A CENTER MANIFOLD TECHNIQUE FOR TRACING VISCOUS WAVES

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ABSTRACT. In this paper we introduce a new technique for tracing viscous travelling profiles. To illustrate the method, we consider a special 2×2 hyperbolic system of conservation laws with viscosity, and show that any solution can be locally decomposed as the sum of 2 viscous travelling profiles. This yields the global existence, stability and uniform BV bounds for every solution with suitably small BV data.

1. INTRODUCTION

This paper is concerned with uniform BV bounds and stability estimates for solutions of a 2×2 hyperbolic system conservation laws in triangular form

$$\begin{cases} u_{1,t} + f(u_1)_x &= u_{1,xx} \\ u_{2,t} + g(u_1, u_2)_x &= u_{2,xx} \end{cases} \tag{1.1}$$

Let $A(u) = Df(u)$ be the Jacobian matrix of f and call $\lambda_1 = \partial f / \partial u_1$, $\lambda_2 = \partial g / \partial u_2$ its eigenvalues. We make the assumption of strict hyperbolicity, so that

$$\lambda_2(u) - \lambda_1(u) \geq c > 0$$

for u in a neighborhood of the origin. The right and left eigenvectors of $A(u)$ will be written as $r_1(u)$, $r_2(u)$ and $l_1(u)$, $l_2(u)$, respectively.

In order to obtain uniform bounds on Tot.Var.$\{u(t, \cdot)\}$ for all $t > 0$, a natural strategy is as follows. We decompose the gradient u_x along a suitable basis of vectors $\tilde{r}_1, \ldots, \tilde{r}_n$, say

$$u_x = \sum_{i=1}^{n} v_i \tilde{r}_i \,. \tag{1.2}$$

Differentiating (1.1), we find that each component v_i satisfies a scalar viscous conservation law with source:

$$v_{i,t} + (\tilde{\lambda}_i v_i)_x - v_{i,xx} = \phi_i \qquad\qquad i = 1, \ldots, n \,. \tag{1.3}$$

This implies

$$\big\| v_i(t, \cdot) \big\|_{L^1} \leq \big\| v_i(0, \cdot) \big\|_{L^1} + \int_0^t \int_{\mathbb{R}} \big| \phi_i(t, x) \big| \, dx dt \,. \tag{1.4}$$

Since the vectors $\tilde{r}_i$ have uniform length, the total variation of u at any time $t > 0$ can be estimated by

$$\text{Tot.Var.}\{u(t, \cdot)\} = \big\| u_x(t, \cdot) \big\|_{L^1} = \mathcal{O}(1) \cdot \sum_i \big\| v_i(t, \cdot) \big\|_{L^1} \,. \tag{1.5}$$

In order to obtain a uniform bound on the total variation, the key step is thus to construct a basis of unit vectors $\{\tilde{r}_1, \ldots, \tilde{r}_n\}$ in (1.2) in a clever way, so that the functions ϕ_i on the right hand side of (1.9) become integrable on the half plane $\{t > 0, \ x \in \mathbb{R}\}$.

As a preliminary we observe that the choice $\tilde{r}_i \doteq r_i(u)$, the i-eigenvector of the matrix $A(u)$, seems quite natural. This was indeed the choice adopted in [4], valid for $n \times n$ hyperbolic systems under the assumption that all shock curves in the state space are straight lines. In this special case, the source functions ϕ_i have the form

$$\phi_i(t, x) = \mathcal{O}(1) \cdot \sum_{j \neq k} v_j v_k + \mathcal{O}(1) \cdot \sum_{j \neq k} v_{j,x} v_k \,,$$

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involving only products of distinct components. This reflects the fact that, due to the straight line assumption, new oscillations can be generated only by interactions among waves of different families. A transversality lemma then shows that the double integral of the terms $v_j v_k$, $v_{j,x} v_k$, with $j \neq k$, is of the same order of magnitude as the product of the L^1 norms of v_j, v_k at the initial time $t = 0$.

In the general case, the choice $\tilde r_i = r_i(u)$ leads to a system of the form

$$v_{i,t} + (\tilde\lambda_i v_i)_x - v_{i,xx} = \phi_i \doteq l_i \cdot \left\{ \sum_{j\neq k} \lambda_j [r_j, r_k] v_j v_k + 2 \sum_{j,k} (r_k \bullet r_j) v_{j,x} v_k + \sum_{j,k,\ell} [r_\ell,\ r_k \bullet r_j] v_j v_k v_\ell \right\}, \tag{1.6}$$

where $r_k \bullet r_j$ is the directional derivative of r_j in the direction of r_k and $[r_j, r_k] \doteq r_j \bullet r_k - r_k \bullet r_j$ denotes a Lie bracket. Assume that the i-th characteristic field is genuinely nonlinear, with shock and rarefaction curves not coinciding, and consider a travelling wave solution

$$u(t,x) = U(x - \sigma t), \tag{1.7}$$

representing a viscous i-shock. Here $U = U(\xi)$ is a smooth function satisfying

$$\begin{aligned} &\big(A(U) - \sigma I\big) U' = U'', \\ &\lim_{\xi\to\pm\infty} U(\xi) = u^\pm , \end{aligned} \tag{1.8}$$

It is then easy to show that the right hand side of (1.6) is not identically zero. Being a travelling wave, the integral

$$\int_{\mathbb{R}} \big|\phi_i(t,x)\big|\, dx \neq 0$$

is constant in time. Hence ϕ_i is certainly not integrable over the half plane $\{t > 0,\ x \in \mathbb{R}\}$.

This lack of integrability can be seen, in particular, for the triangular system (1.1). Here the first equation is autonomous, but the integrals curves of r_1 are not necessarily straight lines. A simple computation shows that is one performs for this system a decomposition of the form

$$u_x = \sum_{i=1}^{n} v_i r_i(u)\,, \tag{1.9}$$

then the source function ϕ_2 contains the term $v_1 v_{1,x}$, and is not integrable. Indeed, the gradient component $v_1 = l_1(u) \cdot u_x$ is $\neq 0$, in general. In turn, the quantity $v_1 v_{1,x}$ does not vanish. Being constant in time (apart from the shift with constant speed σ), it is not integrable in the t-x plane. On the other hand, it is obvious that the total variation of the solution $u(t,\cdot)$ remains bounded. Indeed, it is constant in time.

In this example, it is clear that the decomposition (1.9) is not the best one, in order to study the evolution of the gradient u_x. Instead of the basis of eigenvectors $\{r_1(u), r_2(u)\}$, if we took the projection along another basis, say $\{\tilde r_1(u), \tilde r_2(u)\}$, choosing the first vector so that

$$\tilde r_1\big(U(\xi)\big) = \frac{U'(\xi)}{\big|U'(\xi)\big|} \qquad \xi \in \mathbb{R}, \tag{1.10}$$

then the computations would be much simpler. Indeed, we would have the decomposition

$$u_x = \sum_{j=1,2} \tilde v_j \tilde r_j = \tilde v_1 \tilde r_1,$$

$$\tilde v_1 = \pm |u_x|, \qquad \tilde v_2 = 0.$$

The above example motivates our basic approach. Given a solution $u = u(t,x)$ of the viscous hyperbolic system (1.1), we will derive a-priori bounds on the L^1 norm of the gradient u_x by estimating its components along a suitable basis $\{\tilde r_1, \tilde r_2\}$, choosing the vectors $\tilde r_j$ not as *eigenvectors* of the matrix $A(u)$, but as *gradients of viscous travelling waves* through the state u.

In the special case where the solution u itself is a viscous travelling wave, there is an easy way to choose the basis $\{\tilde r_1, \tilde r_2\}$. Namely, it suffices to satisfy (1.10). However, given a general solution $u = u(t,x)$, it is far from obvious how such a basis (depending on u and possibly on its first and second derivatives) can be constructed.

An appropriate method, based on the center manifold theorem, will be described in the Section 3. We show that there exists smooth functions $\tilde r_i$, $i = 1, 2$, which we call "generalized eigenvectors", depending on the state u and on two additional scalar parameters v_i, σ_i, which are the tangent vectors to a travelling profile passing trough u. Here v_i is related to the strength of the wave profile, while σ_i is the speed. Due to the geometry of the system (1.1), the "generalized eigenvector" $\tilde r_2$ is constant and coincides with the second eigenvector of $A(u)$, i.e. $\tilde r_2 = r_2 = (0,1)$. Moreover, we can normalize $\tilde r_1$ so that its first component is identically equal to 1, i.e. $\tilde r_1(u, v_1, \sigma_1) = \big(1,\ s(u, v_1, \sigma_1)\big)$.

Having constructed this basis of vectors $\tilde r_1, \tilde r_2$, we seek a decomposition of u_x in the form

$$u_x = v_1 \tilde r_1(u, v_1, \sigma_1) + v_2 r_2. \tag{1.11}$$

The two parameters v_1, σ_1 will depend in turn on the first and second derivatives of u. A geometric interpretation of this decomposition is given in Section 4.

In Section 5 we write the evolution equations satisfied by v_1, v_2. Due to the particular geometry of the system, they take the form

$$\begin{cases} v_{1,t} + \big(\lambda_1(u)v_1\big)_x - v_{1,xx} &= \quad 0 \\ v_{2,t} + \big(\lambda_2(u)v_2\big)_x - v_{2,xx} &= \phi_2(t,x), \end{cases} \tag{1.12}$$

where λ_1, λ_2 are the eigenvectors of $A(u)$ and ϕ_2 is the source term for v_2. In particular, by our special choice of the decomposition (1.11), if u coincides with the profile of a travelling 1-wave and if σ_1 coincides with the wave speed, then $\phi_2 \equiv 0$. The speed σ_1 of the profile can be recovered by the relation $\sigma_1 = -u_{1,t}/u_{1,x}$. To handle the general case, since $\tilde r_1(u, v_1, \sigma_1)$ is defined only when v_1 is close to zero and σ_1 is close to the characteristic speed $\lambda_1(u)$, we need to insert a cutoff function and modify the definition of σ_1 whenever the ratio $-u_{1,t}/u_{1,x}$ is far from λ_1. By carefully choosing the parameters v_1, σ_1 as functions of $u, u_{1,x}, u_{1,xx}$ and performing the decomposition (1.2), we will show that the corresponding source ϕ_2 in (1.12) has a particular structure. Namely, it contains only terms of three different types.

(1) source terms due to the cutoff function, effective when $-u_{1,t}/u_{1,x}$ is not close to $\lambda_1(u)$;
(2) source terms due to interactions among two waves both of the first family;
(3) source terms due to interactions between a wave of the first family and one of the second.

The proof of uniform BV bounds for v_2 is worked out in Section 6. Relying on the "length" and "area" functionals introduced in [5], [3] and the viscous interaction potential used in [4], we show that, for small BV initial data, the total variation of the solution remains small for all $t \geq 0$.

A similar estimate can be obtained for the L^1 norm of a first order perturbation h. Indeed, calling

$$u_\varepsilon = u_0 + \varepsilon h + \mathcal{O}(\varepsilon^2)$$

a perturbation of a reference solution u_0, one easily checks that h satisfies the linearized evolution equation

$$h_t + \big(A(u)h\big)_x - h_{xx} = 0.$$

Clearly, both $h = u_t$ and $h = u_x$ are particular solutions. The analysis in Section 7 will establish that

$$\int_{\mathbb{R}} \big|h(t,x)\big| dx \leq L \int_{\mathbb{R}} \big|h(0,x)\big| dx,$$

for some constant L independent of h and uniformly valid for all $t \geq 0$. By a standard homotopy argument, this shows that the flow generated by (1.1) is uniformly Lipschitz continuous w.r.t. the initial data, in the L^1 norm. The above results can be summarized as follows.

Theorem 1.1. *Let the triangular system* (1.1) *have smooth coefficients and satisfy the strict hyperbolicity assumption $\partial f/\partial u_1 \neq \partial g/\partial u_2$. Then there exist $\delta_0, \delta_1 > 0$, and Lipschitz constants L, L' such that the following holds. For every initial data $\bar u \in L^1$ with Tot.Var.$\{\bar u\} \leq \delta_0$, the Cauchy problem has a unique, global solution $u = u(t,x)$, which satisfies*

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} \leq \delta_1 \qquad \forall t \geq 0, \tag{1.13}$$

$$\big\|u(t) - u(s)\big\|_{L^1} \leq L'\big(|t-s|^{1/2} + |t-s|\big) \qquad \forall t, s \geq 0. \tag{1.14}$$

Moreover, given a couple of initial data $\bar u, \bar w$, the corresponding solutions satisfy

$$\big\|u(t) - w(t)\big\|_{L^1} \leq L\,\big\|\bar u - \bar w\big\|_{L^1}, \qquad \forall t \geq 0. \tag{1.15}$$

If the initial data $\bar u$ is smooth, then the solution is uniformly Lipschitz continuous w.r.t. time, as a map from $[0,\infty[\mapsto L^1$. On the other hand, if $\bar u \in BV$ is discontinuous, the solution will be smoothed out during an initial time interval, by parabolic regularization. This accounts for the Hölder continuous time dependence stated in (1.14)

We conclude this section with some remarks. For the corresponding hyperbolic system without viscosity

$$\left\{ \begin{array}{rcl} u_{1,t} + f(u_1)_x & = & 0 \\ u_{2,t} + g(u_1,u_2)_x & = & 0 \end{array} \right. \tag{1.16}$$

uniform BV bounds for weak solutions have been known for a long time [10]. Moreover, in this particular case, the stability of solutions can be proved by deriving a priori estimates on the size of shift differentials and using a homotopy argument to connect pairs of solutions [7]. However, extending these stability results from (1.16) to the general $n\times n$ case is technically very difficult [8] due to the lack of regularity. At present, the stability of small BV solutions is known only under the assumption of genuine nonlinearity or linear degeneracy of each characteristic field [9], or for some 2×2 systems [2], [1].

On the other hand, the presence of viscosity has a regularizing effect on solutions. In this case, by the same techniques used to derive BV estimates, one can obtain bounds on the L^1 norm of first order perturbations. By the smoothness of the solutions, these immediately yield the Lipschitz continuous dependence of solutions on the initial data, via a homotopy argument.

2. Parabolic estimates

In this section we prove some estimates for solutions of the general parabolic system

$$u_t + A(u)u_x - u_{xx} = 0. \tag{2.1}$$

We take here the classical point of view, writing (2.1) in the form of a linear parabolic system with constant coefficients, with a small first order nonlinear perturbation. This approach, based on the representation of solutions via Duhamel's formula, yields two main pieces of information:

(1) For small times $t\in[0,\hat t]$, it determines the rate at which the (possibly discontinuous) initial data is smoothed out, by parabolic regularization.
(2) For large times $t \in [\hat t,\infty[$, it shows that the L^∞ and L^1 norms of all higher order derivatives of the solution are uniformly bounded, as long as the total variation remains small.

For a BV solution u of (2.1) we define

$$u_0 \doteq \lim_{x\to-\infty} u(t,x).$$

It is clear that this value is constant is time. By a translation of coordinates, we can assume $u_0=0$. In the following, we assume that $A(u)$ is strictly hyperbolic, i.e. it admits n real distinct eigenvalues $\lambda_i(u)$ with

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u).$$

For the matrix $A(u)$, we denote with r_i, l_i its right and left eigenvectors respectively, normalized such that

$$\big|r_i\big| = 1, \qquad \big\langle l_j, r_i\big\rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \tag{2.2}$$

We write $\lambda_{i,0}$ and $l_{i,0}$, $r_{i,0}$ for the corresponding eigenvalues and eigenvectors computed at $u_0=0$. The brackets $\langle\cdot,\cdot\rangle$ denote the usual scalar product in $\mathbb{R}^n$. The directional derivative of a function φ along a vector $\omega=(\omega_1,\ldots,\omega_n)\in\mathbb{R}^n$ as

$$\omega\bullet\varphi(u) \doteq \lim_{\epsilon\to 0}\frac{\varphi(u+\epsilon\omega)-\varphi(u)}{\epsilon} = \sum_i \omega_i \frac{\partial\varphi}{\partial u_i}.$$

The Landau notation $\mathcal{O}(1)$ will also be used, to denote a uniformly bounded quantity.

We start by proving some regularity estimates for a solution to the linear parabolic system

$$z_t + A(u)z_x + \big(z\bullet A(u)\big)u_x - z_{xx} = 0, \tag{2.3}$$

under the assumption that $u_x(t)$, $z(t)$ have uniformly bounded L^1 norm, say

$$\int_{\mathbb{R}} \big|u_x(t,x)\big|dx \leq 4\delta_0, \qquad \int_{\mathbb{R}} \big|z(t,x)\big|dx \leq 4\delta_0, \tag{2.4}$$

for all $t \geq 0$, with $\delta_0 > 0$ a small constant. Here and in the sequel it is convenient to measure the norm of a vector $\omega \in \mathbb{R}^n$ in terms of the basis of the eigenvectors of the matrix $A_0 \doteq A(u_0)$. In other words:

$$|\omega| \doteq \sum_i \big|\langle l_{i,0}, \omega\rangle\big|.$$

Notice that (2.3) is the linearized evolution equation satisfied by a first order variation to the solution u of the parabolic system (2.1). In particular, both u_x and u_t satisfy (2.3).

Consider first the linear parabolic system with constant coefficients

$$u_t + A_0 u_x - u_{xx} = 0,$$

with $A_0 = A(u_0) = A(0)$. The corresponding Green kernel G^{A_0} can be written explicitly. Its components $G^{i,0} = \langle l_{i,0}, G^{A_0}\rangle$ are given by

$$G^{i,0}(t,x) = \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{\big(x-\lambda_{i,0}t\big)^2}{4t}\right).$$

In particular, we have the estimates

$$\int_{\mathbb{R}} \big|G_x^{i,0}(t,x)\big|dx = \frac{1}{\sqrt{\pi t}}, \qquad \int_{\mathbb{R}} \big|G_{xx}^{i,0}(t,x)\big| = \sqrt{\frac{2}{\pi e}}\frac{1}{t}.$$

Proposition 2.1. *Let z be a solution of (2.3) satisfying the bounds (2.4). Define the constant $\hat{C}$ and the time $\hat{t}$ as*

$$\begin{aligned} \hat{C} &\doteq \max\Big\{16, 4\Big(\big\|DA\big\|_{L^\infty} + 4\big\|D^2A\big\|_{L^\infty}\delta_0\Big)\Big\}, \\ \sqrt{\hat{t}} &\doteq \frac{1}{64\hat{C}\delta_0} \leq \frac{1}{64\Big(\big\|DA\big\|_{L^\infty} + 4\big\|D^2A\big\|_{L^\infty}\delta_0\Big)4\delta_0}. \end{aligned} \tag{2.5}$$

Then for $0 \leq t \leq \hat{t}$ the following regularity estimates hold:

$$\int_{\mathbb{R}} \big|z_x(t,x)\big| \leq \hat{C}\frac{4\delta_0}{\sqrt{\pi t}}, \qquad \int_{\mathbb{R}} \big|z_{xx}(t,x)\big|dx \leq \hat{C}^2\frac{4\delta_0}{\pi t}. \tag{2.6}$$

Proof. We can represent the function z_x as

$$\begin{aligned} z_x(t,x) = &\int_{\mathbb{R}} G_x^{A_0}(t,x-y)z(0,y)dy \\ &- \int_0^t\int_{\mathbb{R}} G_x^{A_0}(t-s,x-y)\Big(\big(z\bullet A(u)\big)u_x(s,y) + \big(A(u)-A_0\big)z_x(s,y)\Big)dyds. \end{aligned}$$

Note that in particular u_x is a solution to (2.3), so that in the following we will use indifferently the estimates (2.6) for z and u_x: in fact a first step is to prove (2.6) for u_x and then to apply them to z_x.

Using (2.4) and (2.6), we obtain

$$\begin{aligned} &\left\|\int_0^t\int_{\mathbb{R}} G_x^{A_0}(t-s,x-y)\Big\{\big(z\bullet A(u)\big)u_x(s,y) + \big(A(u)-A_0\big)z_x(s,y)\Big\}dyds\right\|_{L^1} \\ &\leq \int_0^t \frac{1}{\sqrt{\pi(t-s)}}\Big\{\big\|u_x\big\|_{L^1}\big\|DA\big\|_{L^\infty}\big\|z(s)\big\|_{L^\infty} + \big\|DA\big\|_{L^\infty}\big\|u_x\big\|_{L^1}\big\|z_x(s)\big\|_{L^1}\Big\}ds \\ &\leq \big\|DA\big\|_{L^\infty}(4\delta_0)\frac{\hat{C}}{\pi}\int_0^t \frac{4\delta_0}{\sqrt{s(t-s)}}ds + \big\|DA\big\|_{L^\infty}\hat{C}(4\delta_0)\frac{\hat{C}}{\pi}\int_0^t \frac{4\delta_0}{\sqrt{s(t-s)}}ds \\ &\leq 2\hat{C}\big\|DA\big\|_{L^\infty}(4\delta_0)^2. \end{aligned}$$

where we used the elementary estimate

$$\int_0^1 \frac{1}{\sqrt{s(1-s)}}ds = \pi.$$

Assume now that the $\|z(t)\|_{L^1}$ is strictly less than $4\hat{C}\delta_0/\sqrt{\pi t}$ in $[0,\tau]$, and $\|z_x(\tau)\|_{L^1} = 4\hat{C}\delta_0/\sqrt{\pi\tau}$, with $\tau \leq t$. Using the above integral estimate we obtain

$$\begin{aligned}\big\|z_x(\tau)\big\|_{L^1} &\leq \frac{4\delta_0}{\sqrt{\pi\tau}}\Big(1+2\sqrt{\pi\hat{t}}\big\|DA\big\|_{L^\infty}\hat{C}\big(4\delta_0\big)\Big)\\ &\leq \frac{4\delta_0}{\sqrt{\pi\tau}}\big(1+\sqrt{\pi}\hat{C}/32\big) < \hat{C}\frac{4\delta_0}{\sqrt{\pi\tau}},\end{aligned}$$

yielding a contradiction. Thus $\tau > \hat{t}$. This argument holds for smooth solutions, because they satisfy (2.6) for small t. Since our estimate depends only on the initial total variation, we can extend it to all BV functions satisfying (2.4).

We now apply the same technique to estimate z_{xx}. Indeed, we can write

$$\begin{aligned}z_{xx}(t,x) = &\int_{\mathbb{R}} G_x^{A_0}(t/2,x-y)z_x(t/2,y)dy\\ &-\int_{t/2}^t\int_{\mathbb{R}} G_x^{A_0}(t-s,x-y)\bigg\{\Big(z\bullet A(u)\Big)u_x(s,y)+\big(A(u)-A_0\big)z_x(s,y)\bigg\}_x dyds.\end{aligned}\tag{2.7}$$

Hence, using again (2.4) and (2.6) we obtain

$$\begin{aligned}\big\|z_{xx}(t)\big\|_{L^1} \leq\ &\frac{1}{\sqrt{\pi t/2}}\cdot\hat{C}\frac{4\delta_0}{\sqrt{\pi t/2}}+\int_{t/2}^t\frac{1}{\sqrt{\pi(t-s)}}\cdot\bigg\{\big\|z_x\bullet A(u)u_x(s)\big\|_{L^1}+\big\|z\bullet(u_x\bullet A(u))u_x(s)\big\|_{L^1}\\ &+\big\|z\bullet A(u)u_{xx}(s)\big\|_{L^1}+\big\|u_x\bullet A(u)z_x(s)\big\|_{L^1}+\big\|\big(A(u)-A_0\big)z_{xx}(s)\big\|_{L^1}\bigg\}ds\\ \leq\ &2\hat{C}\frac{4\delta_0}{\pi t}+\int_{t/2}^t\frac{1}{\sqrt{\pi(t-s)}}\cdot\bigg\{4\delta_0\big\|DA\big\|_{L^\infty}\big\|z_{xx}(s)\big\|_{L^1}+4\delta_0\big\|D^2A\big\|_{L^\infty}\big\|u_{xx}(s)\big\|_{L^1}^2\\ &+4\delta_0\big\|DA\big\|_{L^\infty}\big\|u_{xxx}(s)\big\|_{L^1}+4\delta_0\big\|DA\big\|_{L^\infty}\big\|z_{xx}(s)\big\|_{L^1}+4\delta_0\big\|DA\big\|_{L^\infty}\big\|z_{xx}(s)\big\|_{L^1}\bigg\}ds\\ \leq\ &2\hat{C}\frac{4\delta_0}{\pi t}+\frac{4}{\pi\sqrt{\pi t}}\Big(4\big\|DA\big\|_{L^\infty}+\big\|D^2A\big\|_{L^\infty}\big(4\delta_0\big)\Big)\hat{C}^2\big(4\delta_0\big)^2\\ \leq\ &\hat{C}\frac{4\delta_0}{\pi t}\bigg(2+\frac{4}{\sqrt{\pi}}\sqrt{\hat{t}}\Big(4\big\|DA\big\|_{L^\infty}+\big\|D^2A\big\|_{L^\infty}\big(4\delta_0\big)\Big)\hat{C}\big(4\delta_0\big)\bigg) < \hat{C}^2\frac{4\delta_0}{\pi t}.\end{aligned}$$

An argument by contradiction as the one above concludes the proof. □

Next we observe that, for all $t \geq \hat{t}$, one can repeat the argument on the interval $[t-\hat{t},\, t]$ thus obtaining the following corollary:

Corollary 2.2. *For all $t \geq \hat{t}$ one has*

$$\big\|z_x(t)\big\|_{L^1} \leq \frac{16}{\sqrt{\pi}}\hat{C}^2\big(4\delta_0\big)^2, \qquad \big\|z_{xx}\big\|_{L^1} \leq \frac{256}{\pi}\hat{C}^4\big(4\delta_0\big)^3.\tag{2.8}$$

By integration, this also yields an estimate on the L^∞ norms of z and z_x. The same techniques used in the proof of Proposition 2.1 also yield

Corollary 2.3. *For all $t \geq \hat{t}$ we have the further estimate*

$$\big\|z_{xx}\big\|_{L^\infty} \leq \frac{16^3}{\pi\sqrt{\pi}}\hat{C}^6\big(4\delta_0\big)^4.\tag{2.9}$$

Proof. We first show that, for $t \leq \hat{t}$,

$$\big\|z_{xx}\big\|_{L^\infty} \leq \hat{C}^3\frac{4\delta_0}{\pi t\sqrt{\pi t}}.$$

Indeed, if the estimate is valid for all $\tau < t$, from (2.7) we obtain

$$\begin{aligned}\big\|z_{xx}\big\|_{L^1} &\le \frac{1}{\sqrt{\pi t/2}}\cdot \hat{C}^2\frac{4\delta_0}{\pi t/2} + \int_{t/2}^{t}\frac{1}{\sqrt{\pi(t-s)}}\cdot\Big\{\big\|z_x\bullet A(u)u_x(s)\big\|_{L^\infty} + \big\|z\bullet(u_x\bullet A(u))u_x(s)\big\|_{L^\infty}\\ &\qquad + \big\|z\bullet A(u)u_{xx}(s)\big\|_{L^\infty} + \big\|u_x\bullet A(u)z_x(s)\big\|_{L^\infty} + \big\|\big(A(u)-A_0\big)z_{xx}(s)\big\|_{L^\infty}\Big\}ds\\ &\le 2\sqrt{2}\hat{C}^2\frac{4\delta_0}{\pi t\sqrt{\pi t}} + \frac{2}{\sqrt{\pi}}\sqrt{t}\cdot\Big(4\big\|DA\big\|_{L^\infty} + \big\|D^2A\big\|_\infty(4\delta_0)\Big)(4\delta_0)\cdot\hat{C}^3\frac{2\cdot 4\delta_0}{\pi t\sqrt{\pi t/2}}\\ &\le \hat{C}^2\frac{4\delta_0}{\pi\sqrt{\pi t}\sqrt{t}}\left(2\sqrt{2} + \frac{4\sqrt{2}}{\sqrt{\pi}}\sqrt{\hat{t}}\Big(4\big\|DA\big\|_{L^\infty} + \big\|D^2A\big\|_\infty(4\delta_0)\Big)\hat{C}(4\delta_0)\right) < \hat{C}^3\frac{4\delta_0}{\pi\sqrt{\pi t}\sqrt{t}}.\end{aligned}$$

Hence the estimate holds for all $t \le \hat{t}$. Using now (2.5) and repeating the above argument on the interval $[t-\hat{t},\ t]$ the we obtain (2.9). □

The last proposition gives an estimate of the growth of the L^1 norm of z on the initial interval $[0,\hat{t}]$.

Proposition 2.4. *If the initial data satisfy*

$$\int_{\mathbb{R}}\big|u_x(0,x)\big|dx,\ \int_{\mathbb{R}}\big|z(0,x)\big|dx \le \frac{\delta_0}{4},$$

then at time $\hat{t}$ the following inequality holds:

$$\int_{\mathbb{R}}\big|z(\hat{t},x)\big|dx \le \frac{\delta_0}{2}. \tag{2.10}$$

Proof. Writing the solution as

$$z(t,x) = \int_{\mathbb{R}} G^{A_0}(t,x-y)z(0,y) + \int_0^t\int_{\mathbb{R}} G_x^{A_0}(t-s,x-y)A(u)-A_0\big)z_x(s,y)dsdy,$$

and assuming that $\big\|z(t)\big\|_{L^1} < \delta_0/2$ for $0 \le t < \tau < \hat{t}$ and $\big\|z(\tau)\big\|_{L^1} = \delta_0/2$, we have

$$\big\|z(\tau)\big\|_{L^1} \le \frac{\delta_0}{4} + \frac{2}{\sqrt{\pi}}\sqrt{\hat{t}}\big\|DA\big\|_{L^\infty}(4\delta_0)\frac{\delta_0}{2} < \frac{\delta_0}{4} + \frac{1}{8\sqrt{\pi}}\frac{\delta_0}{4} < \frac{\delta_0}{2},$$

leading to a contradiction. □

Remark 2.5. The numerical value of the constant $\hat{C}$ is irrelevant. What matters is that the higher derivatives of z have norm bounded by powers of δ_0. We recall that δ_0 is the order of magnitude of the total variation of u, which we assume suitably small. This fact will be of help in deriving our future estimates, because terms multiplied by norms of these derivatives will contain powers of δ_0 and hence be very small.

In the following, to simplify the notation, we shall shift the time coordinate and consider a solution defined for $t \in [-\hat{t},\ \infty[$. At time $t = 0$ we can thus assume that our solution $u(0,\cdot)$ is smooth satisfies

$$\big\|u_x(0)\big\|_{L^1} \le \frac{\delta_0}{2},\quad \big\|u_x(0)\big\|_{L^\infty} \le \mathcal{O}(1)\delta_0^2,\quad \big\|u_t(0)-\lambda_{i,0}u_x(0)\big\|_{L^1} \le \mathcal{O}(1)\delta_0^2. \tag{2.11}$$

We recall that, without loss of generality, we are always assuming $u(t,-\infty) = 0$.

3. A decomposition using travelling profiles

In this section we construct a smooth manifold of local travelling wave profiles, in connection with the triangular system

$$\left\{\begin{array}{rcl} u_{1,t} + f(u^1)_x - u^1_{xx} &=& 0\\ u_{2,t} + g(u_1,u_2)_x - u^2_{xx} &=& 0\end{array}\right. \tag{3.1}$$

We assume that there exists an open set $\Omega \subseteq \mathbb{R}^2$ such that the matrix

$$Df(u) = A(u) \doteq \begin{bmatrix}\lambda_1(u_1) & 0\\ g_{u^1}(u_1,u_2) & \lambda_2(u_1,u_2)\end{bmatrix},$$

is uniformly hyperbolic, i.e.

$$\lambda_2(u) - \lambda_1(u') \geq c, \tag{3.2}$$

for some constant $c > 0$ and all $u, u' \in \Omega$. In the following we will denote with r_i, l_i, $i = 1, 2$, the right and left eigenvectors of $A(u)$, respectively, normalized as in (2.2).

A travelling wave profile with speed σ is obtained by solving the second order ODE

$$-\sigma u_x + A(u)u_x - u_{xx} = 0, \tag{3.3}$$

which can be written as a first order system of the form

$$\begin{cases} u_x &= & p \\ p_x &= & \big(A(u) - \sigma I\big)p \\ \sigma_x &= & 0 \end{cases}$$

Linearizing at the point $(0, 0, \lambda_1(0))$ we obtain a linear system of 5 equations described by the matrix

$$\begin{bmatrix} 0 & I & 0 \\ 0 & A(0) - \lambda_1(0)I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The dimension of the null space is 4. Hence, by the center manifold theorem, there exists a 4-dimensional locally invariant manifold $\mathcal{M} \subseteq \mathbb{R}^5$ which contains all the "slow" dynamics in a neighborhood of 0. In particular this manifold contains all the small bounded travelling profiles with speed close to $\lambda_1(0)$.

This center manifold $\mathcal{M}$ can be parametrized in terms of the variables u, $v_1 \doteq u_{1,x}$ and $\sigma_1 \doteq \sigma$. In other words, it is described by the equation $v_2 = \psi(u, v_1, \sigma_1)$. When $v_1 = 0$ we obtain the equilibrium points $(u, 0, \sigma_1)$, so that $\psi(u, 0, \sigma_1) = 0$. As a consequence we can "factor out" the term v_1 and write

$$v_2 = s\big(u, v_1, \sigma_1\big)v_1,$$

where s is a smooth function of its arguments.

We now define the generalized eigenvector $\tilde{r}_1$ as

$$\tilde{r}_1(u, v_1, \sigma_1) \doteq \begin{pmatrix} 1 \\ s(u, v_1, \sigma_1) \end{pmatrix}. \tag{3.4}$$

Using u_1 as independent variable, we can rewrite the system as

$$\begin{cases} u_s &= & \tilde{r}_1(u, v_1, \sigma_1) \\ v_{1,s} &= & \lambda_1(u_1) - \sigma_1 \\ v_1\tilde{r}_{1,s} &= & \big(A(u) - \lambda_1 I\big)\tilde{r}_1 \\ \sigma_{1,s} &= & 0 \end{cases}$$

and obtain the fundamental relation

$$v_1\tilde{r}_1 \bullet \tilde{r}_1 + v_1\big(\lambda_1 - \sigma_1\big) = \big(A(u) - \lambda_1 I\big)\tilde{r}_1. \tag{3.5}$$

Here and throughout the following, the derivatives of $\tilde{r}_1(u, v_1, \sigma_1)$ w.r.t. its arguments are written as

$$\mathbf{v} \bullet \tilde{r}_1 \doteq \lim_{\varepsilon \to 0} \frac{\tilde{r}_1(u + \varepsilon\mathbf{v},\ v_1,\ \sigma_1) - \tilde{r}_1(u,\ v_1,\ \sigma_1)}{\varepsilon},$$

$$\tilde{r}_{1,v} \doteq \frac{\partial}{\partial v_1}\tilde{r}_i(u,\ v_1,\ \sigma_1), \qquad \tilde{r}_{1,\sigma} \doteq \frac{\partial}{\partial \sigma_1}\tilde{r}_1(u,\ v_1,\ \sigma_1).$$

Note that when $v_1 = 0$, form the above equation it follows that $\tilde{r}_1 = r_1$. This implies the further estimates

$$\tilde{r}_{1,\sigma} = \mathcal{O}(1)v_1, \qquad \tilde{r}_{1,\sigma\sigma} = \mathcal{O}(1)v_1, \qquad \tilde{r}_i \bullet \tilde{r}_{1,\sigma} = \mathcal{O}(1)v_1 \qquad i = 1, 2. \tag{3.6}$$

Example 3.1. Consider the following equations

$$\begin{cases} u_{1,t} - u_{1,xx} &= 0 \\ u_{2,t} + \big(-(u_1)^2/2 + u^2\big)_x - u^2_{xx} &= 0. \end{cases} \tag{3.7}$$

The differential equations for a travelling profile of the first family, using $y = u_1$ as independent variable is

$$\begin{cases} -\sigma_1 - v_{1,y} &= 0 \\ -\sigma_1 u_{2,y} + u_{2,y} - y - \big(v_1 u_{2,y}\big)_y &= 0 \end{cases}$$

where we denote with $u_{2,y}$ the function

$$u_{2,y} = \frac{\partial u_2}{\partial u_1} = \frac{v_2}{v_1}.$$

The solution to the previous ODE is

$$\begin{cases} v_1(y) &= \qquad C_1 - \sigma y \\ v^2(y) &= y + \dfrac{C_1 - \sigma y}{1+\sigma} + C_2 \big(C_1 - \sigma y\big)^{-1/\sigma} \end{cases}$$

Since we want the solution to be smooth near $v_1 = 0$, we choose $C_2 = 0$, so that the tangent vector to a profile on the center manifold is now given by

$$\tilde{r}_1(u, v_1, \sigma_1) = \begin{bmatrix} 1 \\ u_1 + v_1/(1+\sigma_1) \end{bmatrix}. \tag{3.8}$$

One can check that (3.5) and all the estimates (3.6) are satisfied. The fact that $\tilde{r}_1$ does not depend on u_2 follows from the linearity of the second equation: if $u_2(t,x)$ is a solution to (3.7), then $u_2(t,x) + \kappa$ is also a solutions for all $\kappa \in \mathbb{R}$.

The function $\tilde{r}_1$ is defined in a neighborhood of the point $(0, 0, \lambda_1(0))$ of the form

$$\Big\{ u, v_1, \sigma_1 \ : \ |u| \leq 2\delta_1, |v_1| \leq 2\delta_1, |\sigma_1 - \lambda_1(u)| \leq 2\delta_1 \Big\} \subseteq \mathbb{R}^4. \tag{3.9}$$

By the regularity estimates (2.11) at the end of Section 2, if δ_0 is sufficiently small, u and v_1 satisfy the first two inequality. Thus, given any speed function σ_1 sufficiently close to $\lambda_1(0)$, the vector $\tilde{r}_1$ is defined for $t \in [0, T]$, $x \in \mathbb{R}$.

One can now decompose the vector u_x along $\tilde{r}_1$ and $r_2 \doteq (0, 1)$:

$$u_x = \begin{pmatrix} u_{1,x} \\ u_{2,x} \end{pmatrix} = v_1 \tilde{r}_1 + v_2 r_2 = v_1 \begin{pmatrix} 1 \\ s(u, v_1, \sigma_1) \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.10}$$

Due to the particular form of the system (3.1), the generalized eigenvector corresponding to the second eigenvalue always coincides with the constant eigenvector $r_2 = (0, 1)$ of the matrix $A(u)$.

Using the estimates of Section 2, see that the components v_i, $i = 1, 2$ satisfy the bounds

$$\big\|v_i(t)\big\|_{L^\infty} \leq \big\|v_{i,x}(t)\big\|_{L^1} \leq \mathcal{O}(1)\delta_0^2, \qquad \big\|v_{i,x}(t)\big\|_{L^\infty} \leq \mathcal{O}(1)\delta_0^3. \tag{3.11}$$

In the following we shall assume, without loss of generality, that $\lambda_1(0) = 0$.

4. Geometric remarks

Before we proceed toward applications, let us pause and describe what has been accomplished by the above construction. Our eventual goal is to decompose the gradient u_x of a smooth function $u : \mathbb{R} \mapsto \mathbb{R}^2$ as a *sum* (not a linear combination!) of gradients of 2 viscous travelling waves. At each point x, we expect this decomposition to depend on the vectors u_x and u_{xx}. Our "data" thus consist of 2+2 parameters. On the other hand, let us look at how many viscous travelling i-waves pass through a given state $u \in \mathbb{R}^2$. If we restrict ourselves to bounded viscous shock profiles, assuming that the i-st field is genuinely nonlinear, we can clearly find a 2-parameter family of such shocks. Namely, we can parametrize such family in terms of the first coordinates of the limit points u^+, u^-. More precisely, given any two numbers σ^-, σ^+ with

$$\sigma^+ < \langle r_i(u),\ u\rangle < \sigma^-,$$

we can find unique states u^-, u^+ in a neighborhood of u such that (fig. 3)

$$\langle r_i(u),\ u^-\rangle = \sigma^-, \qquad\qquad \langle r_i(u^*),\ u^+\rangle = \sigma^+,$$

and such that the viscous shock profile connecting u^- with u^+ passes through the state u. In turn, this 2-parameter family of shock profiles yields a 2-parameter family of gradient vectors $\mathbf{v}_i$, i.e. the gradients of these viscous i-shocks at the point u. Observe that these gradient vectors are nearly parallel to $r_i(u)$, but have opposite direction. We can repeat the construction for all characteristic families $i = 1, 2$. This gives us 2 distinct 2-parameter families of gradient vectors. In all, we have just the right number of parameters $4 = 2 + 2$ to fit the data of the problem. Unfortunately, the set of gradient vectors $\mathbf{v}_i$ thus constructed is not large enough to express an arbitrary gradient in the form

$$u_x = \sum_{i=1}^{n} \mathbf{v}_i. \tag{4.1}$$

Indeed, in the genuinely nonlinear case, each shock gradient $\mathbf{v}_i$ will have negative inner product with $r_i(u)$. Hence if, say, the function u consists of an i-rarefaction wave, a decomposition like (4.1) is not possible. We thus need to extend the 2-parameter family of vectors $\mathbf{v}_i$ to include also gradients of travelling viscous rarefaction i-waves. The problem is that now there are no (globally bounded) viscous rarefaction i-waves. On the other hand, if we look at all viscous travelling waves through a given point $\bar{u}$, then we have to consider all solutions of the system (3.3), with initial data $u(0) = \bar{u}$ but $v(0)$ and σ arbitrary. These form a 3-parameter family of solutions. Too many! We have to trim it down, choosing a 2-parameter subfamily. This is precisely what our center manifold construction has achieved.

Summing up, for each state u and each $i = 1, 2$, we have constructed a 2-parameter family of unit vectors $\tilde{r}_i = \tilde{r}_i(u, v_i, \sigma_i)$, depending on the scalar parameters v_i, σ_i. For each value of these parameters, there exists a viscous travelling i-wave passing through u, with gradient $v_i\tilde{r}_i$ and speed σ_i. In other words, there exists a solution $U = U(\xi)$ of

$$U'' = \big(A(U) - \sigma_i\big)U'$$

with

$$U(0) = u, \qquad\qquad U'(0) = v_i\tilde{r}_i\,.$$

Moreover, the way these vectors $\tilde{r}_i$ change as functions of the parameters is restricted by the fundamental identity (3.5).

The above construction of the tangent vectors $\tilde{r}_i$ allows a new approach to the analysis of viscous waves. Consider first the strictly hyperbolic system

$$u_t + A(u)u_x = 0. \tag{4.2}$$

Given a function $u = u(x)$, at a given point $x \in \mathbb{R}$ we can look at the first order jet $(u,\, u_x) \in \mathbb{R}^{n+n}$. It is natural to regard u_x as the linear superposition of n waves

$$u_x = \sum_{i+1}^{n} v_i r_i(u), \tag{4.3}$$

travelling with speeds $\lambda_1(u), \ldots, \lambda_n(u)$ given by the eigenvalues of the matrix $A(u)$. In connection with a smooth solution of (4.3), the lines in the t-x plane defined by

$$\dot{x}_i(t) = \lambda_i\big(u(t,x)\big) \tag{4.4}$$

are called i-characteristics. In all classical textbooks, the basic analysis of hyperbolic systems relies on the study of how i-waves propagate along characteristics.

Next, consider a hyperbolic system with viscosity:

$$u_t + A(u)u_x = u_{xx}. \tag{4.5}$$

Given a function $u = u(x)$, at a given point $x \in \mathbb{R}$ we now look at the second order jet $(u,\, u_x,\, u_{xx}) \in \mathbb{R}^{n+n+n}$. In an ideal situation, we would like to regard $(u_x,\, u_{xx})$ as the superposition of n viscous travelling waves, travelling with speeds $\sigma_1, \ldots, \sigma_n$. We thus seek solutions U_i of

$$U_i'' = \big(A(U_i) - \sigma_i\big)U_i' \tag{4.6}$$

such that

$$u_x = \sum_i U_i'(x), \qquad\qquad u_{xx} = \sum_i U_i''(x). \tag{4.7}$$

In connection with a smooth solution of (4.5), assume for a moment that a decomposition of the form (4.7) can be achieved at all points in the t-x plane. Moreover, assume that the wave speeds $\sigma_i = \sigma_i(t,x)$ remain within the same range of the corresponding characteristic speeds λ_i. In this case, the curves defined by

$$\dot{x}_i(t) = \sigma_i(t,x) \tag{4.8}$$

can be called *second order i-characteristics.* They trace the positions of the viscous travelling waves that (pointwise) best approximate our solution u.

At this stage, however, two remarks are in order.

Remark 4.1. l As we saw earlier, the family of all viscous travelling waves, i.e., of all solutions of (4.6), is too large. The problem is that we are considering as admissible solutions to (4.6) functions which have nothing to do with the travelling waves of small amplitude of the parabolic system (4.5).

In order that the above decomposition be uniquely determined, we need to restrict ourselves to travelling waves which lie on the center manifolds $\mathcal{M}_i$: by construction it contains all the travelling waves of small amplitude, and for each $i \in \{1, \dots, n\}$, this yields a 2-parameter family of viscous waves. In this case, the decomposition of u_x in (4.7) can be written in the form

$$u_x = \sum_{i=1}^{n} v_i \tilde{r}_i.$$

Remark 4.2. The requirement that the wave speeds σ_i remain inside the range of the characteristic speeds λ_i cannot be fulfilled, in general. For example, consider any viscous scalar conservation law.

$$u_t + f(u)_x = u_{xx}.$$

In this case, the center manifold is the whole space: $\mathcal{M} = \mathbb{R}^3$. Consider a smooth function $u : \mathbb{R} \mapsto [a, b]$. To fix the ideas, assume that $f'(u) \in [\lambda^-, \lambda^+]$ for all $u \in [a, b]$. At any given point x, if $u_x \neq 0$ there exists a unique viscous travelling wave U whose second order jet at x coincides with that of u. Indeed, we find U by solving the Cauchy problem

$$U'' = \big(f'(u) - \sigma\big)U', \qquad \begin{cases} U(x) = u(x), \\ U'(x) = u_x(x), \end{cases}$$

where

$$\sigma = f'\big(u(x)\big) - \frac{u_{xx}(x)}{u_x(x)}.$$

Clearly, if $|u_{xx}/u_x|$ is large, the speed σ will fall far outside the interval $[\lambda^-, \lambda^+]$, i.e., outside the range of the characteristic speeds $f'(u)$.

For the validity of future estimates, it is imperative that the speeds σ_i remain within small intervals, close to the characteristic speeds. Say,

$$\sigma_i \in J_i \doteq [\lambda_i^-, \lambda_i^+],$$

distinct intervals being strictly disjoint.

Therefore, we shall need to insert a cut-off function, forcing the speeds σ_i to remain within J_i. The price to pay is that now only the first identity in (4.7) will be achieved. The local representation of the profile of u as superposition of viscous travelling waves will be always correct up to first order. However, it will hold up to second order only in those cases where the cut-off function is not effective.

5. Decomposition of the derivative

In this section we derive the evolution equations for the components v_1, v_2. To obtain these equations, we start with the identity

$$u_x = v_1 \tilde{r}_1 + v_2 r_2.$$

Differentiating w.r.t t and x one obtains

$$u_{xx} = v_{1,x}\tilde{r}_1 + v_{2,x} r_2 + v_1\big(v_1 \tilde{r}_1 \bullet \tilde{r}_1 + v_{1,x}\tilde{r}_{1,v} + \sigma_{1,x}\tilde{r}_{1,\sigma}\big) + v_1 v_2 r_2 \bullet \tilde{r}_1,$$

$$\begin{aligned} u_t &= u_{xx} - A(u)u_x \\ &= \big(v_{1,x} - \lambda_1 v_1\big)\tilde{r}_1 + \big(v_{2,x} - \lambda_2 v_2\big) r_2 + v_1\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\tilde{r}_{1,v} + v_1 \sigma_{1,x}\tilde{r}_{1,\sigma} + v_1 v_2 r_2 \bullet \tilde{r}_1, \end{aligned}$$

$$
\begin{aligned}
u_{tx} = {} & \Big(v_{1,xx} - \big(\lambda_1 v_1\big)_x\Big)\tilde r_1 + \Big(v_{2,xx} - \big(\lambda_2 v_2\big)_x\Big) r_2 \\
& + \big(v_{1,x} - \lambda_1 v_1\big)\Big(v_1 \tilde r_1 \bullet \tilde r_1 + v_2 r_2 \bullet \tilde r_1 + v_{1,x}\tilde r_{1,v} + \sigma_{1,x}\tilde r_{1,\sigma}\Big) + \Big(v_1\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big)_x \tilde r_{1,v} \\
& + v_1\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big(v_1 \tilde r_1 \bullet \tilde r_{1,v} + v_2 r_2 \bullet \tilde r_{1,v} + v_{1,x}\tilde r_{1,vv} + \sigma_{1,x}\tilde r_{1,v\sigma}\Big) \\
& + \big(v_1\sigma_{1,x}\big)_x \tilde r_{1,\sigma} + v_1\sigma_{1,x}\Big(v_1\tilde r_1 \bullet \tilde r_{1,\sigma} + v_2 r_2 \bullet \tilde r_{1,\sigma} + v_{1,x}\tilde r_{1,v\sigma} + \sigma_{1,x}\tilde r_{1,\sigma\sigma}\Big) + \big(v_1 v_2 r_2 \bullet \tilde r_1\big)_x,
\end{aligned}
$$

$$
\begin{aligned}
u_{xt} &= v_{1,t}\tilde r_1 + v_{2,t} r_2 + v_1\big(u_t \bullet \tilde r_1 + v_{1,t}\tilde r_{1,v} + \sigma_{1,t}\tilde r_{1,\sigma}\big) \\
&= v_{1,t}\tilde r_1 + v_{2,t} r_2 + v_1\Big(\big(v_{1,x} - \lambda_1 v_1\big)\tilde r_1 \bullet \tilde r_1 + \big(v_{2,x} - \lambda_2 v_2\big) r_2 \bullet \tilde r_1 + v_1\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\tilde r_{1,v} \bullet \tilde r_1 \\
&\quad + v_1\sigma_{1,x}\tilde r_{1,\sigma} \bullet \tilde r_1 + v_1 v_2\big(r_2 \bullet \tilde r_1\big) \bullet \tilde r_1\Big) + v_1 v_{1,t}\tilde r_{1,v} + v_1\sigma_{1,t}\tilde r_{1,\sigma}\big).
\end{aligned}
$$

Since $u_{xt} = u_{tx}$, we finally obtain the equations for the components:

$$
v_1 + \big(\lambda_1 v_1\big)_x - v_{1,xx} = 0, \tag{5.1}
$$

$$
\begin{aligned}
v_2 + \big(\lambda_2 v_2\big)_x - v_{2,xx} = {} & \Big[2v_{1,x}\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big) + v_1^2\sigma_{1,x}\Big]\big\langle \tilde l_2, \tilde r_{1,v}\big\rangle \\
& + \Big[\big(v_{1,x} - \lambda_1 v_1\big)\sigma_{1,x} + \big(v_1\sigma_{1,x}\big)_x - v_1\sigma_{1,t}\Big]\big\langle \tilde l_2, \tilde r_{1,\sigma}\big\rangle \\
& + \Big[v_1^2\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big]\big\langle \tilde l_2, \big[\tilde r_1 \bullet \tilde r_{1,v}\big]\big\rangle \\
& + \Big[v_1 v_{1,x}\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big]\big\langle \tilde l_2, \tilde r_{1,vv}\big\rangle \\
& + \Big[v_1\sigma_{1,x}\big(2v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big]\big\langle \tilde l_2, \tilde r_{1,v\sigma}\big\rangle \\
& + \Big[v_1^2\sigma_{1,x}\Big]\big\langle \tilde l_2, \big[\tilde r_1, \tilde r_{1,\sigma}\big]\big\rangle + \Big[v_1\sigma_{1,x}^2\Big]\big\langle \tilde l_2, \tilde r_{1,\sigma\sigma}\big\rangle \\
& + \Big[\big(\lambda_2 - \lambda_1\big)v_1 v_2 + 2v_2 v_{1,x}\Big]\big\langle \tilde l_2, r_2 \bullet \tilde r_1\big\rangle \\
& + \Big[v_1^2 v_2\Big]\big\langle \tilde l_2, \big[\tilde r_1, r_2 \bullet \tilde r_1\big]\big\rangle + \Big[v_1 v_2^2\Big]\big\langle \tilde l_2, r_2 \bullet \big(r_2 \bullet \tilde r_1\big)\big\rangle \\
& + \Big[v_1 v_2\big(2v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big]\big\langle \tilde l_2, r_2 \bullet \tilde r_{1,v}\big\rangle + \Big[2v_1 v_2\sigma_{1,x}\Big]\big\langle \tilde l_2, r_2 \bullet \tilde r_{1,\sigma}\big\rangle \\
= {} & \phi_2(t,x).
\end{aligned} \tag{5.2}
$$

The above equations hold for any speed σ_1. A particular choice of σ_1 will now be specified. We first define w_1 as the *effective flux* of (5.1), given by

$$
w_1 \doteq v_{1,x} - \lambda_1(u) v_1. \tag{5.3}
$$

Next, we set

$$
\sigma_1 = \lambda_1(0) + \theta\left(\big(\lambda_1(u) - \lambda_1(0) - \frac{v_{1,x}}{v_1}\right) = \theta\left(-\frac{w_1}{v_1}\right), \tag{5.4}
$$

Here the cut-off function $\theta : \mathbb{R} \mapsto \mathbb{R}$ is an odd function such that

$$
\theta(x) = \begin{cases} x & |x| \le \delta_1 \\ \text{smooth connection} & \delta_1 \le |x| \le 3\delta_1 \\ 0 & |x| \ge 3\delta_1 \,. \end{cases} \tag{5.5}
$$

Taking for example a cubic interpolation followed by smoothing, we can assume that

$$
\big|\theta(x)\big| \le 2\delta_1, \qquad \big|\theta'(x)\big| \le 12, \qquad \big|\theta''(x)\big| \le 12/\delta_1 \qquad \forall x \in \mathbb{R}.
$$

An easy computation shows that the function w_1 satisfies the same equation of v_1, namely

$$
w_{1,t} + \big(\lambda_1 w_1\big)_x - w_{1,xx} = 0.
$$

Moreover, by the regularity estimates of Section 2, we obtain that

$$
\big\|w_1(t)\big\|_{L^1} \le \mathcal{O}(1)\delta_0^2, \quad \big\|w_1(t)\big\|_{L^\infty}, \big\|w_{1,x}(t)\big\|_{L^1} \le \mathcal{O}(1)\delta_0^3, \quad \big\|w_{1,x}(t)\big\|_{L^\infty} \le \mathcal{O}(1)\delta_0^4. \tag{5.6}
$$

By assuming that the total variation of the initial data $\bar{u}$ is sufficiently small, we can assume that

$$\delta_1 \geq 10\big\|\lambda_1'\big\|_{L^\infty}\delta_0 \geq 5\max_{t,x}\Big\{\big|\lambda_1\big(u_1(t,x)\big)\big|\Big\}, \tag{5.7}$$

where we recall that $\|v_1(t)\|_{L^1} \leq 2\delta_0$. As a consequence of (2.11), at $t=0$ one has

$$\int_{\mathbb{R}}\big|v_1(0,x)\big|dx,\ \int_{\mathbb{R}}\big|v_2(0,x)\big|dx \leq \delta_0. \tag{5.8}$$

Note that with the choice of speed (5.4), when

$$v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1 \neq 0,$$

then one must have

$$\Big|v_{1,x} - \lambda_1(u)v_1\Big| \geq \delta_1\big|v_1\big|,$$

which implies

$$\left|\frac{v_{1,x}}{v_1}\right| \geq \frac{4}{5}\delta_1 \geq 8\big\|\lambda_1'\big\|_{L^\infty}\delta_0 \geq 4\big\|\lambda_1\big\|_{L^\infty}. \tag{5.9}$$

Thus in the regions when w_1/v_1 is large, we can always bound $|v_1|$ with $|v_{1,x}|$. Conversely, when the speed is near the eigenvalue $\lambda_1(0)=0$, $|v_{1,x}|$ is bounded by $|v_1|$: in fact, in the regions where $\theta', \theta'' \neq 0$, we have

$$\left|\frac{v_{1,x}}{v_1}\right| \leq \frac{16}{5}\delta_1 < 4\delta_1. \tag{5.10}$$

Note moreover that by our choice of δ_1 we have

$$\Big|\lambda_1(u) - \sigma_1\Big| \leq \frac{11}{5}\delta_1 < 3\delta_1. \tag{5.11}$$

These estimates will be useful in the sequel.

We consider the left hand side of (5.2), $\phi_2(t,x)$, as the source term of total variation. The aim of this and the next chapter is to prove that this is uniformly bounded.

Remark 5.1. Note that with the speed (5.4), the "generalized eigenvector" $\tilde{r}_1$, considered as

$$\tilde{r}_1\big(u,v_1,w_1\big) = \tilde{r}_1\left(u,v_1,\theta\left(-\frac{w_1}{v_1}\right)\right),$$

is a Lipschitz function of its arguments. Recalling that $w_1 = v_{1,x} - \lambda_1(u)v_1$, the relation $\theta(w_1/v_1) \neq 0$ implies

$$\big|v_{1,x} - \lambda_1(u)v_1\big| \leq 3\delta_1 v_1,$$

so that v_1 is bounded above and below two exponential functions: thus v_1 never reaches 0. As a consequence, if $v_1(x)=0$ at some point $\bar{x}$, then $|w_1/v_1| \to \infty$ as $x \to \bar{x}$. Hence $\theta(w_1/v_1)=0$ in a whole neighborhood. It follows that $\tilde{r}_1$ is actually smooth in the x variable and the source term ϕ_2 is well defined for all t,x.

However there are no uniform bounds on the derivatives of ϕ_2: consider for example the function $v(t,x) = t + x^2/2$ and, using the eigenvector $\tilde{r}_1$ of example 3.1, the term

$$v_{1,x}\sigma_{1,x}\tilde{r}_{1,v\sigma} = -\theta'\frac{v_1 v_{1,x}}{\sigma_1+1}\left(\frac{v_{1,x}}{v_1}\right)_x = -\frac{\theta'}{\sigma_1+1}\left(x - \frac{2x^3}{(x^2+2t)^2}\right).$$

Note that its first derivative is still bounded and continuous for any fixed time t, but the terms like

$$\frac{\theta'}{\sigma_1+1}$$

switch very rapidly as $t \to 0$ from $-1/(-2\delta_1+1)$ to $1/(2\delta_1+1)$ in a time interval $\mathcal{O}(t)$.

Using (5.4), we now reduce the source terms to 4 general categories. Using the definitions of [5], these classes of terms are:

wrong speed: this term arises only when $\sigma_1 \neq -w_1/v_1$. It has the form

$$v_{1,x}\big(w_1 + \sigma_1 v_1\big);$$

change in strength: as in [5], this term has the form

$$w_{1,x}v_1 - v_{1,x}w_1;$$

change in speed: this was also studied in [5].

$$v_1\left[v_1\left(\frac{w_1}{v_1}\right)^2_x\right]\chi\Big\{x : |w_1/v_1| \leq 3\delta_1\Big\};$$

transversal interactions: these are of the form [4]

$$v_1v_2 \qquad v_{1,x}v_2.$$

By $\chi(I)$ we denoted the indicator function of a set I. Observe that the "change in speed" term does not vanish only if $|w_1/v_1| \leq 3\delta_1$.

Using the above definitions and recalling the bounds (3.6), we classify the source terms in (5.2) according to their leading factors:

$\langle\tilde{l}_2, \tilde{r}_{1,v}\rangle$**:** this is the wrong speed term and the change of mass:

$$2v_{1,x}\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big) + v_1^2\sigma_{1,x} = 2\Big[v_{1,x}\big(w_1 + \sigma_1 v_1\big)\Big] + \theta'\Big[v_{1,x}w_1 - v_1 w_{1,x}\Big];$$

$\langle\tilde{l}_2, \tilde{r}_{1,\sigma}/v_1\rangle$**:** this term becomes the change of speed:

$$v_1\Big(\big(v_{1,x} - \lambda_1 v_1\big)\sigma_{1,x} + \big(v_1\sigma_{1,x}\big)_x - v_1\sigma_{1,t}\Big) = \theta'' v_1\Big[v_1\big(w_1/v_1\big)^2_x\Big];$$

$\langle\tilde{l}_2, \tilde{r}_1 \bullet \tilde{r}_{1,v} - \tilde{r}_{1,v} \bullet \tilde{r}_1\rangle$**:** recalling (5.9), this is a higher order term w.r.t. the wrong speed:

$$v_1^2\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big) = v_1\frac{v_1}{v_{1,x}}\Big[v_{1,x}\big(w_1 + \sigma_1 v_1\big)\Big];$$

$\langle\tilde{l}_2, \tilde{r}_{1,vv}\rangle$**:** this is a higher order term w.r.t. the change of strength:

$$v_1 v_{1,x}\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big) = v_1\Big[v_{1,x}\big(w_1 + \sigma_1 v_1\big)\Big];$$

$\langle\tilde{l}_2, \tilde{r}_{1,v\sigma}\rangle$**:** this term can be rewritten as

$$v_1\sigma_{1,x}\big(2v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big) = \theta'\big(2w_1/v_1 + \lambda_1 + \sigma_1\big)\Big[v_{1,x}w_1 - v_1 w_{1,x}\Big];$$

$\langle\tilde{l}_2, (\tilde{r}_1 \bullet \tilde{r}_{1,\sigma} - \tilde{r}_{1,\sigma} \bullet \tilde{r}_1)/v_1\rangle$**:** this is a higher order term w.r.t. the change of strength:

$$v_1\Big[v_1^2\sigma_{1,x}\Big] = v_1\theta'\Big[v_{1,x}w_1 - v_1 w_{1,x}\Big];$$

$\langle\tilde{l}_2, \tilde{r}_{1,\sigma\sigma}/v_1\rangle$**:** this is precisely the change of speed:

$$v_1^2\sigma_{1,x}^2 = \big(\theta'\big)^2 v_1\Big[v_1\big(w_1/v_1\big)^2_x\Big].$$

Thus, collecting the source terms, we can rewrite (5.2) as

(5.12)

$$\begin{aligned}
v_{2,t} + \big(\lambda_2 v_2\big)_x - v_{2,xx} = {} & \Big[v_{1,x}\big(w_1 + \sigma_1 v_1\big)\Big]\Big\{2\langle\tilde{l}_2, \tilde{r}_{1,v}\rangle + \big(v_1\big)^2/v_{1,x}\cdot\langle\tilde{l}_2, [\tilde{r}_1, \tilde{r}_{1,v}]\rangle + v_1\langle\tilde{l}_2, \tilde{r}_{1,vv}\rangle\Big\} \\
& + \Big[v_{1,x}w_1 - w_{1,x}w_1\Big]\Big\{\theta'\langle\tilde{l}_2, \tilde{r}_{1,v}\rangle + \theta'\big(2v_{1,x}/v_1 - \lambda_1 + \sigma_1\big)\langle\tilde{l}_2, \tilde{r}_{1,v\sigma}\rangle \\
& \qquad + v_1\theta'\langle\tilde{l}_2, [\tilde{r}_1, \tilde{r}_{1,\sigma}]/v_1\rangle\Big\} \\
& + v_1\Big[v_1\big(w_1/v_1\big)^2_x\Big]\Big\{\theta''\langle\tilde{l}_2, \tilde{r}_{1,\sigma}/v_1\rangle + \big(\theta'\big)^2\langle\tilde{l}_2, \tilde{r}_{1,\sigma\sigma}/v_1\rangle\Big\} \\
& + \Big[v_1 v_2\Big]\Big\{\big(\lambda_2 - \lambda_1\big)\langle\tilde{l}_2, r_2 \bullet \tilde{r}_1\rangle + v_1\langle\tilde{l}_2, [\tilde{r}_1, r_2 \bullet \tilde{r}_1]\rangle + v_2\langle\tilde{l}_2, r_2 \bullet \big(r_2 \bullet \tilde{r}_1\big)\rangle \\
& \qquad + v_1\big(\sigma_1 - \lambda_1\big)\langle\tilde{l}_2, r_2 \bullet \tilde{r}_{1,v}\rangle + 2\theta' w_{1,x}\langle\tilde{l}_2, r_2 \bullet \tilde{r}_{1,\sigma}/v_1\rangle\Big\} \\
& + \Big[v_{1,x}v_2\Big]\Big\{2\langle\tilde{l}_2, r_2 \bullet \tilde{r}_1\rangle + v_1\langle\tilde{l}_2, r_2 \bullet \tilde{r}_{1,v}\rangle - 2\theta'\Big(w_1/v_1\Big)\langle\tilde{l}_2, r_2 \bullet \tilde{r}_{1,\sigma}\rangle\Big\} \\
= {} & \phi_2(t,x).
\end{aligned}$$

The idea of the proof is the following. Let T be the first time such that

$$\int_0^T \int_{\mathbb{R}} \big|\phi_2(t,x)\big| dxdt = \hat{C}\big(2\delta_0\big)^2,$$

where $\hat{C}$ is a big constant. Then one can show that $\|v_2(t)\|_{L^1} < 2\delta_0$ for $t \in [0,T)$ and $\|v_2(T)\|_{L^1} = 2\delta_0$ if $\delta_0 \leq 1/(4\hat{C})$. This implies $\|u_x(t)\|_{L^1} \leq 4\delta_0$, hence the estimates (2.11) hold for $t \in [0,T]$. We prove that if $\|v_i(t)\|_{L^1} \leq 2\delta_0$ in $[0,T]$, then we have the estimate

$$\int_0^T \int_{\mathbb{R}} \big|\phi_2(t,x)\big| dxdt < \hat{C}\big(2\delta_0\big)^2,$$

yielding a contradiction.

From the estimates in Section 3 it follows

$$\big|\phi_2(t,x)\big| \leq \mathcal{O}(1)\Big\{ \Big|v_{1,x}\big(w_1 + \sigma_1 v_1\big)\Big| + \Big|w_{1,x}v_1 - v_{1,x}w_1\Big| + \Big|v_1^2\big(w_1/v_1\big)_x^2\Big| + \Big|v_1 v_2\Big| + \Big|v_{1,x}v_2\Big|\Big\}. \tag{5.13}$$

where we used the relations (3.11), (5.6).

6. BV estimates

In this section we will prove uniform BV bounds for the solution to (3.1). Toward this result, we introduce three different functionals, which bound some of the terms on the right hand side of (5.12).

Let u_1 be a solution to the scalar conservation law

$$u_{1,t} + f(u_1)_x - u_{1,xx} = 0.$$

Call $\lambda_1 \doteq f'$. At any fixed time t, consider the curve in the plane $x \mapsto \gamma \doteq \big(u_1,\ u_{1,x} - f'(u_1)u_{1,x}\big)$ whose components are the conserved quantity and the flux, respectively. One checks that γ evolves according to the parabolic equation

$$\gamma_t + \lambda_1(u_1)\gamma_x = \gamma_{xx}$$

Since γ moves in the direction of the curvature, there are two Lyapunov functionals that decrease in time. One is the length of the curve, the other is what we call the "area functional", i.e.

$$Q(\gamma) \doteq \frac{1}{2}\int_{x<x'} \big|\gamma_x(x) \wedge \gamma_x(x')\big|\, dxdx'$$

It is convenient to write these functionals using the variables

$$v_1 \doteq u_{1,x}, \qquad w_1 \doteq u_{1,xx} - f'(u_1)u_{1,x}\,.$$

Area functional: we define

$$Q_1(t) \doteq \frac{1}{2}\iint_{x<y} \Big|v_1(t,x)w_1(t,y) - v_1(t,y)w_1(t,x)\Big|\, dxdy = \mathcal{O}(1)\delta_0^3. \tag{6.1}$$

With the same computations as in [3] one has

$$\frac{dQ_1(t)}{dt} \leq -\int_{\mathbb{R}} \Big|w_{1,x}v_1 - v_{1,x}w_1\Big| dx.$$

In particular we have

$$\int_0^T \int_{\mathbb{R}} \Big|w_{1,x}v_1 - v_{1,x}w_1\Big| dxdt = \mathcal{O}(1)\delta_0^3; \tag{6.2}$$

Length functional:

$$L_1(t) \doteq \int_{\mathbb{R}} \sqrt{v_1^2 + w_1^2}\, dx \leq \big\|v_1\big\|_{L^1} + \big\|w_1\big\|_{L^1} \leq 4\delta_0, \tag{6.3}$$

so that as in it shown in [5] one has

$$\begin{aligned} \frac{dL_1}{dt} &\leq -\int_{\mathbb{R}} \frac{1}{\big(1+\sigma_1^2\big)^{3/2}} \Big|v_1 \sigma_{1,x}^2\Big| dx \leq \frac{1}{\big(1+3\delta_1^2\big)^{3/2}} \int_{|\sigma_1|\leq\delta_1} \Big|v_1\sigma_{1,x}^2\Big| dx \\ &\leq \frac{1}{2}\int_{|\sigma_1|\leq 3\delta_1} \left|v_1\left(\frac{w_1}{v_1}\right)_x^2\right| dx, \end{aligned}$$

because we can assume $\delta_1 \le \frac{1}{3}$. Then we have

$$\int_0^T \int_{|\sigma_1| \le 3\delta_1} \left| v_1^2 \left(\frac{w_1}{v_1} \right)_x^2 \right| dxdt \le 2 \big\| v_{1,x} \big\|_{L^1} \cdot \frac{dL_1}{dt} \le \mathcal{O}(1) \cdot \delta_0^3. \tag{6.4}$$

Transversal Interaction functional: let z_1, z_2 be the solutions of the PDE

$$\begin{cases} z_{1,t} + \big(\lambda_1(t,x) z_1\big)_x - z_{1,xx} &= 0 \\ z_{2,t} + \big(\lambda_2(t,x) z_2\big)_x - z_{2,xx} &= 0 \end{cases}$$

with

$$\inf_{t,x} \lambda_2(t,x) - \sup_{t,x} \lambda_1(t,x) \ge c > 0. \tag{6.5}$$

Consider the functional

$$Q_{12}(t) = \int_{\mathbb{R}} P(x-y) \big|z_1(t,x)\big| \big|z_2(t,y)\big| dxdy, \tag{6.6}$$

where P is the weight function defined by

$$P(x) \doteq \begin{cases} 1/c & x \ge 0 \\ 1/c \cdot \exp\{c/2 \cdot x\} & x < 0. \end{cases}$$

Then, as shown in [4], we have

$$\frac{dQ_{12}(t)}{dt} \le - \int_{\mathbb{R}} \big|v_1(t,x)\big| \big|v_2(t,x)\big| \, dx.$$

This implies the estimate

$$\int_0^T \int_{\mathbb{R}} \big|z_1(t,x) z_2(t,x)\big| dxdt \le \frac{1}{c} \big\|z_1(0)\big\|_{L^1} \big\|z_2(0)\big\|_{L^1}. \tag{6.7}$$

As a corollary of (6.7), we have the following result [4]:

Corollary 6.1. *Assume that $z_1(t,x)$, $z_2(t,x)$ are solutions to*

$$\begin{cases} z_{1,t} + \big(\lambda_1(t,x) z_1\big)_x - z_{1,xx} &= \phi_1(t,x) \\ z_{2,t} + \big(\lambda_2(t,x) z_2\big)_x - z_{2,xx} &= \phi_2(t,x) \end{cases}$$

and that (6.5) *holds. Assume moreover that*

$$\big\|z_i(0)\big\|_{L^1} \le \delta_0, \qquad \int_0^t \int_{\mathbb{R}} \big|\phi_i(t,x)\big| dxdt \le \frac{2}{c} \hat{C} \big(2\delta_0\big)^2 \qquad i = 1,2.$$

Then

$$\int_0^T \int_{\mathbb{R}} \big|z_1(t,x)\big| \big|z_2(t,x)\big| dxdt \le \frac{1}{c} \big(2\delta_0\big)^2.$$

Proof. Let $\Gamma_i(t,x)$, $i = 1,2$ the Green kernel of the equation satisfied by z_i. We can write the solution as

$$z_i(t,x) = \int_{\mathbb{R}} \Gamma_i(t, x-y) z(0,y) dy + \int_0^t \int_{\mathbb{R}} \Gamma_i(t-s, x-y) \phi_i(s,y) dyds.$$

From the estimate

$$\int_0^t \int_{\mathbb{R}} \Gamma_1(t,x) \Gamma_2(t-s, x-y) dxdt \le \frac{1}{c} \qquad \forall 0 \le s \le t, x \in \mathbb{R},$$

it now follows

$$\int_0^t \int_{\mathbb{R}} \Big| z_1(t,x) z_2(t,x) \Big| \le \frac{1}{c} \Big(\delta_0 + 2\hat{C} \big(2\delta_0\big)^2 \Big)^2 \le \frac{1}{c} \big(2\delta_0\big)^2.$$

□

Concerning the last term in (5.12), as in [4], we will prove that this is of higher order w.r.t. the previous one.

Proposition 6.2. *Assume that* $\|v_i\|_{L^1} \le 2\delta_0$, *for* $= 1,2$. *Moreover, let the bounds in* (3.11) *and* (6.5) *hold. Then, for all* $0 \le t \le \bar{t}$, *we have*

$$\int_0^T \int_{\mathbb{R}} \big|v_{1,x} v_2\big| dx \le \mathcal{O}(1)\delta_0^3. \tag{6.8}$$

Proof. Define the quantity

$$\mathcal{I}(\alpha) \doteq \sup_{(\tau,z)\in[0,\alpha]\times\mathbb{R}} \int_0^{\alpha-\tau} \int_{\mathbb{R}} \big|v_{1,x}(t,x)\big|\big|v_2(t+\tau,x+z)\big| dxdt.$$

Assume first that $\alpha - \tau < \hat{t}$, where $\hat{t}$ is a small time satisfying

$$\sqrt{\hat{t}} \le \frac{1}{\hat{C}\delta_0},$$

where $\hat{C}$ is a sufficiently big constant. We write $v_{1,x}$ as

$$\begin{aligned} v_{1,x}(t,x) = &\int_{\mathbb{R}} G(t,x-y) v_{1,x}(0,y) dy - \int_0^t \int_{\mathbb{R}} G_x(s,y)\lambda_1\big(u_1\big) v_{1,x}(t-s,x-y) dy \\ &- \int_0^t \int_{\mathbb{R}} G_x(t,x-y)\lambda_1'\big(u_1\big)\big(v_1(t-s,x-y)\big)^2 dx. \end{aligned}$$

We now compute the following integrals (see [4]):

$$\int_0^{\alpha-\tau} \iint_{\mathbb{R}^2} \Big|v_2(t+\tau,x+z) G(t,x-y) v_{1,x}(0,y)\Big| dxdydt \le \mathcal{O}(1)\delta_0^3,$$

$$\int_0^{\alpha-\tau}\int_0^t \iint_{\mathbb{R}^2} \Big|G_x(s,y)\lambda_1\big(u_1\big) v_{1,x}(t-s,x-y) v_2(t+\tau,x+z)\Big| dxdydsdt \le \frac{1}{8}\mathcal{I}(\alpha),$$

$$\int_0^{\alpha-\tau}\int_0^t \iint_{\mathbb{R}^2} \Big|G_x(s,y)\lambda_1'\big(v_1(t-s,x-y)\big)^2 v_2(t+\tau,x+z)\Big| dxdydsdt \le \mathcal{O}(1)\delta_0^3.$$

If now $\alpha - \tau \ge \hat{t}$, we split the integral as

$$\int_0^{\alpha-\tau} \int_{\mathbb{R}} \Big|v_1(t,x) v_2(t+\tau,x+z)\Big| dxdt = \left\{\int_0^{\hat{t}} + \int_{\hat{t}}^{\alpha-\tau}\right\} \int_{\mathbb{R}} \Big|v_1(t,x) v_2(t+\tau,x+z)\Big| dxdt,$$

and we write $v_{1,x}$ as

$$\begin{aligned} v_{1,x}(t,x) = &\int_{\mathbb{R}} G_x(\hat{t},x-y) v_{1,x}(t-\hat{t},y) dy - \int_0^{\hat{t}} \int_{\mathbb{R}} G_x(s,y)\lambda_1\big(u_1\big) v_{1,x}(t-s,x-y) dy \\ &- \int_0^{\hat{t}} \int_{\mathbb{R}} G_x(t,x-y)\lambda_1'\big(u_1\big)\big(v_1(t-s,x-y)\big)^2 dx. \end{aligned}$$

We now compute the following integrals (see [4]):

$$\int_{\hat{t}}^{\alpha-\tau} \iint_{\mathbb{R}^2} \Big|v_2(t+\tau,x+z) G_x(\hat{t},y) v_1(t-\hat{t},x-y)\Big| dxdydt \le \mathcal{O}(1)\delta_0^3,$$

$$\int_{\hat{t}}^{\alpha-\tau}\int_0^{\hat{t}} \iint_{\mathbb{R}^2} \Big|G_x(s,y)\lambda_1\big(u_1\big) v_{1,x}(t-s,x-y) v_2(t+\tau,x+z)\Big| dxdydsdt \le \frac{1}{8}\mathcal{I}(\alpha),$$

$$\int_{\hat{t}}^{\alpha-\tau}\int_0^{\hat{t}} \iint_{\mathbb{R}^2} \Big|G_x(s,y)\lambda_1'\big(v_1(t-s,x-y)\big)^2 v_2(t+\tau,x+z)\Big| dxdydsdt \le \mathcal{O}(1)\delta_0^3.$$

Using the above estimates we have

$$\mathcal{I}(\alpha) \le \mathcal{O}(1)\delta_0^3 + \frac{1}{4}\mathcal{I}(\alpha),$$

from which (6.8) follows. □

The final estimate is an energy estimate. In general, the energy

$$E(t) \doteq \frac{1}{2}\int_{\mathbb{R}} v_{1,x}^2(t,x)\,dx,$$

does not decay in time: consider for example a travelling wave. However we will show that, on the region where the cutoff function is active, the energy does decay. Indeed, in this region the evolution is dominated by the dissipative effect due to viscosity. We thus expect the same decay estimates valid for the heat equation to hold here.

Consider a scalar viscous conservation law

$$u_{1,t} + f(u_1)_x - u_{1,xx} = 0,$$

and let $v_1 \doteq u_{1,x}$. If $v_{1,x} \geq 2\|\lambda_1\|_{L^\infty}|v_1|$, for example near a local maximum, then the equation for u_1 is

$$u_{1,t} = v_{1,x} - \lambda_1(u_1)v_1 \leq \frac{1}{2}u_{1,xx},$$

which is a heat equation.

We consider then the following equation

$$v_{1,t} + \big(\lambda_1(u_1)v_1\big)_x - v_{1,xx} = 0, \tag{6.9}$$

where u_1 is the integral of v_1. We assume $v_1(0,\cdot) \in L^1$ and smooth. We recall that also the effective flux w_1,

$$w_1 \doteq v_{1,x} - \lambda_1(u)v_1 \tag{6.10}$$

satisfies the same equation of v_1:

$$w_{1,t} + \big(\lambda_1(u)w_1\big)_x - w_{1,xx} = 0.$$

Define now a cut-off function $\hat\theta$,

$$\hat\theta(x) \doteq \begin{cases} 0 & |x| \leq 3\delta_1/5 \\ \text{smooth connection} & 3\delta_1/5 \leq |x| \leq 4\delta_1/5 \\ 1 & |x| \geq 4\delta_1/5 \end{cases} \tag{6.11}$$

We can always assume that $\delta_1|\hat\theta'|$, $\delta_1^2|\hat\theta''| \leq 16$. If we multiply (6.9) by $v_1\theta(w_1/v_1)$ and we integrate by parts, we obtain

$$\int_{\mathbb{R}} \left\{ \left(\frac{v_1^2}{2}\hat\theta\right)_t - \frac{v_1^2}{2}\big(\hat\theta_t + 2\lambda\hat\theta_x - \hat\theta_{xx}\big) + 2v_1v_{1,x}\hat\theta_x + \hat\theta v_{1,x}\big(v_{1,x} - \lambda_1 v_1\big)\right\} dx = 0.$$

We now compute

$$\hat\theta_t + \lambda\hat\theta_x - \hat\theta_{xx} = -\hat\theta''\left(\frac{w_1}{v_1}\right)_x^2 - 2\frac{v_{1,x}}{v^!}\hat\theta_x,$$

Using (5.7), in the regions where $\hat\theta > 0$ we have

$$|v_{1,x}| \geq \frac{3}{5}\delta_1|v_1| - \|\lambda_1\|_{L^\infty}|v_1| \geq 2\|\lambda_1\|_{L^\infty}|v_1|.$$

Therefore, we finally obtain the following energy estimate:

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{2}v_{1,x}^2\hat\theta dx \leq & -\frac{d}{dt}\int_{\mathbb{R}}\left(\frac{v_1^2}{2}\hat\theta\right)dx + \int_{\mathbb{R}}\Big|\hat\theta'\big(3\lambda_1/2 - w_1/v_1\big)\Big|\Big|w_{1,x}v_1 - v_{1,x}w_1\Big|dx \\
& + \int_{\mathbb{R}}\left|\hat\theta''\frac{v_1^2}{2}\left(\frac{w_1}{v_1}\right)_x^2\right|dx \\
\leq & -\frac{d}{dt}\int_{\mathbb{R}}\left(\frac{v_1^2}{2}\hat\theta\right)dx + \mathcal{O}(1)\int_{\mathbb{R}}\Big|w_{1,x}v_1 - v_{1,x}w_1\Big|dx \\
& + \mathcal{O}(1)\int_{|w_1/v_1|\leq\delta_1}\left|v_1^2\left(\frac{w_1}{v_1}\right)_x^2\right|dx.
\end{aligned} \tag{6.12}$$

Note that the two last source terms in the left hand side are already present in the the source of (5.12).

Now we are ready to prove uniform BV bounds. Using energy estimates we obtain

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}}\Big|v_{1,x}(w_1-\sigma_1 v_1)\Big|dxdt &\leq \int_0^t\int_{\mathbb{R}}\Big[v_{1,x}^2+\|\lambda_1\|_{L^1}|v_1v_{1,x}|\Big]dxdt \\
&\leq \left(1+\frac{4\|\lambda_1\|_{L^\infty}}{5\delta_1}\right)\int_0^t\int_{\mathbb{R}}\big(v_{1,x}\big)^2dxdt \\
&\leq \mathcal{O}(1)\left\{\int_{\mathbb{R}}\frac{v_1^2(0,x)}{2}\hat\theta dx+\delta_0^3\right\}\leq \mathcal{O}(1)\delta_0^3.
\end{aligned}
\tag{6.13}
$$

Using the above estimates, we conclude

$$
\int_0^T\int_{\mathbb{R}}\big|\phi_2(t,x)\big|dxdt\leq \mathcal{O}(1)\big(2\delta_0\big)^2\Big\{1+\delta_0+\delta_0^2+\delta_0^4\Big\}<\hat C\big(2\delta_0\big)^2,
$$

provided that δ_0 is sufficiently small and $\hat C$ large enough. Here C_i, $i=1,2,3$ denote suitable constants, depending only on δ_1, C_0 and $\hat C$. In particular, at time $\bar t$, we have

$$
\int_{\mathbb{R}}\big|v_2(t,x)\big|dx<\delta_0+2\hat C\big(2\delta_0\big)^2\leq 2\delta_0,
$$

contradicting the assumption $\|v_2(\bar t)\|_{L^1}=2\delta_0$. Therefore the total variation of the solution remains $\leq 2\delta_0$ for all $t\in\mathbb{R}^+$ and all the estimates proved in the previous sections are valid.

This concludes the proof of the uniform BV bounds.

7. Stability estimates

We now consider the linearized evolution equation for a first order variation h:

$$
h_t+\big(A(u)h\big)_x-h_{xx}=0,
\tag{7.1}
$$

where we recall that

$$
A(u)=\begin{bmatrix}\lambda_1(u_1) & 0\\ g_{u_1}(u_1,u_2) & \lambda_2(u_1,u_2)\end{bmatrix}.
$$

We consider the same decomposition as in (3.10), i.e.

$$
h=h_1\tilde r_1+h_2r_2.
\tag{7.2}
$$

Since (7.7) is linear, using the rescaling $h\mapsto h\delta_0/\|h\|_{L^1}$ we can always assume that the L^1 norm of h is of the order of the L^1 norm of v. We will prove that in this case its L^1 norm can at most be twice the initial value.

The proof relies on the same techniques used for the BV estimate. We write the equations for the components, which will be of the form

$$
\begin{cases}
h_{1,t}+\big(\lambda_1(u)h_1\big)_x-h_{1,xx} &= 0\\
h_{2,t}+\big(\lambda_2(u)h_2\big)_x-h_{2,xx} &= \psi_2(t,x)
\end{cases}
$$

Assume that there exists a first time T such that

$$
\int_0^T\int_{\mathbb{R}}\big|\psi_2(t,x)\big|dxdt=\hat C\big(2\delta_0\big)^2.
$$

As a consequence we have $\|h_i(t)\|_{L^1}<2\delta_0$, $i=1,2$ for $t\in[0,T[$ and moreover $\|h_2(T)\|=2\delta_0$, where we assumed $\delta_0\leq 1/(4\hat C)$. We will prove that $\|h_2(t)\|\leq 2\delta_0$ for all $t\in[0,T]$ implies

$$
\int_0^T\int_{\mathbb{R}}\big|\psi_2(t,x)\big|dxdt<\hat C\big(2\delta_0\big)^2,
$$

reaching a contradiction.

Using the regularity estimates of Section 2, if $\|h_i(t)\|_{L^1}\leq 2\delta_0$, $i=1,2$, we obtain the following estimates for $t\in[0,T]$:

$$
\big\|h_i(t)\big\|_{L^\infty},\big\|h_{i,x}(t)\big\|_{L^1}\leq\mathcal{O}(1)\delta_0^2,\qquad \big\|h_{i,x}(t)\big\|_{L^\infty}\leq\mathcal{O}(1)\delta_0^3.
\tag{7.3}
$$

With similar computations as the ones in Section 5, we derive the equations

$$
h_x=h_{1,x}\tilde r_1+h_1\big(v_1\tilde r_1\bullet\tilde r_1++v_2r_2\bullet\tilde r_1+v_{1,x}\tilde r_{1,v}+\sigma_{1,x}\tilde r_{1,\sigma}\big)+h_{2,x}r_2,
$$

$$\begin{aligned} h_x - Df(u)h = &\ \big(h_{1,x} - \lambda_1 h_1\big)\tilde r_1 + \big(h_{2,x} - \lambda_2 h_2\big) r_2 \\ &+ h_1\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\tilde r_{1,v} + h_1 \sigma_{1,x}\tilde r_{1,\sigma} + h_1 v_2 r_2 \bullet \tilde r_1, \end{aligned}$$

$$\begin{aligned} h_{xx} - \big(Df(u)h\big)_x = &\ \Big(h_{1,xx} - \big(\lambda_1 h_1\big)_x\Big)\tilde r_1 + \Big(h_{2,xx} - \big(\lambda_2 h_2\big)_x\Big) r_2 \\ &+ \big(h_{1,x} - \lambda_1 h_1\big)\big(v_1 \tilde r_1 \bullet \tilde r_1 + v_2 r_2 \bullet \tilde r_1 + v_{1,x}\tilde r_{1,v} + \sigma_{1,x}\tilde r_{1,\sigma}\big) \\ &+ \Big(h_1\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big)_x \tilde r_{1,v} + \Big(h_1\sigma_{1,x}\Big)_x \tilde r_{1,\sigma} \\ &+ h_1\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\big(v_1 \tilde r_1 \bullet \tilde r_{1,v} + v_2 r_2 \bullet \tilde r_{1,v} + v_{1,x}\tilde r_{1,vv} + \sigma_{1,x}\tilde r_{1,v\sigma}\big) \\ &+ h_1\sigma_{1,x}\big(v_1 \tilde r_1 \bullet \tilde r_{1,\sigma} + v_2 r_2 \bullet \tilde r_{1,\sigma} + v_{1,x}\tilde r_{1,v\sigma} + \sigma_{1,x}\tilde r_{1,\sigma\sigma}\big) + \big(h_1 v_2 r_2 \bullet \tilde r_1\big)_x, \end{aligned}$$

$$\begin{aligned} h_t = &\ h_{1,t}\tilde r_1 + h_1\big(u_t \bullet \tilde r_1 + v_{1,t}\tilde r_{1,v} + \sigma_{1,t}\tilde r_{1,\sigma}\big) + h_{2,t} r_2 \\ = &\ h_{1,t}\tilde r_1 + h_{2,t} r_2 + h_1\big(v_{1,x} - \lambda_1 v_1\big)\tilde r_1 \bullet \tilde r_1 + h_1\big(v_{2,x} - \lambda_2 v_2\big) r_2 \bullet \tilde r_1 \\ &+ h_1 v_1\big(v_{1,x} - \lambda_1 v_1 + v_1\sigma_1\big)\tilde r_{1,v} \bullet \tilde r_1 + h_1 v_1 \sigma_{1,x}\tilde r_{1,\sigma} \bullet \tilde r_1 \\ &+ h_1 v_1 v_2 \big(r_2 \bullet \tilde r_1\big) \bullet \tilde r_1 + h_1 v_{1,t}\tilde r_{1,v} + h_1 \sigma_{1,t}\tilde r_{1,\sigma}, \end{aligned}$$

$$\begin{aligned} u_x \bullet A(u)h - h \bullet A(u)u_x = &\ \big(r_2 \bullet A(u)\tilde r_1 - \tilde r_1 \bullet A(u) r_2\big)\big(h_1 v_2 - v_1 h_2\big) \\ = &\ \begin{bmatrix} 0 \\ \partial g/\partial u_2 - \partial\lambda_2/\partial u_1 \end{bmatrix} \big(h_1 v_2 - v_1 h_2\big), \end{aligned}$$

so that finally

$$h_{1,t} + \big(\lambda_1 h_1\big)_x - h_{1,xx} = 0, \tag{7.4}$$

while for the second component we have

(7.5)

$$\begin{aligned} h_{2,t} + \big(\lambda_2 h_2\big)_x - h_{2,xx} = &\ \Big[h_{1,x} v_1 - h_1 v_{1,x}\Big]\big\langle \tilde l_2, \tilde r_1 \bullet \tilde r_1\big\rangle \\ &+ \Big[h_{1,x}\big(v_{1,x} - \lambda_1 v_1 + v_1\sigma_1\big) + v_{1,x}\big(h_{1,x} - \lambda_1 h_1 + h_1\sigma_1\big) + h_1 v_1 \sigma_{1,x}\Big]\big\langle \tilde l_2, \tilde r_{1,v}\big\rangle \\ &+ \Big[\big(h_{1,x} - \lambda_1 h_1\big)\sigma_{1,x} + \big(h_1\sigma_{1,x}\big)_x - h_1\sigma_{1,t}\Big]\big\langle \tilde l_2, \tilde r_{1,\sigma}\big\rangle \\ &+ \Big[h_1 v_1\big(v_{1,x} - \lambda_1 v_1 + v_1\sigma_1\big)\Big]\big\langle \tilde l_2, \big[\tilde r_1, \tilde r_{1,v}\big]\big\rangle \\ &+ \Big[h_1 v_{1,x}\big(v_{1,x} - \lambda_1 v_1 + v_1\sigma_1\big)\Big]\big\langle \tilde l_2, \tilde r_{1,vv}\big\rangle \\ &+ \Big[h_1\sigma_{1,x}\big(2v_{1,x} - \lambda_1 v_1 + v_1\sigma_1\big)\Big]\big\langle \tilde l_2, \tilde r_{1,v\sigma}\big\rangle \\ &+ \Big[v_1 h_1 \sigma_{1,x}\Big]\big\langle \tilde l_2, \big[\tilde r_1, \tilde r_{1,\sigma}\big]\big\rangle + \Big[h_1\big(\sigma_{1,x}\big)^2\Big]\big\langle \tilde l_2, \tilde r_{1,\sigma\sigma}\big\rangle. \\ &+ \Big[\big(\lambda_2 - \lambda_1\big) h_1 v_2 + 2h_{1,x} v_2\Big]\big\langle \tilde l_2, r_2 \bullet \tilde r_1\big\rangle \\ &+ \Big[v_1 h_1 v_2\Big]\big\langle \tilde l_2, \big[\tilde r_1, r_2 \bullet \tilde r_1\big]\big\rangle + \Big[h_1\big(v_2\big)^2\Big]\big\langle \tilde l_2, r_2 \bullet \big(r_2 \bullet \tilde r_1\big)\big\rangle \\ &+ \Big[h_1 v_2\big(2v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big]\big\langle \tilde l_2, r_2 \bullet \tilde r_{1,v}\big\rangle + \Big[2h_1 v_2 \sigma_{1,x}\Big]\big\langle \tilde l_2, r_2 \bullet \tilde r_{1,\sigma}\big\rangle \\ &+ \Big[h_1 v_2 - v_1 h_2\Big]\big\langle \tilde l_2, r_2 \bullet A(u)\tilde r_1 - \tilde r_1 \bullet A(u) r_2\big\rangle \end{aligned}$$

$$\doteq \psi_2(t,x) \tag{7.6}$$

As speed σ_1, we again adopt the choice (5.4).

Remark 7.1. Recalling Remark 5.1, in this case the source is still smooth, but not uniformly bounded in L^∞. In any case the source ψ_2 is still well defined.

As in Section 6, the source terms can be classified as follows:

wrong speed:

$$h_{1,x}\big(w_1 + \sigma_1 v_1\big);$$

change in mass: in this case we have 2 areas, namely

$$h_{1,x}v_1 - v_{1,x}h_1 \quad \text{and} \quad h_{1,x}w_1 - w_{1,x}h_1;$$

change in speed: this has the form

$$h_1 \left[v_1 \left(\frac{w_1}{v_1} \right)^2_x \right];$$

transversal terms: as in [4], they are of the form

$$h_1 v_2 \qquad h_{1,x} v_2.$$

We now collect these terms in the source $\psi_2(t,x)$:

$\langle \tilde{l}_2, \tilde{r}_1 \bullet \tilde{r}_1 \rangle$**:** terms due to the fact that h_1 is not distributed as v_1:

$$\Big[h_{1,x}v_1 - h_1 v_{1,x} \Big];$$

$\langle \tilde{l}_2, \tilde{r}_{1,v} \rangle$**:** wrong speed and change in strength terms:

$$2\Big[h_{1,x}(w_1 + \sigma_1 v_1)\Big] + \big(\lambda_1 - \sigma_1 - \theta' w_1/v_1\big)\Big[h_{1,x}v_1 - h_1 v_{1,x}\Big] + \theta'\Big[h_{1,x}w_1 - h_1 w_{1,x}\Big];$$

$\langle \tilde{l}_2, \tilde{r}_{1,\sigma}/v_1 \rangle$**:** we have the shortening here and a mixed term, which can be reduced easily to the change in speed and change in strength, using the inequality $ab \leq (a^2+b^2)/2$ with $a=1$:

$$\begin{aligned} &2\Big[\big(h_{1,x}v_1 - v_{1,x}h_1\big)\sigma_{1,x}\Big] + \theta'' h_1 \Big[v_1 \big(w_1/v_1\big)^2_x\Big] \leq \\ &\qquad \Big[\big(h_{1,x}v_1 - v_{1,x}h_1\big)\Big] + \Big(\theta'/2 \cdot \big(\big(h_{1,x} - \lambda_1 h_1\big) - h_1 w_1/v_1\big) + \theta'' h_1\Big)\Big[v_1 \big(w_1/v_1\big)^2_x\Big]; \end{aligned}$$

$\langle \tilde{l}_2, \tilde{r}_1 \bullet \tilde{r}_{1,v} - \tilde{r}_{1,v} \bullet \tilde{r}_1 \rangle$**:** with our choice of speed, this is a higher order term w.r.t. the wrong speed:

$$\Big[h_1 v_1 \big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big] = \frac{h_1}{\delta_1}\Big[v_{1,x}\big(w_1 + \sigma_1 v_1\big)\Big];$$

$\langle \tilde{l}_2, \tilde{r}_{1,vv} \rangle$**:** this is a higher order term w.r.t. the wrong speed

$$\Big[h_1 v_{1,x}\big(v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big] = h_1 \Big[v_{1,x}\big(w_1 + \sigma_1 v_1\big)\Big];$$

$\langle \tilde{l}_2, \tilde{r}_{1,v\sigma} \rangle$**:** this term can be rewritten as

$$\begin{aligned} \Big[h_1 \sigma_{1,x}\big(2v_{1,x} - \lambda_1 v_1 + \sigma_1 v_1\big)\Big] &= \Big[\theta'\Big(\big(h_{1,x}w_1 - h_1 w_{1,x}\big) - w_1/v_1\big(h_{1,x}v_1 - h_1 v_{1,x}\big)\Big)\big(2v_{1,x}/v_1 - \lambda_1 + \sigma_1\big)\Big] \\ &= \theta'\big(2v_{1,x}/v_1 - \lambda_1 + \sigma_1\big)\Big[h_{1,x}w_1 - h_1 w_{1,x}\Big] \\ &\quad + \theta' w_1/v_1 \cdot \big(2v_{1,x}/v_1 - \lambda_1 + \sigma_1\big)\Big[h_{1,x}v_1 - h_1 v_{1,x}\Big]; \end{aligned}$$

$\langle \tilde{l}_2, (\tilde{r}_1 \bullet \tilde{r}_{1,\sigma} - \tilde{r}_{1,\sigma} \bullet \tilde{r}_1)/v_1 \rangle$**:** this is a higher order term w.r.t. the change in mass:

$$v_1 \Big[h_1 v_1 \sigma_{1,x}\Big] = h_1 \theta' \Big[v_{1,x}w_1 - v_1 w_{1,x}\Big];$$

$\langle \tilde{l}_2, \tilde{r}_{1,\sigma\sigma}/v_1 \rangle$**:** this is the change in speed:

$$h_1 \Big[v_1 \big(\sigma_{1,x}\big)^2\Big] = h_1 \big(\theta'\big)^2 \Big[v_1 \big(w_1/v_1\big)^2_x\Big].$$

Finally we can collect all the terms and write the equation for the second component (7.5) as

$$
\begin{aligned}
(7.7)\quad h_{2,t} + \big(\lambda_2 h_2\big)_x - h_{2,xx} = & \Big[h_{1,x}\big(w_1 + \sigma_1 v_1\big)\Big]\Big\{2\big\langle \tilde l_2, \tilde r_{1,v}\big\rangle\Big\} \\
& + h_1\Big[v_{1,x}\big(w_1 + \sigma_1 v_1\big)\Big]\Big\{\big\langle \tilde l_2, [\tilde r_1, \tilde r_{1,v}]\big\rangle/\delta_1 + \big\langle \tilde l_2, \tilde r_{1,vv}\big\rangle\Big\} \\
& + \Big[h_{1,x} v_1 - h_{1,x} v_1\Big]\Big\{\big\langle \tilde l_2, \tilde r_1 \bullet \tilde r_1\big\rangle + \big(\lambda_1 - \sigma_1 + \theta' w_1/v_1\big)\big\langle \tilde l_2, \tilde r_{1,v}\big\rangle \\
& \qquad\qquad + \theta' w_1\big(2v_{1,x}/v_1 - \lambda_1 + \sigma_1\big)\big\langle \tilde l_2, \tilde r_{1,v\sigma}\big\rangle/v_1\Big\} \\
& + \Big[h_{1,x} w_1 - h_1 w_{1,x}\Big]\Big\{\theta'\big\langle \tilde l_2, \tilde r_{1,v}\big\rangle + \theta'\big(2v_{1,x}/v_1 - \lambda_1 + \sigma_1\big)\big\langle \tilde l_2, \tilde r_{1,v\sigma}\big\rangle\Big\} \\
& + h_1\Big[w_{1,x} v_1 - v_{1,x} w_1\Big]\Big\{\theta'\big\langle \tilde l_2, [\tilde r_1, \tilde r_{1,\sigma}]\big\rangle\Big\} \\
& + \big(h_{1,x} - \lambda_1 h_1\big)\Big[v_1\big(w_1/v_1\big)_x^2\Big]\Big\{\theta'\big\langle \tilde l_2, \tilde r_{1,\sigma}/v_1\big\rangle\Big\} \\
& + h_1\Big[v_1\big(w_1/v_1\big)_x^2\Big]\Big\{\big(\theta'' - \theta' w_1/2v_1\big)\big\langle \tilde l_2, \tilde r_{1,\sigma}/v_1\big\rangle + \big(\theta'\big)^2\Big\} \\
& + \Big[h_1 v_2\Big]\Big\{\big(\lambda_2 - \lambda_1\big)\big\langle \tilde l_2, r_2 \bullet \tilde r_1\big\rangle + v_1\big\langle \tilde l_2, \big[\tilde r_1, r_2 \bullet \tilde r_1\big] \\
& \qquad + v_2\big\langle \tilde l_2, r_2 \bullet \big(r_2 \bullet \tilde r_1\big)\big\rangle + \big(2w_1 + \lambda_1 + \sigma_1\big)\big\langle \tilde l_2, r_2 \bullet \tilde r_{1,v}\big\rangle \\
& \qquad + \big(w_{1,x} - w_1 v_{1,x}/v_1\big)\big\langle \tilde l_2, r_2 \bullet \tilde r_{1,\sigma}/v_1\big\rangle \\
& \qquad + \big\langle \tilde l_2, r_2 \bullet A(u)\tilde r_1 - \tilde r_1 \bullet A(u) r_2\big\rangle\Big\} \\
& + \Big[2h_{1,x} v_2\Big]\Big\{2\big\langle \tilde l_2, r_2 \bullet \tilde r_1\big\rangle\Big\} + \Big[h_2 v_1\Big]\Big\{\big\langle \tilde l_2, \tilde r_1 \bullet A(u) r_2 - r_2 \bullet A(u)\tilde r_1\big\rangle\Big\} \\
= & \ \psi_2(t,x).
\end{aligned}
$$

Using the regularity estimates (7.3) and after some computations, we obtain the following bound for $\psi_2(t,x)$:

(7.8)

$$
\begin{aligned}
\big|\psi_2(t,x)\big| \le \mathcal{O}(1)\bigg\{ & \Big|h_{1,x}\big(w_1 + \sigma_1 v_1\big)\Big| + \Big|h_{1,x} v_1 - h_{1,x} v_1\Big| + \Big|h_{1,x} w_1 - h_1 w_{1,x}\Big| + \Big|h_1\big(w_{1,x} v_1 - v_{1,x} w_1\big)\Big| \\
& + \Big|\big(h_{1,x} - \lambda_1 h_1\big)\big(v_1\big(w_1/v_1\big)_x^2\big)\Big| + \Big|h_1\big(v_1\big(w_1/v_1\big)_x^2\big)\Big| + \Big|h_1 v_2\Big| + \Big|h_{1,x} v_2\Big| + \Big|h_2 v_1\Big|\bigg\}.
\end{aligned}
$$

In the following section we prove that the L^1 norm of $h(t)$ is bounded by a constant times its initial L^1 norm. First of all, we prove an energy estimate similar to (6.12) for the solution to the parabolic PDE

$$h_{1,t} + \big(\lambda_1(u) h_1\big)_x - h_{1,xx} = 0.$$

Define the function ι as

$$\iota_1(t,x) \doteq h_{1,x}(t,x) - \lambda_1(u_1) h_1(t,x). \tag{7.9}$$

Arguing as in the case of w_1, one proves that

$$\big\|\iota_1(t)\big\|_{L^1} \le \mathcal{O}(1)\delta_0^2, \quad \big\|\iota_1(t)\big\|_{L^\infty}, \big\|\iota_1(t)_x\big\|_{L^1} \le \mathcal{O}(1)\delta_0^3. \tag{7.10}$$

With easy computations one finds that ι_1 satisfies the equation

$$\iota_{1,t} + \big(\lambda_1(u_1)\iota_1\big)_x - \iota_{1,xx} = \lambda_1'\big(v_1 h_{1,x} - v_{1,x} h_1\big).$$

Multiplying by $h_1\hat\theta(\iota_1/h_1)$, where $\hat\theta$ is defined n (6.11), and integrating by parts we obtain

$$\int_{\mathbb{R}} \bigg\{\bigg(\frac{h_1^2}{2}\hat\theta\bigg)_x - \frac{h_1^2}{2}\big(\hat\theta_t + 2\lambda_1\hat\theta_x - \hat\theta_{xx}\big) + v_1 v_{1,x}\hat\theta_x + \hat\theta h_{1,x}\big(h_{1,x} - \lambda_1 h_1\big)\bigg\}\, dx = 0.$$

After some computations we obtain

$$\hat\theta_t + \lambda_1\hat\theta_x - \hat\theta_{xx} = \lambda_1'\frac{\hat\theta'}{h_1}\big(v_1 h_{1,x} - h_{1,x} v_1\big) - \hat\theta''\bigg(\frac{\iota_1}{h_1}\bigg)_x^2 + 2\frac{h_{1,x}}{h_1}\hat\theta_x.$$

Since in the regions where $\hat\theta \neq 0$ we have $|h_{1,x}| \geq 2\|\lambda\|_{L^\infty}|h|$, we conclude that

$$\int_{\mathbb{R}} \frac{1}{2} h_{1,x}^2 \hat\theta dx \leq -\frac{d}{dt}\int_{\mathbb{R}} \left(\frac{h_1^2}{2}\hat\theta\right)_x dx + \int_{\mathbb{R}} \left|\hat\theta'\right|\left|3\lambda_1/2 - \iota_1/h_1\right|\left|h_1\iota_{1,x} - h_{1,x}\iota_1\right| dx$$
$$+ \int_{\mathbb{R}} \left|\hat\theta'\lambda_1' h_1/2\right|\left|v_1 h_{1,x} - v_{1,x}h_1\right| dx + \int_{\mathbb{R}} \left|\theta'' \frac{h_1^2}{2}\left(\frac{\iota_1}{2}\right)_x^2\right| dx$$
$$\leq -\frac{d}{dt}\int_{\mathbb{R}} \left(\frac{h_1^2}{2}\hat\theta\right)_x dx + \mathcal{O}(1)\int_{\mathbb{R}} \left|h_1\iota_{1,x} - h_{1,x}\iota_1\right| dx$$
$$+ \mathcal{O}(1)\delta_0^2 \int_{\mathbb{R}} \left|v_1 h_{1,x} - v_{1,x}h_1\right| dx + \mathcal{O}(1)\int_{|\iota_1/h_1|\leq\delta_1} \left|\left(h_1\right)^2\left(\frac{\iota_1}{h_1}\right)_x^2\right| dx. \tag{7.11}$$

We now introduce some functionals to control the source term ψ_2.

First, we consider the following three **Area functionals**.

$$Q_1^h(t) \doteq \frac{1}{2}\iint_{x<y}\left|h_1(t,x)v_1(t,y) - h_1(t,y)v_1(t,x)\right| dxdy \leq \left\|h_1\right\|_{L^1}\cdot\left\|v_1\right\|_{L^1} \leq \left(2\delta_0\right)^2, \tag{7.12}$$

$$Q_2^h(t) \doteq \frac{1}{2}\iint_{x<y}\left|h_1(t,x)w_1(t,y) - h_1(t,y)w_1(t,x)\right| dxdy \leq \left\|h_1\right\|_{L^1}\cdot\left\|w_1\right\|_{L^1} = \mathcal{O}(1)\delta_0^3, \tag{7.13}$$

$$Q^{h,\iota}(t) \doteq \frac{1}{2}\iint_{x<y}\left|h_1(t,x)\iota_1(t,y) - h_1(t,y)\iota_1(t,x)\right| dxdy \leq \left\|\iota_1\right\|_{L^1}\left\|h_1\right\|_{L^1} = \mathcal{O}(1)\delta_0^3, \tag{7.14}$$

where in the last one we have used (7.10). With the same computations as in [3] one finds

$$\frac{dQ_1^h}{dt} \leq -\int_{\mathbb{R}}\left|h_{1,x}v_1 - h_1 v_{1,x}\right| dx,$$
$$\frac{dQ_2^h}{dt} \leq -\int_{\mathbb{R}}\left|h_{1,x}w_1 - h_1 w_{1,x}\right| dx,$$
$$\frac{dQ^{h,\iota}}{dt} \leq -\int_{\mathbb{R}}\left|h_{1,x}\iota_1 - h_1\iota_{1,x}\right| dx + \int_{\mathbb{R}}\left|\lambda_1' h_1\left(h_{1,x}v_1 - v_{1,x}h_1\right)\right| dx,$$

so that for all $t \leq \bar t$ we have the estimates

$$\int_0^t\int_{\mathbb{R}}\left|h_1 v_{1,x} - h_{1,x}v_1\right| dx \leq \left(2\delta_0\right)^2, \tag{7.15}$$

$$\int_0^t\int_{\mathbb{R}}\left|h_1 w_{1,x} - h_{1,x}w_1\right| dx \leq \mathcal{O}(1)\delta_0^3, \tag{7.16}$$

$$\int_0^t\int_{\mathbb{R}}\left|h_1\iota_{1,x} - h_{1,x}\iota_1\right| dx \leq \mathcal{O}(1)\delta_0^3 + \mathcal{O}(1)\delta_0^4 \leq \mathcal{O}(1)\delta_0^3, \tag{7.17}$$

if δ_0 is sufficiently small.

Next, we introduce the **length functional**

$$L_1^h(t) \doteq \int_{\mathbb{R}}\sqrt{h_1^2 + \iota_1^2}\, dx \leq \left\|h_1\right\|_{L^1} + \left\|\iota_1\right\|_{L^1} \leq 4\delta_0. \tag{7.18}$$

As in Section 6, we have

$$\frac{dL_1^h}{dt} \leq -\int_{\mathbb{R}}\frac{1}{1+\left(h_1/v_1\right)^2}\left|h_1\left(\frac{\iota_1}{h_1}\right)_x^2\right| dx + \int_{\mathbb{R}}\left|\lambda_1'\left(h_{1,x}v_1 - v_{1,x}h_1\right)\right| dx$$
$$\leq \frac{1}{2}\int_{|\iota_1/h_1|\leq\delta_1}\left|h_1\left(\frac{\iota_1}{h_1}\right)_x^2\right| dx + \hat C\int_{\mathbb{R}}\left|h_{1,x}v_1 - v_{1,x}h_1\right| dx.$$

This yields the estimate

$$\int_0^t\int_{|\iota_1/h_1|\leq\delta_1}\left|h_1^2\left(\frac{\iota_1}{h_1}\right)_x^2\right| dx \leq \mathcal{O}(1)\delta_0^3, \tag{7.19}$$

if δ_0 is sufficiently small.

Using Corollary 6.1 and Proposition 6.2, the transversal terms can be bounded as follows.

$$\int_0^t\int_{\mathbb{R}}\left|h_1 v_2\right|dxdt,\ \int_0^t\int_{\mathbb{R}}\left|h_2 v_1\right|dxdt \le \frac{1}{c}\big(2\delta_0\big)^2, \tag{7.20}$$

$$\int_0^t\int_{\mathbb{R}}\left|h_{1,x} v_2\right|dxdt \le \mathcal{O}(1)\delta_0^3.$$

We now evaluate the other terms in the source ψ_2. First we observe that

$$\int_0^t\int_{\mathbb{R}}\left|h_{1,x}\big(w_1+\sigma_1 v_1\big)\right| = \int_0^t\left\{\int_{|\iota_1/h_1|\ge 4\delta_1/5} + \int_{|\iota_1/h_1|\le 4\delta_1/5}\right\}\left|h_{1,x}\big(w_1+\sigma_1 v_1\big)\right|dx = I_1+I_2.$$

In the region I_1, both h_1 and v_1 have a speed much larger than λ_1. Thus, using the inequality $2ab\le a^2+b^2$ and (7.11), we can write

$$I_1 \le \frac{1}{2}\int_0^t\int_{|\iota_1/h_1|\ge 4/5\delta_1}\left|h_{1,x}\right|^2dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}}\left|v_{1,x}+\big(\sigma_1-\lambda_1\big)v_1\right|^2dx \le \mathcal{O}(1)\delta_0^3.$$

In the region I_2, we note that they have a different speed,

$$\left|\frac{\iota_1}{h_1}\right| \le \frac{4}{5}\delta_1 \le \frac{4}{5}\left|\frac{w_1}{v_1}\right|,$$

so that we can write the sequence of inequalities:

$$\left|h_{1,x}v_1 - v_{1,x}h_1\right| = \left|\iota_1 v_1 - w_1 h_1\right| = \left|h_1 v_1\right|\left|\frac{\iota_1}{h_1}-\frac{w_1}{v_1}\right| \ge \frac{1}{5}\left|h_1 w_1\right| \ge \frac{1}{5\delta_1}\left|h_{1,x}w_1\right| \le \frac{1}{5}\left|h_{1,x}v_1\right|.$$

Note that we have used the fact that $|h_{1,x}/h_1|\le\delta_1$, $|w_1/v_1|\ge\delta_1$. We have thus the estimate

$$\int_0^t\int_{I_2}\left|h_{1,x}\big(w_1+\sigma_1 v_1\big)\right|dxdt \le 15\delta_1\int_0^t\int_{\mathbb{R}}\left|h_{1,x}v_1 - v_{1,x}h_1\right|dxdt \le 5\big(2\delta_0\big)^2.$$

Adding all terms, one can prove that, if $\|h^i(0)\|_{L^1}=\delta_0$, $i=1,2$, then

$$\int_0^t\int_{\mathbb{R}}\left|\psi_2(t,x)\right|dxdt \le \mathcal{O}(1)\big(2\delta_0\big)^2\Big(1+\delta_0+\delta_0^2+\delta_0^3\Big) < \hat{C}\big(2\delta_0\big)^2,$$

if δ_0 is sufficiently small, so that we have

$$\big\|h(t)\big\|_{L^1} \le 2\delta_0. \tag{7.22}$$

Using now the rescaling we obtain that for a general perturbation we have

$$\big\|h(t)\big\|_{L^1} \le 2\big\|h(0)\big\|_{L^1}.$$

By a homotopy argument, this establishes the uniform stability of solutions, completing the proof of the theorem.

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