{\alpha_i}\big)\Big) : (\alpha_1, \dots, \alpha_n) \in \Gamma\Big\} \quad (\alpha_{n+1} = 1)\\
&= \Big\{\tilde\mu\Big(\bigcup_{\substack{1\le i\le n\\ i \text{ odd}}} (\alpha_i, \alpha_{i+1}]\Big) : (\alpha_1, \dots, \alpha_n) \in \Gamma\Big\} = \mathcal{R}(\tilde\mu). \quad \square
\end{aligned}$$

References

1. C.D. Aliprantis, O. Burkinshaw, *Principles of Real Analysis*, Academic Press, London, 1990.
2. S. Bianchini, R. Cerf, C. Mariconda, *Two dimensional zonoids and Chebyshev measures*, J. Math. Anal. Appl. **211** (1997), 512–526.
3. E.D. Bolker, *A class of convex bodies*, Trans. Amer. Math. Soc. **145** (1969), 323–345.
4. R. Cerf, C. Mariconda, *On bang–bang constrained solutions of a control system*, S.I.A.M. J. Control Optim. **33** (1995), 554–567.
5. R. Cerf, C. Mariconda, *Oriented measures with continuous densities and the bang–bang principle*, J. Funct. Anal. **126** (1994), 476–505.

6. R. Cerf, C. Mariconda, *Oriented measures*, J. Math. Anal. Appl. **197** (1996), 925–944.
7. R. Cerf, C. Mariconda, *Chebyshev measures*, Proc. Amer. Math. Soc. (1997) (to appear).
8. A. Fryszkowski, *Continuous selections for a class of non–convex multivalued maps*, Studia Math. **76** (1983), 163–174.
9. H. Halkin, *On a generalization of a Theorem of Lyapunov*, J. Math. Anal. Appl. **10** (1965), 325–329.
10. C.J. Himmelberg, *Measurable relations*, Fund. Math. **87** (1975), 53–72.
11. S. Karlin, W.J. Studden, *Tchebycheff systems: with applications in analysis and statistics*, Pure and applied mathematics, John Wiley & Sons, New York, 1966.
12. C. Olech, *The Lyapunov theorem: its extensions and applications*, in Methods of Nonconvex Analysis **1446**, Springer–Verlag, Berlin, 1989.
13. N. W. Rickert, *Measures whose range is a ball*, Pacific J. Math. **23** (1967), 361–371.
14. R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.
15. W. Rudin, *Functional Analysis*, Mc Graw-Hill, New York, 1991.
16. R. Schneider, *Measures in convex geometry*, Workshop di Teoria della Misura e Analisi Reale 1995, Rend. Istit. Mat. Univ. Trieste (1996) (to appear).
17. R.L. Wheeden, A. Zygmund, *Measure and Integral*, Dekker, New York, 1977.

S. Bianchini, Scuola Internazionale Superiore di studi Avanzati, Via Beirut 2/4, 34013 Trieste, Italy

C. Mariconda, Dipartimento di Matematica Pura e Applicata, via Belzoni 7, 35131 Padova, Italy

E-mail address: bianchin@sissa.it - maricond@math.unipd.it