ON THE EULER-LAGRANGE EQUATION FOR A VARIATIONAL PROBLEM

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Abstract. In this paper we prove the existence of a solution in $L^\infty_{\mathrm{loc}}(\Omega)$ to the Euler-Lagrange equation for the variational problem

$$\inf_{\bar u + W_0^{1,\infty}(\Omega)} \int_\Omega (\mathbb{I}_D(\nabla u) + g(u)) dx, \tag{0.1}$$

with D convex closed subset of $\mathbb{R}^n$ with non empty interior. We next show that if D^* is strictly convex, then the Euler-Lagrange equation can be reduced to an ODE along characteristics, and we deduce that the solution to Euler-Lagrange is different from 0 a.e. and satisfies a uniqueness property. Using these results, we prove a conjecture on the existence of variations on vector fields [5].

1. Introduction

We consider the existence of a solution to the Euler-Lagrange equation for the minimization problem

$$\inf \Big\{ g(u), u \in \bar u + W_0^{1,\infty}(\Omega), \nabla u \in D \Big\}, \tag{1.1}$$

where $g : \mathbb{R} \mapsto \mathbb{R}$ strictly monotone increasing and differentiable, Ω open set with compact closure in $\mathbb{R}^n$, and D convex closed subset of $\mathbb{R}^n$. Under the assumption that $\nabla \bar u \in D$ a.e. in Ω, there is a unique solution u to (1.1) and we can actually give an explicit representation of u is terms of a Lax-type formula. The solution is clearly Lipschitz continuous because $\nabla u \in \partial D$ a.e. in Ω.

The Euler-Lagrange equation for (1.1) can be written as

$$\mathrm{div}(\pi(x)) = g'(u(x)), \quad \pi(x) \cdot \nabla u(x) = \max \Big\{ \pi(x) \cdot d, d \in D \Big\} \tag{1.2}$$

where π is a measurable function. The first equation is considered in the distribution sense, and the second relation follows by using the subdifferential to the convex function

$$\mathbb{I}_D(x) = \begin{cases} 0 & x \in D \\ +\infty & x \notin D \end{cases}$$

in the standard formulation of the Euler-Lagrange equations. It means that the vector $\pi(x)$ lies in the convex support cone of ∂D at the point $\nabla u(x)$.

In [6], the authors prove that under the assumption $D = B(0,1)$ (in which case u is basically the solution to the eiconal equation), there is a solution to the Euler-Lagrange equation (1.2), which can be rewritten as

$$\mathrm{div}\big(p(x) \nabla u(x)\big) = g'(u(x)), \quad p \geq 0. \tag{1.3}$$

The main point in the proof is that in the region $\Omega \setminus J$, where J is the singularity set of u, the solution u is $C^{1,1}$, and thus the above equation can be reduced to an ODE for p along the characteristics. We recall that in this case u is locally semi convex, so that ∇u has many properties of monotone functions (see for example [1] for a survey on monotone functions).

Simple examples show that such differentiability properties do not hold for general sets D. To prove the existence of a solution π, we thus obtain some continuity properties of ∇u, which do not depend on the particular structure of D. The basic property follows by the Lax-type representation of u stated in Section 2:

$$u(x) = \max \Big\{ u(\bar x) - |\bar x - x|_{D^*}, \bar x \in \partial\Omega, \alpha x + (1-\alpha)\bar x \in \Omega \ \forall \alpha \in (0,1) \Big\}. \tag{1.4}$$

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Dedicated to Prof. Arrigo Cellina, in the occasion of his 65th birthday.

Define in fact the function

$$x \mapsto \mathcal{D}(x) = \left\{ \frac{\bar{x} - x}{|\bar{x} - x|_{D^*}} : \bar{x} \in \partial\Omega, u(x) = u(\bar{x}) - |\bar{x} - x|_{D^*} \right\} \subset \partial D^*, \tag{1.5}$$

i.e. the set of directions of maximal growth of u. As it is shown also in [6] for the case $D = B(0,1)$, this function is closed graph. We prove that this set valued function has continuity properties similar to the properties of the subdifferential of a convex function.

Using this property and the weak convergence of approximate solutions u_i, we can prove our first result:

Theorem 1.1. *The Euler-Lagrange equation* (1.2) *has a solution for the minimization problem* (1.1).

With only the assumption that D is convex, the solution can in general be very complicated, as one can show by taking particular selections $d(x)$ in the convex cone $\partial D(\nabla u(x))$ at $\nabla u(x)$. A special case is when D^* is strictly convex, because it implies that $\partial D(\nabla u(x)) = d(x)$ is single valued in each differentiability point of u, i.e. almost everywhere. The Euler-Lagrange equation (1.2) thus becomes

$$\operatorname{div}\big(p(x)d(x)\big) = g'(u(x)), \quad d(x) = \partial D(\nabla u(x)) \tag{1.6}$$

and $p \geq 0$ a.e. in Ω. We thus study the vector field $d \in L^\infty$, and prove the second result:

Theorem 1.2. *The divergence of d is a locally bounded measure, whose singular part is non negative. The set E where d is discontinuous is $n-1$ rectifiable, and*

$$\operatorname{div}\, d|_E = (d_2 - d_1) \cdot n(x) d\mathcal{H}^{n-1}\Big|_E, \tag{1.7}$$

where $n(x)$ is the normal to E defined $\mathcal{H}^{n-1}$-a.e..

For a comparison with regularity results for singular sets of solutions to Hamilton-Jacobi equation see [4].

Using a refined divergence formula for d and a decomposition of Ω as in [6], we thus show that any solution to the Euler-Lagrange equation can be rewritten as an ODE along almost everywhere segment $x + td(x)$, and that as a consequence $p(x)$ is greater than 0 outside the singular set J: the main result is then

Theorem 1.3. *The solution $p(x)$ to the Euler-Lagrange* (4.1) *is absolutely continuous on almost every segment $x + td(x)$ and satisfies*

$$\frac{d}{dt} p(x + td(x)) + p(x + td(x))(\operatorname{div}\, d)_{a.c.}(x + td(x)) = g'(u(x + td(x))). \tag{1.8}$$

Conversely, if p is an $L^\infty_{\text{loc}}(\Omega)$ solution of the above equation with initial data $p(a(x)) = 0$, then it is a weak solution of the Euler-Lagrange equation.

We also give an optimal uniqueness result.

As an application, we consider the following conjecture stated in [5]: if $u \in W^{1,\infty}(\Omega)$ with $\nabla u \in D$ a.e., then

(1) either there exists a function $\eta \in W^{1,\infty}_0(\Omega)$ such that $\nabla\eta + \nabla u \in D$;
(2) or there exists a divergence free vector $\pi \in (L^1_{\text{loc}}(\Omega))^n$ such that $\pi \neq 0$ and

$$\pi(x) \cdot \nabla u(x) = \max_{k \in D} \big\{ \pi \cdot k \big\} \tag{1.9}$$

a.e. in Ω.

The ODE formulation of Euler-Lagrange equation yields that 2) $\Longrightarrow$ 1) when D^* is strictly convex. The proof that 1) $\Longrightarrow$ 2) is given in [5], but for completeness we give a shorter proof.

The next sections are organized as follows.

In Section 2 we prove formula (1.4), and in Section 3 we study the regularity of ∇u and $\mathcal{D}$. These results are used in Section 4 to prove the existence of a solution to the Euler-Lagrange equation, thus proving Theorem 1.1.

From Section 5 we restrict to the case D^* strictly convex. In Sections 5, 6 we prove Theorem 1.2, together with a divergence formula for d and a decomposition of Ω into suitable disjoint sets on which the vector field d can be linearized by means of a change of variable. Finally we prove Theorem 1.3 and the application to the Bertone-Cellina conjecture in Section 7.

Some questions remains open.

The most important one is the generalization of the above results to the D^* not strictly convex case. We formulate this question as:

is there a measurable selection $d(x) \in \mathcal{D}(x)$ for which we can prove results similar to Theorems 1.2, 1.3, when D^* is not strictly convex?

It is possible to construct by hand such a vector field d in specific examples, but we do not have an idea of a general procedure to select $d(x) \in \mathcal{D}(x)$.

The second question is if the measure div d has a positive Cantor part. This is in part related to the divergence formula (1.8), and to the SBV regularity for solution of Hamilton-Jacobi equation proved in [3]. The absence of Cantor part in div d can be seen as a kind of SBV regularity, because in our case d is not BV.

2. A singular minimization problem

We consider the following minimization problem

$$\inf_{\bar{u}+W_0^{1,\infty}} \int_\Omega (\mathbb{I}_D(\nabla u) + g(u))dx, \tag{2.1}$$

with $g : \mathbb{R} \mapsto \mathbb{R}$ strictly monotone increasing and differentiable, Ω open set with compact closure in $\mathbb{R}^n$. The function $\mathbb{I}_A$ is the inicative function of a set $A \subset \mathbb{R}^n$,

$$\mathbb{I}_A(x) = \begin{cases} 0 & x \in A \\ +\infty & x \notin A \end{cases} \tag{2.2}$$

Moreover, to have a finite infimum in (2.1), we assume that the function $\bar{u}$ satisfies

$$\nabla \bar{u} \in D. \tag{2.3}$$

As a consequence, the infimum is finite and it is attained.

To avoid degeneracies, in the following we assume that D is a bounded convex closed subset of $\mathbb{R}^n$, with non empty interior, and without loss of generality we suppose that

$$B(0,r) = \Big\{ x \in \mathbb{R}^n, |x| \leq r \Big\} \subset D. \tag{2.4}$$

We then denote the dual convex set D^* by

$$D^* = \Big\{ d \in \mathbb{R}^n : d \cdot \ell \leq 1 \ \forall \ell \in D \Big\}, \tag{2.5}$$

where the scalar product of two vectors $x, y \in \mathbb{R}^n$ is $x \cdot y$. The set D^* is closed, convex and $D^{**} = D$. We will write the support set at $\bar{\ell} \in \partial D$ as

$$\delta D(\bar{\ell}) = \Big\{ d \in D^* : d \cdot \bar{\ell} = \sup_{\ell \in D} d \cdot \ell \Big\}. \tag{2.6}$$

Let $|\cdot|_D$ be the pseudo-norm given by the Minkowski functional

$$|x|_D = \inf \big\{ k \in \mathbb{R} : x \in kD \big\} = \sup \big\{ d \cdot x, d \in D^* \big\}, \tag{2.7}$$

and define the dual pseudo-norm by

$$|x|_{D^*} = \inf \big\{ k \in \mathbb{R} : x \in kD^* \big\} = \sup \big\{ \ell \cdot x, \ell \in D \big\}. \tag{2.8}$$

Note that due to convexity the triangle inequality holds,

$$|x + y|_{D^*} \leq |x|_{D^*} + |y|_{D^*}, \quad x, y \in R^n, \tag{2.9}$$

and that $|\cdot|_D$, $|\cdot|_{D^*}$ are the Legendre transforms of $\mathbb{I}_{D^*}$, $\mathbb{I}_D$ respectively.

In the following, we denote with $\mathcal{H}^{n-1}$ the $n-1$ dimensional Hausdorff measure [7]: for any $\Omega' \subset \Omega$,

$$|\Omega'|_{\mathcal{H}^{n-1}} = \mathcal{H}^{n-1}(\Omega') = \kappa \sup_{\delta > 0} \bigg(\inf \bigg\{ \sum_{i \in I} |\mathrm{diam}(B_i)|^{n-1}, \mathrm{diam}(B_i) \leq \delta, \Omega \subset \bigcup_{i \in I} B_i \bigg\} \bigg), \tag{2.10}$$

where κ is the constant such that $\mathcal{H}^{n-1}$ is equivalent to the Lebesgue measure on $n-1$ dimensional planes:

$$\kappa = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(1+\frac{n-1}{2})}, \quad \Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt.$$

The first proposition is the explicit representation of the solution by a Hopf-Lax type formula.

Proposition 2.1. *The solution of* (2.1) *is given explicitly by*

$$u(x) = \max\Big\{u(\bar{x}) - |\bar{x}-x|_{D^*}, \bar{x} \in \partial\Omega, \alpha x + (1-\alpha)\bar{x} \in \Omega \ \forall \alpha \in (0,1)\Big\}. \tag{2.11}$$

Moreover, u is Lipschitz continuous and $\nabla u \in D$ a.e..

The formula requires that the line connecting x to the boundary point $\bar{x}$ lies inside the domain, i.e. each fixed point $x \in \Omega$ sees only the boundary data in the domain

$$\partial\Omega_x = \Big\{z \in \partial\Omega, \alpha x + (1-\alpha)z \in \Omega \ \forall \alpha \in (0,1)\Big\} \subset \partial\Omega.$$

As we will see from the proof, this follows from the fact that also u is Lipschitz continuous on the boundary $\partial\Omega$. As a remark, observe that this function is also the unique viscosity solution to the Hamilton-Jacobi equation

$$1 - |\nabla u|_{D^*} = 0. \tag{2.12}$$

Proof. From the Lipschitz constraint $\nabla u \in D$, the solution to

$$\inf_{\bar{u}+W_0^{1,\infty}(\Omega)} \int_\Omega (\mathbb{I}_D(\nabla u) + g(u))dx$$

has to be above all functions

$$u(\bar{x}) - |\bar{x}-x|_{D^*}, \quad \alpha x + (1-\alpha)\bar{x} \in \Omega,$$

so that the function defined in (2.11),

$$u(x) = \sup\Big\{u(\bar{x}) - |\bar{x}-x|_{D^*}, \bar{x} \in \partial\Omega, \alpha x + (1-\alpha)\bar{x} \in \Omega \ \forall \alpha \in (0,1)\Big\}. \tag{2.13}$$

gives a lower bound. In fact the cone

$$f(x) = |x|_{D^*}$$

is Lipschitz continuous with derivative in ∂D a.e.. Note that, since we assume Ω to be a bounded set and that the infimum of the functional is not $+\infty$, then u is not $-\infty$ and $u \leq \bar{u}$, i.e. this supremum is not $+\infty$.

Let now $\bar{x}_n \in \partial\Omega$ be a maximizing sequence of (2.13), and define the sequence of directions

$$\xi_n = \frac{\bar{x}_n - x}{|\bar{x}_n - x|_{D^*}} \in \partial D^*.$$

Up to subsequences, we can assume that $\bar{x}_n$ converges to a point $\bar{x}$, to which it corresponds the direction $\bar{\xi} = (\bar{x}-x)/|\bar{x}-x|_{D^*}$. Consider the segment

$$(0,1) \ni \alpha \mapsto \alpha x + (1-\alpha)\bar{x} \in \bar{\Omega},$$

and, starting from x, let $\hat{x}$ be the first point of this segment belonging to $\partial\Omega$. On the direction $\bar{\xi}$, formula (2.13) and the convexity of $|\cdot|_{D^*}$ yield

$$u(\alpha x + (1-\alpha)\bar{x}) = u(x) + \Big|\alpha x + (1-\alpha)\bar{x} - x\Big|_{D^*},$$

so that

$$u(\hat{x}) = u(x) + |\hat{x} - x|_{D^*}.$$

This proves that the supremum is attained. Moreover we obtain that for each $x \in \Omega$ there is at least a direction of maximal growth: it follows that at the points where ∇u exists one has $|\nabla u|_D \geq 1$. By the fact that $\nabla \bar{u} \in D$, and $\bar{u} \geq u$, then it also follows that

$$u(x) \leq u(\bar{x}) + |x - \bar{x}|_{D^*} \tag{2.14}$$

for all $\bar{x} \in \partial\Omega$. It remains to show that the function u defined in (2.11) is Lipschitz with derivative in D.

Given now two point x, y in Ω such that $\alpha x + (1-\alpha) y \in \Omega$, $\alpha \in [0,1]$, let $\bar{x}$, $\bar{y}$ be any two points such that the maximum in (2.11) is assumed for x, y respectively. Let γ be the shortest curve in $\bar{\Omega}$ connecting $\bar{x}$ to y, with

$$\|\gamma\| = \int |\gamma'(s)|_{D^*} ds.$$

Since $\bar{\Omega}$ is compact, this curve exists, but in general is not unique. We assume that the curve γ is parametrized by arc length, with $\gamma(0) = x$.

The curve γ touches the boundary $\partial\Omega$ in some sets $A \subset [0, \|\gamma\|]$, and let $\gamma(\bar{z}) \in A$ be the first point on $\partial\Omega$ of γ starting from y. By the definition of the curve γ and the Lipschitz continuity of u, we have

$$u(\bar{z}) \geq u(\bar{x}) - \int_{\bar{z}}^{\bar{x}} |\gamma'(s)|_{D^*} ds.$$

Clearly from $\bar{z}$ to y, γ is a straight line, so that from the triangle inequality

$$u(y) \geq u(\bar{z}) - |\bar{z} - y|_{D^*} \geq u(\bar{x}) - \|\gamma\| \geq u(\bar{x}) - |\bar{x} - x|_{D^*} - |x - y|_{D^*} = u(x) - |x - y|_{D^*}.$$

In a similar way, considering the curve connecting $\bar{y}$ to x, one can prove

$$u(x) \geq u(y) - |y - x|_{D^*}.$$

It follows that

$$-|y - x|_{D^*} \leq u(x) - u(y) \leq |x - y|_{D^*}.$$

Thus the function u of (2.11) is Lipschitzian, and thus it is the solution.

Finally, if $\nabla u(z)$ does not belong to ∂D for a fixed $z \in \Omega$, then by replacing the solution with

$$\tilde{u}(x) = \min\Big\{ u(x), u(z) + |x - z|_{D^*} - \epsilon \Big\},$$

with ϵ sufficiently small in such a way that $\tilde{u} = u$ for all $|x - z| \geq \delta$, we obtain that u is not the minimum of (2.1). □

In the previous proof we can single out the following general principle, which is well known in Calculus of Variations:

Lemma 2.2. *If u_α is a family of functions such that $\nabla u_\alpha \in K$, K compact set in $\mathbb{R}^n$, then the gradient ∇u of $u = \inf_\alpha u_\alpha$ belongs to the convex envelope of K.*

3. Regularity estimates

In this section, we prove some elementary regularity estimates on u, which follow from the explicit form of the solution (2.11). The idea is to consider the set valued functions

$$\mathcal{B}(x) = \Big\{ \bar{x} \in \partial\Omega : u(x) = u(\bar{x}) - |\bar{x} - x|_{D^*} \Big\} \subset \partial\Omega, \tag{3.1}$$

$$x \mapsto \mathcal{D}(x) = \left\{ \frac{\bar{x} - x}{|\bar{x} - x|_{D^*}}, \bar{x} \in \mathcal{B}(x) \right\} \subset \partial D^*. \tag{3.2}$$

Thus $\mathcal{D}(x)$ is the set of directions where u has the maximal growth in the norm $|\cdot|_{D^*}$. From Proposition 2.1, both sets $\mathcal{B}(x)$, $\mathcal{D}(x)$ are closed not empty subset of $\partial\Omega$, ∂D^*, respectively. The normalization in (3.2) implies that

$$u(x + td) = u(x) + t \tag{3.3}$$

for all $x \in \Omega$, $d \in \mathcal{D}(x)$. We can say that $\mathcal{B}(x)$ is the set where the half lines $x + td(x)$, with $d \in \mathcal{D}(x)$ and $t \geq 0$, intersect $\partial\Omega$.

The following lemma on monotonicity properties of $\mathcal{B}(x)$ follows from the explicit formula of the solution (2.11).

Lemma 3.1. *Under the assumption that Ω is convex, the map $x \mapsto \mathcal{B}(x)$ is D^*-cyclically monotone, i.e. for all $(x_i, b_i) \subset (x_i, \mathcal{B}(x_i))$, it holds*

$$\sum_i |b_i - x_i|_{D^*} \leq \sum_i |b_{i-1} - x_i|_{D^*}. \tag{3.4}$$

The two functions $u|_{\partial\Omega}$ and $-u|_\Omega$ are D^*-conjugate functions: the D^*-superdifferential of $-u$ is $\mathcal{B}(x)$.

We recall that if X, Y are non empty sets and $c : X \times Y \mapsto \mathbb{R}$, a function $u : X \mapsto \mathbb{R}$ is said to be c-concave if

$$u(x) = \inf_{y \in Y} \Big\{ c(x,y) - v(y) \Big\} \tag{3.5}$$

for some function $v : Y \mapsto \mathbb{R}$. The c-superdifferential is the set of points $(x,y) \subset X \times Y$ such that

$$u(z) \leq u(x) + c(z,y) - c(x,y) \tag{3.6}$$

for all z. Two functions u, v are said to be c-conjugate if (3.5) holds and

$$v(y) = \inf_{x \in X} \Big\{ c(x,y) - u(x) \Big\}.$$

A subset Z of $X \times Y$ is c-cyclically monotone if for all $\{x_i, y_i\}_{i=1}^I \subset Z$ the following holds

$$\sum_{i=1}^I c(x_i, y_i) \leq \sum_{i=1}^I c(x_i, y_{i-1}), \quad y_0 = y_I. \tag{3.7}$$

In our setting $c(x,y) = |y - x|_{D^*}$.

A noteworthy case is when $X = Y = \mathbb{R}^n$ and $c(x,y) = |x-y|^2$: in this case $|x|^2$-concave functions are of the form $|x|^2 - \Xi(x)$, with $\Xi : \mathbb{R}^n \mapsto \mathbb{R}^n$ convex.

Proof. From the definition of solution (2.11) we have that for all $b_i \in \mathcal{B}(x_i)$

$$u(x_i) = u(b_i) - |b_i - x_i|_{D^*} \geq u(b_{i-1}) - |b_{i-1} - x_i|_{D^*},$$

so that

$$u(b_i) - u(b_{i-1}) \geq |b_i - x_i|_{D^*} - |b_{i-1} - x_{i-1}|_{D^*}.$$

The first part of lemma follows by adding up the above inequality. The second directly from the definition (3.5), (3.6). □

In general, apart from the $|x|^2$-conjugate functions, it is difficult to give general characterizations of c-conjugate functions. In this particular problem, the explicit formula (2.11) yields some compactness properties of $\mathcal{D}$, very similar to the properties of monotone functions. In Section 6 we show that the discontinuity set of D has similar rectifiability properties of maximal monotone functions. However, ∇u (and the restriction d of $\mathcal{D}$ to the set where it is single valued) is not in general a BV function, see Remark 3.7. In particular d is not quasi monotone.

The first result is a first uniform continuity of the "subdifferential" $\mathcal{D}(x)$.

Proposition 3.2. *The function $\mathcal{D}(x)$ is closed graph: more precisely*

$$\lim_{x \to y} \mathcal{D}(x) = \Big\{ d : \exists\ \mathcal{D}(x_i) \ni d_i(x_i) \to d \Big\} \subset \mathcal{D}(y). \tag{3.8}$$

Moreover the map $x \mapsto \mathcal{D}(x)$ is uniformly continuous in the sense that for all $\Omega' \subset\subset \Omega$, for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mathcal{D}(x) \subset \mathcal{D}(y) + B(0, \epsilon) \tag{3.9}$$

for $x \in y + B(0, \delta)$.

Proof. Fixed the point y, by rescaling we can restrict to the set of points distant 1 from y

$$D^*(y, 1) = \Big\{ x : |x - y|_{D^*} \Big\},$$

and we can assume that $u(y) = 0$. By the explicit formula of solutions, the set $\mathcal{D}(y)$ is given by

$$\mathcal{D}(y) = \Big\{ z - y : |z - y|_{D^*} = 1,\ u(z) = 1 \Big\},$$

so that it follows from Lipschitz continuity that for all ϵ there is a δ such that

$$u(z) < 1 - \epsilon \qquad \forall z : |z - y|_{D^*} = 1, \mathrm{dist}(z, \mathcal{D}(y)) > \delta.$$

We thus have that for all x such that $|x - y|_{D^*} \leq \epsilon/2$

$$u(x) \geq -\epsilon/2 > u(z) - 1 + \epsilon/2.$$

Thus the set $\mathcal{D}(z)$ for such a z has a distance from $\mathcal{D}(y)$ less than $\mathcal{O}(\delta + \epsilon)$. □

The same result can be said for the the function $\mathcal{B}(x)$. Clearly, if the convex set D, the ambient set Ω and the boundary data $\bar{u}|_{\partial\Omega}$ do not have any special structure, we cannot give any uniform continuity estimate of $\mathcal{D}$.

Proposition 3.3. *The set valued functions $\mathcal{D}(x)$, $\mathcal{B}(x)$ are measurable.*

Proof. If $\mathcal{B}(x)$ is measurable, so $\mathcal{D}(x)$ is, since this function is obtained from $\mathcal{B}(x)$ by means of algebraic operations (projecton on ∂D). Take a closed set $\bar{O}$ on the boundary $\partial\Omega$. The measurability of $\mathcal{B}^{-1}(\bar{O})$ is trivial for the function

$$u_0 = \max\Big\{u(\bar{x}) - |\bar{x} - x|_{D^*}, \bar{x} \in \bar{O}, \alpha x + (1-\alpha)\bar{x} \in \Omega \ \forall \alpha \in [0,1)\Big\}.$$

Then, one only observes that

$$\mathcal{B}^{-1}(\bar{O}) = \Big\{x : u_0(x) = u(x)\Big\},$$

where $u(x)$ is the solution to (2.1). □

Clearly ∇u is strictly related to the function $\mathcal{D}(x)$. In fact, by means of the explicit solution, if $d \in \mathcal{D}(x)$, then we have

$$u(x + td) = u(x) + t,$$

and thus ∇u belongs to the face of D which corresponds to the direction d,

$$\nabla u \in \partial D^*(d) = \Big\{\ell \in D : \ell \cdot d = 1\Big\}. \tag{3.10}$$

Equivalently,

$$\mathcal{D}(x) \subset \partial D(\nabla u) = \Big\{d \in D^* : d \cdot \nabla u = 1\Big\}. \tag{3.11}$$

Thus, if $\mathcal{D}(x)$ contains at least two vectors not belonging to the same extremal face of D^*, then u is not differentiable in x. We thus have proved that

Lemma 3.4. *If u is differentiable in x, then $\mathcal{D}(x)$ is contained in an extremal subset of D^*.*

The converse does not hold, because we do not assume any special regularity on D, D^*, and it can be seen with the example

$$\Omega = B(0,1), \quad u(x) = -\max_{i=1,\dots,n} |e_i \cdot x|,$$

with e_i the unit vector on the i direction.

Assume now D strictly convex. From Proposition 3.3 and the above lemma it follows that

Corollary 3.5. *If D is strictly convex, the functions $x \mapsto \nabla u(x)$ is continuous w.r.t. the inherited topology on the differentiability set of u, i.e. the topology inherited by $\{x \in \Omega : \exists \nabla u(x)\}$ from $(\mathbb{R}^n, |\cdot|_{D^*})$.*

By the above corollary, when D is strictly convex, we can introduce the following multifunction:

$$\Omega \ni x \mapsto \partial u = \bigcap_{\epsilon > 0} \Big[\text{closure of } \Big\{\nabla u(y) : y \in B(x,y)\Big\}\Big], \tag{3.12}$$

i.e. the set of limits of gradients of u on sequences converging to x. This multifunction is closed graph, and by the above observation it coincides on the differentiability point with ∇u. Clearly Proposition 3.2 holds also for ∂u.

A corollary of Propositions 3.2 and Corollary 3.5 is that ∇u depends smoothly also w.r.t. the boundary data and convolution with a smooth kernel. We skip the proof, which can be obtained by adapting the proof of Proposition 3.2.

Corollary 3.6. *If D is strictly convex, the following holds:*

1. *If $u_i(\partial\Omega) \to u(\partial\Omega)$ in $L^\infty(\partial\Omega)$, then $u_i \to u$ in $W^{1,p} \cap L^\infty(\Omega')$, for all $p \in [1,\infty)$, $\Omega' \subset\subset \Omega$.*
2. *If ρ_ϵ is a convolution kernel, then $\rho_\epsilon * u$ converges to u in $W^{1,p} \cap L^\infty(\Omega')$, for all $p \in [1,\infty)$, $\Omega' \subset\subset \Omega$.*

(3) *If D_i is a sequence of convex compact sets such that $D \subset D_i$ and converging to D in Hausdorff metric, then the solution u_i to*

$$\inf_{\bar{u}+W_0^{1,\infty}(\Omega)} \int_\Omega (\mathbb{I}_{D_i}(\nabla u) + g(u))dx, \tag{3.13}$$

converges to u in $W^{1,p} \cap L^\infty(\Omega')$, for all $p \in [1,\infty)$, $\Omega' \subset\subset \Omega$.

Observe that the choice $D \subset D_i$ assures the existence of a solution to (3.13) with the same initial data of the original problem (2.1).

Remark 3.7. We observe that, apart from the case of uniform quadratic convexity of D or when $\Omega \subset \mathbb{R}^2$, the functions ∇u, d are in general not BV. Consider the convex sets in $\mathbb{R}^3$

$$D = \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_1|,|x_2|,|x_3| \leq 1\big\}, \quad D^* = \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_1|+|x_2|+|x_3| \leq 1\big\}, \tag{3.14}$$

and the function

$$u(x) = \max\Big\{-|x-\xi_i|_{D^*}, -|x-\eta_i|, \xi_i = 2^{-i}e_3, \eta_i = e_1+e_2+2^{-i}e_3, i \in \mathbb{N}\Big\}. \tag{3.15}$$

Clearly we have that the set where d jumps contains

$$\Big\{x_3 = 3\cdot 2^{-i-2}, x_1 \geq 1, x_2 \leq 0\Big\},$$

and since the jump of d on this set is of order 1, d is not BV.

As an example of non strictly convex D for which Corollary 3.6 does not hold, we can consider the set

$$D = \big\{x = (x_1,x_2) \in \mathbb{R}^2 : |x_1|,|x_2| \leq 1\big\}, \quad D^* = \big\{y = (y_1,y_3) \in \mathbb{R}^2 : |y_1|+|y_2| \leq 1\big\},$$

with the boundary data

$$u(x,x) = \frac{1}{2}x^2\sin(1/x).$$

In this case the derivative of u oscillates between $(-3/4, 3/4)$ for x close to 0, and one can check that Corollary 3.5 and part (2) of Corollary 3.6 are false.

3.1. **The strictly convex case.** Let D^* be strictly convex (so that D is differentiable), and x a differentiability point of u. From (3.11) it follows that $\mathcal{D}(x)$ reduces to a single vector, and thus the minimum in (2.11) is assumed in a single point on the boundary $\partial\Omega$, i.e. $\mathcal{B}$ is single valued in each differentiability point.

We can thus consider the set of lines

$$\Sigma(x) = \bigcup_{d\in\mathcal{D}(x)} \Big\{x+td : t \in \mathbb{R}, u(x+td) = u(x)+t\Big\}. \tag{3.16}$$

The set $\mathcal{B}(x)$ is the set of end points for $t \geq 0$, while by considering the end points for $t \leq 0$, we define the function

$$a(x) = \Big\{x - td : t \geq 0, d \in \mathcal{D}, u(x-td) = u(x)-t, u(x-(t+\epsilon)d) > u(x)-(t+\epsilon)\ \forall\epsilon > 0\Big\}. \tag{3.17}$$

From the strict convexity of D^*, it follows in fact that $a(x)$ is single valued: if $\mathcal{D}(x)$ contains two different directions, then $a(x) = x$. This is clearly not the case when D^* is not strictly convex.

As a corollary of the explicit form of the solution and the above definitions, we have

Corollary 3.8. *If D^* is strictly convex, the function $a(x)$ is single valued and, on the differentiability set of u, the functions $\mathcal{B}(x)$, $\mathcal{D}(x)$ are single valued.*

Moreover, the solution u can be written as

$$\begin{aligned} u(x) = {} & \min\Big\{u(\bar{x}) + |x-\bar{x}|_{D^*}, \bar{x} \in \bigcup_{y\in\Omega} a(y), \alpha x + (1-\alpha)\bar{x} \in \Omega\ \forall \alpha \in [0,1)\Big\}, \\ u(x) = {} & \max\Big\{u(\bar{x}) - |\bar{x}-x|_{D^*}, \bar{x} \in \bigcup_{y\in\Omega} \mathcal{B}(y), \alpha x + (1-\alpha)\bar{x} \in \Omega\ \forall \alpha \in [0,1)\Big\}. \end{aligned} \tag{3.18}$$

In the set where $\mathcal{B}(x)$, $\mathcal{D}$ are single valued, we will use the notation

$$\mathcal{B}(x)\Big|_S = b(x), \quad \mathcal{D}(x)\Big|_S = d(x). \tag{3.19}$$

We note moreover that the function

$$\mathbb{R}^n \ni x \mapsto \nabla |x|_D \subset \delta D^*(x) \tag{3.20}$$

is monotone, because it is the derivative of the convex function $|\cdot|_D$. In particular $\nabla u(x) \mapsto d(x) = \delta D(\nabla u)$ satisfies

$$(d(x) - d(y)) \cdot (\nabla u(x) - \nabla u(y)) \geq 0$$

in each differentiability point of u. However, unless $D = B(0,1)$, the function d is not quasi monotone. In the following example, we prove that it is not even BV.

Example 3.9. The idea is to perturb the set D of Remark 3.7 in such a way that D^* is strictly convex, but $d \notin BV$. Consider a strictly convex set D^*, symmetric w.r.t. the origin $(x_1, x_2, x_3) = (0,0,0)$, such that near the direction $e_1 = (1,0,0)$ the norm $|\cdot|_{D^*}$ can be written as

$$|(x_1,x_2,x_3)|_{D^*} = |x_3| + \sqrt[N]{x_1^N + x_2^N + x_3^N}, \quad (x_1,x_2,x_3) \in \bigcup_{\alpha \geq 0} B(e_1, 1/3).$$

The data are

$$u(x_1,x_2,x_3) = 0 \quad \forall x \in \big\{x_1 = 0, x_2 = (-1)^i/\sqrt{i}, x_3 = i^{-2}\big\}.$$

We study the surfaces delimiting the basins of attraction of the boundary points $(0,(-1)^i/i, 1/i^2)$ at the point $(1,0,0)$: these surfaces can be described as a subset of the intersections

$$\begin{aligned}\left|x_3 - \frac{1}{i^2}\right| &+ \sqrt[N]{x_1^N + \left(x_2 - \frac{(-1)^i}{\sqrt{i}}\right)^N + \left(x_3 - \frac{1}{i^2}\right)^N} \\ &= \left|x_3 - \frac{1}{j^2}\right| + \sqrt[N]{x_1^N + \left(x_2 - \frac{(-1)^j}{\sqrt{j}}\right)^N + \left(x_3 - \frac{1}{j^2}\right)^N}.\end{aligned}$$

for $i \neq j$. These surfaces can be approximated uniformly by the constant functions

$$x_3 = \frac{1}{2}\left(\frac{1}{i^2} + \frac{1}{j^2}\right) + \mathcal{O}(i+j)^{-N/2}$$

in a neighborhood of $(1,0,0)$ with $x \in [3/4, 5/4]$, $y \in [-\min\{\sqrt{i},\sqrt{j}\}, \min\{\sqrt{i},\sqrt{j}\}]$. Thus the jump set of d is uniformly close to the set $z_i = \{x_3 = (i^{-1} + (i+1)^{-1})/2\}$, and the vector d has a jump of order $1/\sqrt{i}$. Since the area of z_i is of order $1/\sqrt{i}$, d is not BV. Observe that for $N \to \infty$ we just recover the set $\{x_1 + |x_3| \leq 1\}$, which is the same used in Remark 3.7.

From the fact that in each differentiability point $\mathcal{D}(x)$ is single valued, by Proposition 3.2 we obtain the analog of Corollaries 3.5 and 3.6:

Proposition 3.10. *The function $x \mapsto d(x)$ is continuous w.r.t. the inherited topology on the differentiability set of u. Moreover, it holds*

1. *If $u_i(\partial\Omega) \to u(\partial\Omega)$ in $L^\infty(\partial\Omega)$, then $d_i(x) \to d(x)$ in $L^p_{\mathrm{loc}}(\Omega)$, for all $p \in [1,\infty)$, where $d_i = \delta D(\nabla u)$.*
2. *If ρ_ϵ is a convolution kernel, then $\rho_\epsilon * d$ converges to d in $L^p_{\mathrm{loc}}(\Omega)$, for all $p \in [1,\infty)$.*
3. *If D_i is a sequence of convex sets converging to D w.r.t. the Hausdorff distance, with D_i^* strictly convex and $D \subset D_i$, then the vector field $d_i(x)$ corresponding to the solution u_i to*

$$\inf_{\bar{u} + W_0^{1,\infty}(\Omega)} \int_\Omega (\mathbb{1}_{D_i}(\nabla u) + g(u))dx, \tag{3.21}$$

converges to the vector field d corresponding to u in $L^p_{\mathrm{loc}}(\Omega)$, for all $p \in [1,\infty)$.

To end this section, we prove this simple semicontinuity result for the function $a(x)$.

Lemma 3.11. *If $x_i \to x$, and $x_i, a(x_i) \in \Omega'$, where Ω' is a convex subset of Ω, then we have*

$$|x - a(x)|_{D^*} \geq \limsup_{i \to \infty} |x_i - a(x_i)|_{D^*}. \tag{3.22}$$

Proof. Since by definition of $a(x_i)$ we have

$$u(x_i) = u(a(x_i)) + |x_i - a(x_i)|_{D^*},$$

by passing to the limit we obtain

$$u(x) = u(a) + |x - a|_{D^*},$$

where a is the limit of a subsequence $a(x_i)$ converging to the supremum. The result follows from the definition of $a(x)$ in (3.17). □

4. Existence of a solution to Euler-Lagrange equation

In this section we prove that a solution to Euler-Lagrange equation can be constructed by weak* compactness. We recall that the Euler-Lagrange equation for (2.1) is

$$\operatorname{div}(\pi(x)) = g'(u(x)), \quad \pi(x) \in \delta D(\nabla u(x)) \subset R^n, \tag{4.1}$$

in the sense of distribution. If D is strictly convex, we can use the function $d(x)$ to rewrite (4.1) as

$$\operatorname{div}(p(x)d(x)) = g'(u(x)), \quad 0 \leq p(x) \in \mathbb{R}. \tag{4.2}$$

We start by showing that in the strictly convex case there is a solution.

Proposition 4.1. *If D^* is strictly convex, then there exists a positive function $p(x)$ in $L^\infty_{\mathrm{loc}}(\Omega)$ satisfying the Euler-Lagrange* (4.2).

Proof. Consider the functions

$$u_I(x) = \max\Big\{u(\bar{x}_i) - |\bar{x}_i - x|_{D^*} : i = 1, \dots, I, \alpha x + (1-\alpha)\bar{x}_i \in \Omega \ \forall \alpha \in (0,1)\Big\}. \tag{4.3}$$

for a dense sequence of points $\{\bar{x}_i\}_{i=1}^\infty$ in $\partial\Omega$ (it suffices that $\{\bar{x}_i\}_{i=1}^\infty$ is dense in $\cup_x \mathcal{B}(x)$).

We can split the set Ω into at most I open regions Ω_i, $i = 1, \dots, I$, which are defined by

$$\Omega_i = \text{interior of } \Big\{x : u_I(x) = u(\bar{x}_i) - |\bar{x}_i - x|_{D^*}\Big\},$$

together with the negligible set

$$\Upsilon_I = \bigcup_{i \neq j}\Big(\bar{\Omega}_i \cap \bar{\Omega}_j\Big) = \Big\{x : \exists i, j, \ i \neq j, u_I(x) = u(\bar{x}_i) - |\bar{x}_i - x|_{D^*} = u(\bar{x}_j) - |\bar{x}_j - x|_{D^*}\Big\}.$$

It follows from the definition that Υ_I is a piecewise smooth hypersurface: in fact, for each point $x \in \Upsilon_I$

$$\Big|\nabla|\bar{x}_i - x|_{D^*} - \nabla|\bar{x}_j - x|_{D^*}\Big| \geq \kappa\left|\frac{\bar{x}_i - x}{|\bar{x}_i - x|_{D^*}} - \frac{\bar{x}_j - x}{|\bar{x}_i - x|_{D^*}}\right|,$$

so that Υ_I is piecewise Lipschitz continuous. The strictly positive contant κ depends on the strict convexity of D^*.

In each open region Ω_i, the function $d_I(x)$ is given by

$$d_I(x) = \frac{x_i - x}{|x_i - x|_{D^*}},$$

and (4.2) can be rewritten as

$$d_I \cdot \nabla p_I + p_I \operatorname{div}\, d_I = g'(u_I),$$

or equivalently

$$\frac{dp_I}{dt} + p_I \operatorname{div}\, d_I = g'(u_I) \tag{4.4}$$

on the line $x + td_I(x)$. This implies that for all $\phi \in C_c(\Omega)$

$$\int_{\Omega_i} p_I(x) d_I(x) \cdot \nabla\phi(x) dx = \int_{\Omega_i} g'(u)\phi(x) dx = \int_\Omega g'(u)\phi(x) dx, \tag{4.5}$$

because the boundary data on Υ_I is 0.

By standard ODE theory we can solve for p_I, with initial data $p_I = 0$ on $\Upsilon_I \cap \Omega_i$ or on the boundary $\partial\Omega$, if the line $x + td_I(x)$ has both end points on the boundary. From the explicit form of d_I, its divergence in locally bounded in each Ω_i, and moreover

$$\operatorname{div}\, d_I(x) \geq -\frac{\varrho}{|x - x_i|} \tag{4.6}$$

for all $x \in \Omega_i$, where $\varrho > 0$ depends only on D^* (for $D = B(0,1)$, $\varrho = n-1$). Integrating (4.4) and taking into account the above estimate and $g' \geq 0$, it follows that p_I is locally bounded:

$$0 \leq p_I(x) \leq \|g'(u)\|_{L^\infty} \exp\left\{ \frac{\varrho \mathrm{diam}(\Omega)}{\mathrm{dist}(x, \partial\Omega)} \right\}. \tag{4.7}$$

Moreover, $p_I d_I$ is locally Lipschitz continuous. From $p_I = 0$ on Υ_I it follows

$$\begin{aligned} \int_\Omega \mathrm{div}(d_I(x)p_I(x))\phi(x)dx &= \sum_{i=1}^I \int_{\Omega_i} \mathrm{div}(d_I(x)p_I(x))\phi(x)dx = \sum_{i=1}^I \int_{\Omega_i} p_I(x)d_I(x) \cdot \nabla\phi(x)dx \\ &= \sum_{i=1}^I \int_{\Omega_i} g'(u)\phi(x)dx = \int_\Omega g'(u)\phi(x)dx. \end{aligned}$$

Thus is a solution to (4.2) for the boundary data $u_I|_{\partial\Omega}$.

The general case follows by taking a weak* converging sequence $p_I \rightharpoonup^* p$ in $L^\infty_{\mathrm{loc}}(\Omega)$ and using the strong convergence of d_I to d stated in Proposition 3.10. □

To show that there is a solution in the general case, we consider a sequence of strictly convex sets D_i converging to D: natural candidates are the sets D_i obtained by the inf-convolution,

$$D_i = D \square B(0,1/i) = \Big\{ x : \exists x_1 \in D, x_2 \in B(0,1/i), x = x_1 + x_2 \Big\}. \tag{4.8}$$

By construction, D_i is smooth and its Legendre transform is

$$|x|_{D_i^*} = |x|_{D^*} + \frac{1}{i}|x|_B. \tag{4.9}$$

We thus have that D_i^* is strictly convex, and $D_i^* \subset D^*$. By computations similar to the proof of Proposition 3.2, it follows that for all $x \in \Omega' \subset\subset \Omega$, $\epsilon > 0$,

$$\mathcal{D}_i(x) \subset \mathcal{D}(x) + B(0,\epsilon), \tag{4.10}$$

if i sufficiently large. In particular it follows that the solution $p_i d_i$ to

$$\mathrm{div}(p(x)d_i(x)) = g'(u_i(x)), \quad p(x) \geq 0, \tag{4.11}$$

is contained in the cone

$$\left\{ \pi \in L^\infty_{\mathrm{loc}} : \pi(x) \in \bigcup_{\alpha \geq 0} \alpha\big(\mathcal{D}(x) + B(0,\epsilon)\big) \text{ a.e.}, \in \Omega' \right\} \tag{4.12}$$

if i is sufficiently large. We can thus prove the following theorem:

Theorem 4.2. *The Euler-Lagrange equation* (4.1) *for the minimization problem* (2.1) *has a solution.*

Proof. Let $n(x)$ be a vector such that

$$n(x) \cdot z \leq 0 \quad \forall z \in \mathcal{D}(x).$$

By (4.10) and the continuity of $\mathcal{D}$, it follows that for any $\epsilon \ll 1$, $\bar{x} \in \Omega$ there is a δ such that

$$n(\bar{x}) \cdot z \leq 2\epsilon \quad \forall z \in \mathcal{D}(y) + B(0,\epsilon),\ y \in B(\bar{x},\delta).$$

Thus, by defining the $(L^1(\Omega))^n$ function

$$\phi(x) = n(\bar{x}) \frac{\chi_{B(\bar{x},\delta)}(x)}{|B(0,\delta)|},$$

we obtain that

$$\int_\Omega p_i(x)d_i(x) \cdot \phi(x)dx \leq 2\|p_i\|_{L^\infty(\Omega')}\epsilon \leq 2C\epsilon,$$

where C depends only on $\mathrm{dist}(\bar{x}, \partial\Omega)$. As a consequence, taking a weak* limit π of the sequence $p_i d_i$,

$$\int_\Omega \pi(x) \cdot \phi(x)dx \leq 2C\epsilon.$$

Since ϵ is arbitrary and the weak limit of a solution is a solution, this implies that

$$\pi(x) \in \bigcup_{\alpha \geq 0} \alpha\mathcal{D}(x)$$

for a.e. $x \in \Omega$, so that the proof is complete. □

5. Analysis of the vector field $d(x)$ in the strictly convex case

In this section we show some basic properties of the vector field d when D^* is strictly convex. The main results are that div d is locally a measure in Ω, and that that the set

$$J = \bigcup_{x \in \Omega} a(x) \tag{5.1}$$

has measure 0. Moreover on the complementary set $\Omega \setminus J$ a divergence formula holds. Finally we obtain a decomposition of $\Omega \setminus J$ into disjoint sets A_k on which we can integrate L^1 functions along characteristics. Observe that in general J is not closed, and its closure has in general positive Lebesgue measure, see Remark 5.2.

Consider the function u_I constructed in (4.3). It is easy to check that if $\Omega' \subset\subset \Omega$, from (4.6) and the monotonicity of δD it follows

$$\text{div } d_I \big|_{\Omega'} + \frac{\varrho}{\text{dist}(\Omega', \partial\Omega)} \geq 0, \tag{5.2}$$

in the sense of distributions. In fact, on the set $\Upsilon_I = J_I \cap \Omega$, where d_I jumps, by construction we have that if $n_I(x)$ is the unit normal vector to J_I in the point x, then a.e.-$\mathcal{H}^{n-1}$

$$n_I(x) = \frac{\nabla u_I(x^+) - \nabla u_I(x^-)}{|\nabla u_I(x^+) - \nabla u_I(x^-)|},$$

where $\nabla u_I(x^+)$, $\nabla u_I(x^-)$ are the limits of ∇u_I on the side $n_I(x) \cdot (y - x) > 0$, $n_I(x) \cdot (y - x) < 0$, respectively. Since $d_I(x) = \delta D(\nabla u_I(x))$ is a strictly monotone function of ∇u_I, then it follows

$$(d_I(x^+) - d_I(x^-)) \cdot n_I(x) = (d_I(x^+) - d_I(x^-)) \cdot \frac{\nabla u_I(x^+) - \nabla u_I(x^-)}{|\nabla u_I(x^+) - \nabla u_I(x^-)|} > 0. \tag{5.3}$$

By passing to the limit $d_I \to d$, we obtain that the same relation holds for d in the strictly convex case.

Proposition 5.1. *The divergence of the vector field $d(x)$ is a positive locally finite Radon measure, satisfying*

$$\text{div } d(x) + \frac{\varrho}{\text{dist}(\Omega', \partial\Omega)} \geq 0 \tag{5.4}$$

for all $x \in \Omega' \subset\subset \Omega$. Moreover, we have the estimate

$$|\text{div } d|(B(x,r)) \leq |\partial B(0,r)| + \frac{2\varrho |B(x,r)|}{\text{dist}(B(x,r), \partial\Omega)}, \quad B(x,r) \subset\subset \Omega. \tag{5.5}$$

Finally, the singular part is strictly positive in Ω.

Proof. The first inequality follows by the convergence of d_I to d in distributions, and the fact that positive definite distributions are positive locally finite Radon measures. The inequality (5.4) implies that the singular part is positive.

It is clear that since $d \in L^\infty(\Omega)$, for a.e. $r \in (0, \infty)$

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{B(x,r+\delta) \setminus B(x,r)} d(y) \cdot \frac{y}{|y|} dy = \int_{\partial B(x,r)} d(y) \cdot \frac{y}{|y|} dy \in L^\infty(\mathbb{R}),$$

so that we can write by taking the limit of the test function $\rho_\delta * B(x,r)$

$$\text{div}(B(x,r)) = \text{div}^+(B(x,r)) - \text{div}^-(B(x,r)) = \int_{\partial B(x,r)} d(y) \cdot \frac{y}{|y|} dy \leq |\partial B(x,r)|$$

for a.e. $r > 0$. From the first estimate (5.4), we have

$$\text{div } d^-(B(x,r)) \leq \frac{\varrho |B(x,r)|}{\text{dist}(B(x,r), \partial\Omega)},$$

so that (5.5) follows. □

Using the fact that d can be also constructed starting from the set $J = \cup_x a(x)$, we obtain an estimate on the Lebesgue absolutely continuous part of div d on the complementary set of J. We will write the decomposition of div d as

$$\text{div } d = (\text{div } d)_{\text{a.c.}} \mathcal{H}^n + (\text{div } d)_{\text{s}}, \tag{5.6}$$

with $\mathcal{H}^n$ coinciding with the Lebesgue measure on $\mathbb{R}^n$.

The following observation clarifies the aim of this section.

Remark 5.2. Since the solution u can be constructed also by the formula

$$u(x) = \max\Big\{ u(\bar{x}) + |x - \bar{x}|_{D^*}, \bar{x} \in \bigcup_{x \in \Omega} a(x), \alpha x + (1-\alpha)\bar{x} \in \Omega \ \forall \alpha \in [0,1) \Big\},$$

by a symmetric argument we can prove that if

$$\Omega' \text{ open}, \ \Omega' \subset\subset \Omega \setminus \Big(\bigcup_{x \in \Omega} a(x) \Big),$$

then the vector field $d(x)$ satisfies (5.4) and

$$\text{div } d(x) - \frac{\varrho}{\text{dist}(\Omega', \cup_x a(x))} \leq 0, \qquad -\mathcal{H}^{n-1}(\Omega') \leq (\text{div } d)_{\text{s}}(\Omega') \leq 0. \tag{5.7}$$

In particular the singular part is 0, and one can then show that d is single valued and thus continuous. The difficulty is clearly that the closure of $J = \cup_x a(x)$ in general is a closed set with positive measure. This shows that J is different from the case considered in [3], where the boundary is smooth and the boundary data regular.

For example, one can consider the construction of a Cantor set on $[0,1]$ with positive measure: for example by removing a sequence of intervals $I_i = a_{ij} + [-5^{-i}, 5^{-i}]$ of length $2 \cdot 5^{-i}$ with the same procedure as for the standard 1/3 Cantor set (i.e. each a_i in the center of the remaining intervals). If the boundary data is then

$$u(x,0) = \min\Big\{ 0, |x - a_{ij}| - \frac{1}{5^i}, i \in \mathbb{N}, j = 0, \dots, 2^{i-1} \Big\}, \tag{5.8}$$

then the singular set J is

$$\Big\{ x = a_{ij}, i \geq 0, 0 \leq j \leq 2^{i-1} \Big\},$$

which is dense in the complementary of

$$\bigcup_{ij} \Big(a_{ij} + [-5^{-i}, 5^{-i}] \Big).$$

The measure of this set is 2/3, which is less than 1. Thus the closure of J has measure 1/3.

Let x be a point in $\Omega \setminus J$, and assume without any loss of generality that $x = 0$ and

$$\Big\{ td(0), t \in (-\epsilon, h + \epsilon') \Big\} \in \Omega \setminus J, \quad d(0) = e_1,$$

where e_1 is the unit vector in the direction of x_1 and $h > 0$. Consider then the subset of $\Omega \setminus J$ defined by

$$S^{\epsilon, h+\epsilon'} = \Big\{ x : x + td(x) \notin J \cup \partial\Omega \ \forall t \in (-\epsilon, h + \epsilon') \Big\}.$$

These are the points in Ω which satisfies

$$u(x + td(x)) = u(x) + t, \qquad \forall t \in [-\epsilon, h + \epsilon'].$$

Lemma 5.3. *The set $S^{\epsilon, h+\epsilon'}$ is compact in*

$$K = \Big\{ x : \text{dist}(x, \partial\Omega) \geq \max\{\epsilon, h + \epsilon'\} \Big\}.$$

We have to consider K as above because Ω may not be convex.

Proof. Let $x_i \in K \cap S^{\epsilon, h+\epsilon'}$, $i \in \mathbb{N}$, be a sequence converging to x, and let $a(x_i)$ be the sequence of end points of x. By Lemma 3.11, it follows that if the limit a of a subsequence of $a(x_i)$ belongs to Ω then $|a - x|_{D^*} \geq \epsilon$. If $a \in \partial\Omega$, then $|a - x|_{D^*} \geq \epsilon$ by the definition of K. The last observation holds for the end point $b(x)$. □

Note that since $\partial\Omega$ is compact, the condition that $x+td(x) \notin \partial\Omega$ can always be satisfied for all $x \in \Omega'$, $\Omega' \subset\subset \Omega$ and t sufficiently small, uniform in Ω'.

Let Z be the set of points in $\{x \cdot e_1 = 0\} \cap S^{\epsilon,h+\epsilon'}$ sufficiently close to $x=0$,

$$Z = S^{\epsilon,h+\epsilon'} \cap \Big\{ x \cdot e_1 = 0, |x| \leq r \Big\}. \tag{5.9}$$

Due to the continuity of $d(x)$, it follows

$$d_1(x) = d(x) \cdot e_1 \geq 1 - \alpha, \tag{5.10}$$

for r sufficiently small and $\alpha \in (0,1)$. We moreover define the compact sets

$$Z(s) = \bigg\{ x + s\frac{d(x)}{d_1(x)}, x \in Z \bigg\}, \quad C(s) = \bigg\{ x + t\frac{d(x)}{d_1(x)}, t \in (0,s), x \in Z \bigg\} \tag{5.11}$$

for $s \in [0,h]$. In the following, we will call the set $C(s)$ a cylinder of base Z. Clearly

$$Z(0) = Z, \quad C(s) = \bigcup_{0<s'<s} Z(s'),$$

and we have have supposed ϵ', $h+\epsilon$ sufficiently small, such that we can apply Lemma 5.3. We observe that by means of Proposition 3.2, the map $x \mapsto x + td(x)$ is a homeomorphism from $Z(s)$ to $Z(s+t)$, if $0 \leq s, s+t \leq h$.

The first result is an estimate of the dilatation of $Z(s)$ compared with $Z = Z(0)$. The key part of the proof is suggested by Remark 5.2.

Lemma 5.4. *If $|Z|_{\mathcal{H}^{n-1}}$, $|Z(s)|_{\mathcal{H}^{n-1}}$ are the $\mathcal{H}^{n-1}$ measures of Z, $Z(s)$ respectively, then*

$$(1-\alpha)|Z|_{\mathcal{H}^{n-1}} \exp\bigg\{ -\frac{\varrho}{(1-\alpha)\epsilon'} s \bigg\} \leq |Z(s)|_{\mathcal{H}^{n-1}} \leq \frac{|Z|_{\mathcal{H}^{n-1}}}{1-\alpha} \exp\bigg\{ \frac{\varrho}{(1-\alpha)\epsilon} s \bigg\}, \tag{5.12}$$

for $s \in [0,h]$, and α is the constant of (5.10).

Proof. Fixed Z, the sets $Z(s)$, $C(s)$ can be constructed in 2 ways. In fact, if we define

$$A = \bigcup_{x \in Z} a(x), \quad B = \bigcup_{x \in Z} b(x),$$

then we have that the two functions

$$\begin{aligned} u_A(x) &= \min\Big\{ u(\bar{x}) + |x - \bar{x}|_{D^*}, \bar{x} \in A, \alpha x + (1-\alpha)\bar{x} \in \Omega \ \forall \alpha \in (0,1) \Big\}, \\ u_B(x) &= \max\Big\{ u(\bar{x}) - |\bar{x} - x|_{D^*}, \bar{x} \in B, \alpha x + (1-\alpha)\bar{x} \in \Omega \ \forall \alpha \in (0,1) \Big\}, \end{aligned} \tag{5.13}$$

coincide with the solution u on $C(h)$. Moreover, by construction, the function $d_A(x)$, $d_B(x)$ defined by

$$d_A(x) = \delta D(\nabla u_A(x)), \quad d_B(x) = \delta D(\nabla u_B(x)),$$

coincide with $d(x)$ on $C(s)$. Finally the sets A, B are at a finite distance from Z, so that we can use Remark 5.2.

Consider a neighborhood O_β of $C(h)$ such that $d(x) \cdot e_1 \geq (1-\beta)/2$ a.e. $x \in O_\beta$, for a fixed $\alpha < \beta < 1$. Let δ be sufficiently small, so that the convolutions

$$d_{A,\delta} = d_A * \rho_\delta, \quad d_{B,\delta} = d_B * \rho_\delta, \quad d_\delta = d * \rho_\delta$$

satisfy

$$(d_{A,\delta})_1 = e_1 \cdot d_{A,\delta} \geq 1 - \beta, \quad (d_{B,\delta})_1 = e_1 \cdot d_{B,\delta} \geq 1 - \beta, \quad (d_\delta)_1 = e_1 \cdot d_\delta \geq 1 - \beta,$$

for all $x \in O_\beta$. Consider first the ODE

$$\frac{dX_A(s',x)}{ds'} = \frac{d_{\delta,A}(X_A(s',x))}{(d_{\delta,A})_1(X_A(s',x))}, \quad X_A(s,x) = x,$$

and construct the sets $Z_{A,\delta} = Z_{A,\delta}(0)$, $C_{A,\delta}(s)$ by

$$Z_{A,\delta}(s') = \Big\{ X_A(s',x), x \in Z(s) \Big\} \cap O_\beta, \qquad C_{A,\delta}(s) = \bigcup_{0<s'<s} Z_{A,\delta}(s'). \tag{5.14}$$

Similar definitions for $Z_{B,\delta}(s)$, $C_{B,\delta}(s)$ using the vector field $d_{B,\delta}$ and the flow of the ODE

$$\frac{dX_B(s',x)}{ds'} = \frac{d_{B,\delta}(X_B(s',x))}{(d_{B,\delta})_1(X_B(s',x))}, \quad X_B(0,x) = x.$$

Since the vector fields $d_{A,\delta}$, $d_{B,\delta}$ are smooth in O_β, then there is no problem in the definitions of $Z_{A,\delta}(s')$, $Z_{B,\delta}(s')$, $0 \le s' \le s$. Moreover, by using the continuity of d stated in Proposition 3.2, for δ sufficiently small $Z_{A,\delta}(s) \subset O_\beta$.

Using the smoothness of $d_{A,\delta}$ and estimate (5.7),

$$\text{div } d_A(\Omega') - \frac{\varrho}{\text{dist}(\Omega', A)} \le 0, \qquad \Omega' \subset\subset \Omega \setminus A,$$

we obtain that by means of $d_{A,\delta}$ and the divergence formula that

$$\begin{aligned}\frac{d}{ds'}\int_{Z_{A,\delta}(s')} (d_{A,\delta})_1(x) d\mathcal{H}^{n-1}(x) &= \int_{Z_{A,\delta}(s')} \text{div } d_{A,\delta} d\mathcal{H}^{n-1}(x) \le \frac{\varrho}{\epsilon - \delta}|Z_{A,\delta}(s')|_{\mathcal{H}^{n-1}} \\ &\le \frac{\varrho}{(1-\beta)(\epsilon-\delta)}\int_{Z_{A,\delta}(s')} (d_{A,\delta})_1(x) d\mathcal{H}^{n-1}(x).\end{aligned} \tag{5.15}$$

It thus follows that

$$\begin{aligned}|Z_{A,\delta}|_{\mathcal{H}^{n-1}} \ge \int_{Z_{A,\delta}} (d_{A,\delta})_1(x) d\mathcal{H}^{n-1}(x) &\ge \exp\left\{-\frac{\varrho}{(1-\beta)(\epsilon-\delta)}s\right\}\int_{Z_{A,\delta}(s)} (d_{A,\delta})_1(x) d\mathcal{H}^{n-1}(x) \\ \ge (1-\beta)\exp\left\{-\frac{\varrho}{(1-\beta)(\epsilon-\delta)}s\right\}|Z_A(s)|_{\mathcal{H}^{n-1}}.\end{aligned}$$

Similarly, the estimate (5.4) of Proposition 5.1 implies the opposite inequality for u_B,

$$\begin{aligned}\frac{d}{ds'}\int_{Z_{B,\delta}(s')} (d_{B,\delta})_1(x) d\mathcal{H}^{n-1}(x) &= \int_{Z_{B,\delta}(s')} \text{div } d_{B,\delta} d\mathcal{H}^{n-1}(x) \ge -\frac{\varrho}{\epsilon' - \delta}|Z_{A,\delta}(s)|_{\mathcal{H}^{n-1}} \\ &\ge -\frac{(1-\beta)\varrho}{\epsilon' - \beta}\int_{Z_{B,\delta}(s')} (d_{B,\delta})_1(x) d\mathcal{H}^{n-1}(x),\end{aligned}$$

so that

$$|Z_{B,\delta}(s)|_{\mathcal{H}^{n-1}} \ge (1-\beta)\exp\left\{-\frac{(1-\beta)\varrho}{(\epsilon'-\delta)}s\right\}|Z|_{\mathcal{H}^{n-1}}.$$

We now use the following lower semicontinuity property of Hausdorff metric: if the Hausdorff distance $\text{dist}(K_i, K) \to 0$, K compact, then

$$\limsup_{i\to\infty} \mu(K_i) \le \mu(K) \tag{5.16}$$

for every localy bounded positive Borel measure μ. In fact, let O_ϵ be an open set such that $K \subset O_\epsilon$ and $\mu(O_\epsilon) \le \mu(K) + \epsilon$. Since K is compact, then $\text{dist}(K, R^n \setminus O_\epsilon) > \epsilon'$, so that $K_i \subset O_\epsilon$ for i sufficiently large.

Passing to the limit as $\delta \to 0$, by the uniform convergence of $d_{A,\delta}$, $d_{B,\delta}$ to d on the set C (Proposition 3.2), it follows that $C_{A,\delta}$, $C_{B,\delta}$ converge to C. Hence we obtain

$$(1-\beta)|Z|_{\mathcal{H}^{n-1}}\exp\left\{-\frac{\varrho}{(1-\beta)\epsilon'}s\right\} \le |Z(s)|_{\mathcal{H}^{n-1}} \le \frac{|Z|_{\mathcal{H}^{n-1}}}{1-\beta}\exp\left\{\frac{\varrho}{(1-\beta)\epsilon}s\right\}.$$

Since $\beta \in (\alpha, 1)$, (5.12) follows. □

An equivalent (and actually more precise) estimate which can be deduced from the proof is

$$e^{-\frac{\varrho s}{(1-\beta)\epsilon'}}\int_{Z_{A,\delta}} (d_{A,\delta})_1(x) d\mathcal{H}^{n-1}(x) \le \int_{Z_{A,\delta}(s)} (d_{A,\delta})_1(x) d\mathcal{H}^{n-1}(x) \le e^{\frac{\varrho s}{(1-\beta)\epsilon'}}\int_{Z_{A,\delta}} (d_{A,\delta})_1(x) d\mathcal{H}^{n-1}(x). \tag{5.17}$$

Note that this estimate is sharp, as one can check with the solution $u = |x|_{D^*}$, with $\Omega = D^*$ and $u|_{\partial\Omega} = 1$.

Using the above lemma we show that the set J is negligible, by using an argument similar to the one in [6]. The key point is that, by letting $\epsilon \to 0$ in (5.12), we obtain a lower estimate on the area of $|Z(s)|$:

$$(1-\alpha)|Z|_{\mathcal{H}^{n-1}}\exp\left\{-\frac{1}{(1-\alpha)\epsilon'}s\right\} \le |Z(s)|_{\mathcal{H}^{n-1}}. \tag{5.18}$$

Proposition 5.5. *The set J has Lebesgue measure* 0.

Proof. Let x be a Lebesgue density point of J. Without any loss of generality, assume that $x = 0$, u is differentiable in $x = 0$ and $d(x) = e_1$. Consider the set $Q_\delta = \{|x_i| \leq \delta, i = 1, \dots, n\}$, and the planes $\{x_1 = s\}$. Due to the continuity of d, we have that for all $\epsilon > 0$ there exists δ such that $\mathcal{D}(x) \in e_1 + B(0, \epsilon)$ for all $x \in Q_\delta$. Due to the Lebesgue density in $x = 0$, it follows that for s on a set of measure $\delta(1-\epsilon)$ in $(0, \delta)$

$$\mathcal{H}^{n-1}\Big(\{x \in Q_\delta : x_1 = s\} \cap J\Big) \geq 2^{n-1}\delta^{n-1}(1-\epsilon). \tag{5.19}$$

For $s \geq \delta/2$, let $Z(s)$ be the set of points in $\{x \in Q_\delta : x_1 = s\}$ such that $a(x) \in \{x \in Q_\delta : x_1 = \bar{s}\}$, where $\bar{s}$ is one of the points satisfying (5.19). From the continuity of d, it follows that the segment $[a(x), x]$ for all points x in the closure of $Z(s)$ in $\{x \in Q_{\delta(1-2\epsilon)} : x_1 = s\}$ have to pass through $\{x \in Q_\delta : x_1 = s\}$.

By passing to the limit $\epsilon \to 0$ in (5.12), we obtain that

$$|Z(s)|_{\mathcal{H}^{n-1}} \geq \frac{1}{C}|Z(\bar{s})|_{\mathcal{H}^{n-1}}, \tag{5.20}$$

with C not depending on δ, for δ sufficiently small. It follows that

$$\frac{2^{n-2}}{C}\delta^n(1-\epsilon) \leq \frac{\delta}{2C}|Z(\bar{s})|_{\mathcal{H}^{n-1}} \leq \int_{\delta/2}^{\delta} |Z(s)|_{\mathcal{H}^{n-1}} dx,$$

and we reach a contradition with (5.19). $\square$

We now improve the result of Lemma 5.4. Consider the vector field $d_\delta(x) = \rho_\delta * d(x)$, and let $X_\delta(t, s, \cdot)$ be the flow of the vector field $d_\delta/d_{\delta,1}$,

$$\frac{dX_\delta(t,s,x)}{dt} = \frac{d_{B,\delta}(X_\delta(t,s,x))}{(d_{B,\delta})_1(X_\delta(t,s,x))}, \quad X_B(s,s,x) = x \in Z(s).$$

As in Lemma 5.5, for δ sufficiently small define the sets $Z_\delta(s)$ and the cylinders $C_\delta(s)$ by

$$Z_\delta(s') = \bigcup_{x \in Z(s)} X_\delta(s', s, x), \quad C_\delta(s) = \bigcup_{0 \leq s' \leq s} Z_\delta(s').$$

and let $Z(s)$, $C(s)$ be the sets defined in (5.11).

Lemma 5.6. *The sets $Z_\delta(s')$, $0 \leq s' \leq s$ converge in $\mathcal{H}^{n-1}$-measure to $Z(s')$. The set $C_\delta(s)$ converges in $L^1(\Omega)$ to $C(s)$.*

Proof. Since $X_\delta(s', s, \cdot)$ is a homeomorphism, we have that $Z_\delta(0)$ is compact and by the continuity of $\mathcal{D}$, $Z_\delta(0)$ converges uniformly to $Z = Z(0)$, so that we conclude by (5.16) in the proof of Lemma 5.4 that

$$|Z(0)|_{\mathcal{H}^{n-1}} \geq \limsup_{\delta \to 0} |Z_\delta(0)|_{\mathcal{H}^{n-1}}. \tag{5.21}$$

The same construction can be performed for the image of $Z(0)$ by $X_\delta(s, 0, \cdot)$, i.e.

$$\mathrm{dist}(X_\delta(s,0)Z(0), Z(s)) \to 0.$$

Assume by contradiction that the two sides of (5.21) differ by $\epsilon > 0$. Let $O(s)$ be a neighborhood of $Z(s)$ such that

$$|O(s) \setminus Z(s)|_{\mathcal{H}^{n-1}} = |O(s)|_{\mathcal{H}^{n-1}} - |Z(s)|_{\mathcal{H}^{n-1}} \leq \epsilon',$$

ϵ' small. Let $O_\delta(0)$ be the image of $O(s)$ by $X(0, s, \cdot)$.

Then it follows that for some constant C, depending only on the distance of $Z(s)$ from the boundary $\partial\Omega$, and for ϵ' sufficiently small,

$$\begin{aligned} |Z(0) \setminus O_\delta(0)|_{\mathcal{H}^{n-1}} &\geq |Z(0)|_{\mathcal{H}^{n-1}} - |O_\delta(0)|_{\mathcal{H}^{n-1}} = |Z(0)|_{\mathcal{H}^{n-1}} - |Z_\delta(0)|_{\mathcal{H}^{n-1}} - |O_\delta(0) \setminus Z_\delta(0)|_{\mathcal{H}^{n-1}} \\ &\geq \epsilon - C|O(s) \setminus Z(s)|_{\mathcal{H}^{n-1}} \geq \epsilon - C\epsilon' \geq \epsilon/2. \end{aligned}$$

This means that a non empty part of $Z(0)$ is outside $O_\delta(0)$, for all δ. As a consequence, the image of $Z(0)$ by $X_\delta(s, 0, \cdot)$ cannot converge to $Z(s)$, because $\mathrm{dist}(\partial O(s), Z(s)) > 0$ ($\partial O(s)$ is taken in the $n-1$ dimensional hyperplane $x_1 = s$). This yields a contradiction.

The above computation implies that $Z_\delta(0)$ converges to $Z(0)$ in $L^1(\{x\cdot e_1=0\},\mathcal{H}^{n-1})$. In fact, fixed a neighborhood $O(0)$ of $Z(0)$ with $|O(0)\Delta Z(0)|_{\mathcal{H}^{n-1}}<\epsilon$, it follows that $Z_\delta(0)\subset O(0)$ and

$$\begin{aligned}|Z(0)\Delta Z_\delta(0)|_{\mathcal{H}^{n-1}} &\leq |O(0)\Delta Z_\delta(0)|_{\mathcal{H}^{n-1}}+|O(0)\Delta Z(0)|_{\mathcal{H}^{n-1}}=|O(0)|_{\mathcal{H}^{n-1}}-|Z_\delta(0)|_{\mathcal{H}^{n-1}}+\epsilon\\ &\leq |Z(0)|_{\mathcal{H}^{n-1}}-|Z_\delta(0)|_{\mathcal{H}^{n-1}}+2\epsilon.\end{aligned}$$

Hence by Fubini also $C_\delta(s)$ converges to $C(s)$ in measure. □

By passing to the limit in the divergence formula for d_δ and using the fact that d is continuous, we obtain

$$\int_{Z(s)} d_1(x)d\mathcal{H}^{n-1}(x)-\int_{Z(0)} d_1(x)d\mathcal{H}^{n-1}(x)=\lim_{\delta\to 0}\bigg(\int_{C_\delta(s)}\text{div } d_\delta(x)dx\bigg). \tag{5.22}$$

Since $C(s)$ is compact and $C_\delta(s)$ converges to $C(s)$ in $L^1(\Omega)$, an easy argument shows that the limit in (5.22) satisfies

$$\int_{C(s)}(\text{div } d)_{\text{a.c.}}dx\leq\lim_{\delta\to 0}\bigg(\int_{C_\delta(s)}\text{div } d_\delta(x)dx\bigg)\leq\int_{C(s)}(\text{div } d)_{\text{a.c.}}dx+(\text{div } d)_{\text{s}}(C(s)),$$

where we used the fact that the singular part of the measure is positive.

To find an explicit form of the r.h.s. of (5.22), let us denote with ω the modulus of continuity of d on $C(h)\subset S^{\epsilon,h+\epsilon}$. In the following we also denote the cylinder contained in $C(h)$ with starting point in $B(\bar{x},r)\cap\{x_1=\bar{x}_1\}\cap S^{\epsilon,h+\epsilon}$ and length $t-s$ by

$$C(\bar{x},r,s,t)=\Big\{x+s'd(x):x\in B(\bar{x},r)\cap\{x_1=\bar{x}_1\}\cap S^{\epsilon,h+\epsilon},x_1=s,s'\in[s,t]\Big\}. \tag{5.23}$$

By means of the uniform continuity, it follows that

$$B(\bar{x}+re_1/2,r-r\omega(r))\cap S^{\epsilon,h+\epsilon}\subset C(\bar{x},r,s,s+r)\subset B(\bar{x}+re_1/2,r+r\omega(r))\cap S^{\epsilon,h+\epsilon},$$

so that for r sufficiently small,

$$B(\bar{x}+re_1/2,r/2)\cap S^{\epsilon,h+\epsilon}\subset C(\bar{x},r,s,s+r)\subset B(\bar{x}+re_1/2,2r)\cap S^{\epsilon,h+\epsilon}. \tag{5.24}$$

From the estimate of the area of $Z(t)$ we have that

$$\big|Z(t)\cap B(te_1,r)\big|_{\mathcal{H}^{n-1}}\geq\big|Z(s)\cap B(se_1,r)\big|_{\mathcal{H}^{n-1}}(1-C|t-s|)(1-|t-s|\omega(r))^{n-1},$$

where C depends only on ϵ. This means that $n-1$ dimensional Lebesgue points of $Z(s)$ are also Lebesgue points of $Z(t)$ for all t. Moreover sets $A\subset C(h)$ of measure 0 remains of measure 0 after the change of coordinates $(t,x)\mapsto x+td(x)$.

From (5.17), passing to the limit $\delta\to 0$ and using the continuity of d and the convergence in measure of Z_δ, we have

$$\bigg|\int_{Z(h)} d_1(x)d\mathcal{H}^{n-1}(x)-\int_{Z(0)} d_1(x)d\mathcal{H}^{n-1}(x)\bigg|\leq|Z(0)|_{\mathcal{H}^{n-1}}\Big(e^{\varrho h/((1-\beta)\epsilon)}-1\Big)\leq C'|C(h)|.$$

It thus follows that repeating the argument for all $C(\bar{x},r,s,s+r)$

$$\lim_{\delta\to 0}\int_{C_\delta(\bar{x},r,s,s+r)}\text{div } d_\delta(x)dx\leq C'|C(\bar{x},r,s,s+r)|<\infty. \tag{5.25}$$

Let $K\subset C(h)$ be a compact set with $|K|\leq\epsilon$, $(\text{div } d)_{\text{s}}(K)>(\text{div } d)_{\text{s}}(C(h))-\epsilon$. By means of (5.24), we can cover it by a finite union of sets $C_i=C(\bar{x}_i,r_i,s_i,s_i+r_i)$, in such a way that

$$\sum_i\Big|C(\bar{x}_i,r_i,s_i,s_i+r_i)\Big|\leq C\epsilon,$$

with C fixed constant depending only on the distance from the distance from A and B. For any $C(\bar{x}_i,r_i,s_i,s_i+r_i)$, let $\tilde{C}_i$ be the set

$$\tilde{C}_i=\Big\{x+td(x),x\in C_i\cap\{x_1=s_i\},t\in[0,h]\setminus(s_i,s_i+r)\Big\}.$$

This means that the set

$$W=C(h)\setminus\bigg(\bigcup_i C_i\bigg)$$

satisfies $(\text{div } d)_{\rm s}(W) \leq \epsilon$. Denoting with ∂W the contact surface of W with $\cup_i C_i$ transverse to d (i.e. with normal $\pm e_1$), one can add finitely many equations (5.22) to obtain that

$$\int_{Z(h)} d_1(x) d\mathcal{H}^{n-1}(x) - \int_{Z(0)} d_1(x) d\mathcal{H}^{n-1}(x) + \int_{\partial W} d_1(x) d\mathcal{H}^{n-1}(x) = \int_{C(h)} (\text{div } d)_{\rm a.c.} dx + \mathcal{O}(\epsilon). \tag{5.26}$$

On the other hand, by using (5.25) it follows that for the set $\cup_i C_i$

$$\left| \int_{\partial W} d_1(x) d\mathcal{H}^{n-1}(x) \right| \leq \sum_i \left| \lim_{\delta \to 0} \int_{C_{i,\delta}} \text{div } d_\delta(x) dx \right| \leq C' \epsilon, \tag{5.27}$$

so that we obtain

$$\int_{Z(h)} d_1(x) \mathcal{H}^{n-1}(x) - \int_Z d_1(x) \mathcal{H}^{n-1}(x) = \int_{C(h)} (\text{div } d)_{\rm a.c.}(x) dx + \mathcal{O}(\epsilon) \tag{5.28}$$

for all $\epsilon > 0$. We have thus proved the first part of the following theorem:

Theorem 5.7. *The Gauss-Green formula holds for the cylinder $C(h) \subset S^{\epsilon, h+\epsilon}$ in the form:*

$$\int_{Z(s)} d_1(x) dx - \int_{Z(0)} d_1(x) dx = \int_{C(s)} (\text{div } d)_{a.c.}(x) dx. \tag{5.29}$$

If $\alpha(s,x)$ is the push forward of the $\mathcal{H}^{n-1}$ measure on $Z(0)$ to $Z(s)$, $0 \leq s \leq h$, then it satisfies

$$\frac{d}{ds} \alpha(s,x) = (\text{div } d)_{a.c.}(x + sd(x)) \alpha(s,x) \tag{5.30}$$

for almost every line $x + sd(x)$.

Proof. The density $\alpha(s,x)$ is bounded by Lemma 5.4 and the maps $(x,s) \mapsto x + sd(x)$ are uniformly continuous on $C(s)$, so that we can write

$$\int_{C(s)} (\text{div } d)_{\rm a.c.}(x) dx = \int_Z \int_0^s (\text{div } d)_{\rm a.c.}(x + td(x)/d_1(x)) \alpha(t,x) dt d\mathcal{H}^{n-1}(x).$$

From the continuity of d, for almost every $x \in Z$ we can write

$$\alpha(s,x) = 1 + \int_0^s \alpha(t,x) (\text{div } d)_{\rm a.c.}(x + td(x)/d_1(x)) dt,$$

which is equivalent to the ODE (5.29) along almost every line $x + sd(x)$. $\square$

As a corollary of the above theorem and Lemma 5.5, it follows that a.e. in $S^{\epsilon,\epsilon'}$

$$(\text{div } d)_{\rm a.c.}(x) \in \left[-\frac{\varrho}{\epsilon'}, \frac{\varrho}{\epsilon} \right]. \tag{5.31}$$

It thus follows from (5.30) that α is bounded and Lipschitz continuous on the line $x + td(x)$ by

$$\alpha(t,x) \leq \varrho \max \left\{ \frac{|b(x) - x|}{\epsilon'}, \frac{|x - a(x)|}{\epsilon} \right\}. \tag{5.32}$$

Define the set S the complementary of J,

$$S = \Omega \setminus J = \bigcup_{i=0}^{\infty} S^{1/i,1/i}. \tag{5.33}$$

Proposition 5.5 yields that S is of full Lebesgue measure, i.e. $|\Omega| = |S|$, and by Proposition 3.10 d is continuous in S.

Following [6], we now introduce a decomposition of Ω into sets A_i as follows. First decompose $S^{1/i,1/i}$ by defining

$$S_1^{1/i,1/i} = \Big\{ x \in S^{1/i,1/i}, d(x) \cdot e_1 \geq 1/2 \Big\},$$
$$S_j^{1/i,1/i} = \Big\{ x \in S^{1/i,1/i}, d(x) \cdot e_j \geq 1/2 \Big\} \setminus \big(S_1^{1/i,1/i} \cup \cdots \cup S_{i-1}^{1/i,1/i} \big),$$

with $j = 1, \dots, n$. Consider $S_j^{1/i,1/i}$ with positive measure: from Fubini's theorem we have that

$$|S_j^{1/i,1/i}| = \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big(S_j^{1/i,1/i} \cap \{x \cdot e_j = z\}\Big) dz.$$

Fix $\bar{z}$ such that the $\mathcal{H}^{n-1}$-measure of the integrand is strictly positive. From Lemma 5.4, it thus follows that the set

$$A_{ij\bar{z}} = \Big\{ y \in (a(x), b(x)) : x \in S_j^{1/i,1/i} \cap \{x \cdot e_j = \bar{z}\}\Big\} \tag{5.34}$$

is measurable and has strictly positive measure. Moreover, we have by using the push forward α of the $\mathcal{H}^{n-1}$-measure on $S_j^{1/i,1/i} \cap \{x_j = \bar{z}\}$ that its measure is

$$|A_{ij\bar{z}}| = \int_{S_i^{1/j,1/j} \cap \{x \cdot e_i = z\}} \bigg(\int_{a(x)}^{b(x)} \alpha(x,t) dt \bigg) d\mathcal{H}^{n-1}(x). \tag{5.35}$$

In fact the above equation holds for all cylinder $C \subset S_j^{1/i,1/i}$, and $A_{ij\bar{z}}$ can be covered by a countable number of disjoint cylinders.

We can thus find a finite disjoint decompositions of S into sets $A_k = A_{i_k j_k \bar{z}_k}$ of the form (5.34), to which there correspond measurable sets $Z_k = \{x_{j_k} = \bar{z}\} \cup S_{j_k}^{1/i_k,1/i_k}$ contained in a $n-1$ dimensional plane transverse to d. By the definition of the sets A_k push forward measure α, we have the following theorem:

Theorem 5.8. *For all functions f in $L^1(\Omega)$ we can write*

$$\int_\Omega f(x) dx = \sum_{k=1}^\infty \int_{Z_k} \bigg(\int_{a(x)}^{b(x)} f(x + td(x)) \alpha(x + td(x)) dt \bigg) d\mathcal{H}^{n-1}(x). \tag{5.36}$$

Proof. Clearly the above formula holds in any cylinder $C \subset S$, because we know that sets of measure 0 remains of measure 0 in the coordinates $(x,t) \mapsto x + td(x)$. The general case follows by the countable covering of S, and the fact that $|\Omega \setminus S| = 0$. □

Note that $f(x + td(x))|_S$ is measurable, but in general it is not in $L^1(S)$. It is the product $f(x + td(x))\alpha(x + td(x))$ which is in $L^1(\Omega)$. Clearly the decomposition of Theorem 5.8 is not unique.

6. Rectifiability of the singular set

In this section we show some rectifiability properties of the set E where $\mathcal{D}$ is multi valued. Clearly E is a strict subset of the singular set J. We begin with this simple lemma.

Lemma 6.1. *The following estimate holds:*

$$0 \le (\text{div } d)_s(\Omega') \le \mathcal{H}^{n-1}(\Omega'), \tag{6.1}$$

with $\Omega' \subset\subset \Omega$.

Proof. From (5.5) is follows that for all δ

$$(\text{div } d)_s(\Omega') \le \bigg(1 + \frac{2\delta}{n \text{dist}(\Omega', \Omega)}\bigg) \inf\bigg\{ \sum_{i \in I} |\partial B(x_i, r_i)|, r_i \le \delta, \Omega' \subset \bigcup_{i \in I} B(x_i, r_i)\bigg\},$$

so that, by passing to the limit $\delta \to 0$ and observing that the spherical Hausdorff measure is bigger that the Hausdorff measure, we recover (6.1). □

Let J_N^m be the set of points in J such that

$$J_N^m = \Big\{ x \in J : \exists d_i \in \mathcal{D}(x), i = 1, \dots, N, \text{dist}\big(d_i, \text{span}\{d_j, j \neq i\}\big) \ge 1/m \Big\}. \tag{6.2}$$

These are the points x where in $\mathcal{D}(x)$ there are at least N directions uniformly linear independent, $1/m$ being the uniform separation. We can also say that, if D is also strictly convex, the set J_N^m the set valued function ∂u defined in (3.12) has at least N linearly independent values, separated uniformly by κ/m, where κ depends on the strict convexity of D^*. From the continuity of $\mathcal{D}$ proved in Proposition 3.2, it follows that J^m_N is locally compact in Ω, and that the complementary of

$$E = \bigcup_{m \in \mathbb{N}} J^m_2 \tag{6.3}$$

in J is the set of J where u is differentiable.

We begin with a simple geometrical lemma. Given a compact convex set K with $0 \notin K$, define

$$C^+(K) = \Big\{ x \in \mathbb{R}^n : x \cdot \ell > 0 \, \forall \ell \in K \Big\}, \quad C(K) = C^+(K) \cup C^+(-K) = \Big\{ x \in \mathbb{R}^n : x \cdot \ell \neq 0 \, \forall \ell \in K \Big\}. \tag{6.4}$$

Clearly $C(K)$ is an open non empty cone, because K is convex and compact and does not contains the origin.

Lemma 6.2. *Let $d_1, d_2 \in \mathcal{D}(x_0)$, $d_1 \neq d_2$, and assume that x_i is a sequence of points converging to x_0 such that there exists $d_i \in \mathcal{D}(x_i)$ with $d_i \to d_1$ (this is always true up to subsequences). Then, if Y is the contact set of $\frac{x_i - x}{|x_i - x|_{D^*}}$,*

$$Y \cap C^+(\delta D^*(d_2) - \delta D^*(d_1)) = \emptyset, \tag{6.5}$$

where $\delta D^*(x)$ is the support cone of D^* at x.

We recall that the contact set of a sequence a_i is the closed set of limits of all subsequences.

Proof. Without any loss of generality, assume $x_0 = 0$, $u(x_0) = 0$ and consider the set $D^* = \{|x|_{D^*} = 1\}$ of D^*-radius 1, so that $u(d_1) = u(d_2) = 1$. Moreover $u \leq 1$ on $\partial D^* \setminus \mathcal{D}(x)$.

Since d_i is close to d_1, then for some $\epsilon > 0$ it holds

$$1 - |y - x_i|_{D^*} \geq 1 - |d_2 - x_i|_{D^*},$$

where y is the closest point to x in the ϵ neighborhood of d_1.

We can approximate the two distances as

$$|d_2 - x_i|_{D^*} = -\ell_i(d_2) x_i + o(x_i), \quad |y - x|_{D^*} = -\ell_i(y) x_i + o(x_i), \tag{6.6}$$

with $\ell_i(d_2) \in \delta D^*(d_2)$, $\ell_i(y) \in \delta D^*(y)$. From continuity of $\delta D^*(d)$, it follows that $\mathcal{D}(y) \subset \mathcal{D}(d_1) + B(0, \delta)$, with $\delta \to 0$ as $y \to d_1$, so that $\delta D^*(y) \cap \delta D^*(d_2) = \emptyset$ for ϵ sufficiently small. It thus follows that

$$x_i \notin C^+(\delta D^*(d_2) - \delta D^*(y)).$$

Using again the continuity of δD^* as the derivative of a convex function, we obtain (6.5). □

Remark 6.3. The set $C^+(K)$ can be written by means of Legendre transform. In fact, consider the convex cone

$$\tilde{C} = \bigcup_{\alpha > 0} \alpha K.$$

Since K is convex and does not contain the origin, by Hahn-Banach theorem it is contained in a half space, which we can suppose to be $\{x_1 > 0\}$. Denoting $\tilde{x}$ the $n-1$ dimensional vector of the decomposition $(x_1, \tilde{x})$, we can assume that $\tilde{C}$ is written as

$$\tilde{C} = \Big\{ x : x_1 \geq |\tilde{x}|_{\tilde{D}} \Big\},$$

for some convex set $\tilde{D}$ (which is precisely the radial projection of K on $\{x_1 = 1\}$). The Legendre transform of $\tilde{D}$ is

$$f(\tilde{y}) = \sup_{\tilde{x} \in \tilde{D}} \{\tilde{x} \cdot \tilde{y}\}.$$

From the definition of $C^+(K)$, we have

$$0 < (y_1, \tilde{y}) \cdot (1, \tilde{x}) \leq y_1 + \tilde{y} \cdot \tilde{x},$$

which implies that $C^+(K)$ is the set $\{y_1 > f(y)\}$.

Up to subsequences x_{i_j}, we can assume that the vectors $\ell_i(d_2)$, $\ell_i(y)$ converge to some limits $\ell(d_2)$, $\ell(d_1)$, so that it follows that the sequence of x_{i_j} asymptotically belongs to the half space $\{x \cdot (\ell(d_2) - \ell(d_1)) \leq 0\}$.

If we have a sequence $x_i \to x$ in J^m_N, it follows that we can extract a subsequence such that $(x_i - x)/|x_i - x|$ and the vectors $\ell_i(d_k)$ defined by

$$|d_k - x_i|_{D^*} = u(x) - \ell_{ik}(x_i - x) + o(|x_i - x|_{D^*}),$$

converge to some vectors e, $\ell(d_k)$, where d_k, $k = 1, \dots, N$, are independent directions in $\mathcal{D}(x)$ in the sense of the definition (6.2). From Lemma 6.2 we have that any limit

$$\begin{aligned} \lim_{i\to\infty} \frac{x_i - x}{|x_i - x|_{D^*}} &= e \in \bigcap_{k_1,k_2=1}^{N} \Big\{ x : (\ell(d_{k_1}) - \ell(d_{k_2})) \cdot (x - x_0) \leq 0 \Big\} \\ &= \bigcap_{k=1}^{N} \Big\{ x : (\ell(d_{k+1}) - \ell(d_k)) \cdot (x - x_0) \leq 0, d_{N+1} = d_1 \Big\}. \end{aligned} \tag{6.7}$$

This is the intersection of N planes. Given N pairwise disjoint convex sets $K_1, \dots, K_N$ not containing the origin and satisfying

$$\mathrm{dist}\Big(K_i, \mathrm{span}\big\{K_j, j \neq i\big\}\Big) \geq c > 0, \tag{6.8}$$

we thus introduce the set

$$C(K_1, \dots, K_N) = \mathbb{R}^n \setminus \bigg(\bigcup_{\ell_1 \in K_1, \dots, \ell_N \in K_N} \bigcap_{i=1}^{N} \Big\{ x : \ell_i \cdot x = 0 \Big\} \bigg). \tag{6.9}$$

From (6.8) it follows that $C(K_1, \dots, K_N)$ contains a N dimensional cone, i.e. there exists $\pi_N : \mathbb{R}^n \mapsto \mathbb{R}^{n-N}$ such that the set

$$\Big\{ y \neq 0 : \big|y - \pi_N y\big| \geq M |\pi_N y| \Big\} \subset C(K_1, \dots, K_N), \tag{6.10}$$

for M sufficiently large.

It follows by the same proof of Lemma 6.2 that if $J^m_N \ni x_i \to x \in J^m_N$, then the contact set Y of the sequence $\{(x_i - x)/|x_i - x|\}$ satisfies

$$Y \cap C\Big(\delta D^*(d_2) - \delta D^*(d_1), \dots, \delta D^*(d_N) - \delta D^*(d_{N-1})\Big) = \emptyset, \tag{6.11}$$

where $d_1, \dots, d_N$ are the separated directions in $\delta D(x)$. Defining the set

$$J_N = \bigcup_{m \in \mathbb{N}} J^m_N, \tag{6.12}$$

by a rectifiability criterion stated in [2] we have

Proposition 6.4. *The set J_N is $n - N + 1$ rectifiable, i.e. $J_N = \cup L_i$, where L_i are Lipschitz continuous graphs, with uniform Lipschitz constant in each $\Omega' \subset\subset \Omega$.*

The last part of the proposition follows from the fact that D^* is compact.

By Lemma 6.1, we conclude that the measure div d has a singular $\mathcal{H}^{n-1}$-rectifiable part supported on the set $E = \cup_m J^m_2$. Moreover, on this set, it follows that for almost all $x \in E$ the measure theoretical normal $n(x)$ satisfies $n(x) \in C(\delta D^*(d_2) - \delta D^*(d_1))$, where $\mathcal{D}(x) = \{d_1, d_2\} \subset D^*$. Note that under some regularity assumptions on D^* one can show that $n(x) = \frac{\nabla u(x^+) - \nabla u(x^-)}{|\nabla u(x^+) - \nabla u(x^-)|}$.

It remains to compute the density of $(\mathrm{div}\, d)_{\mathrm{s}}$ w.r.t. the $\mathcal{H}^{n-1}$-measure. Let $x \in E$ be such that the normal n to E exists. Due to uniform continuity, for all y close to x the direction of the vector field $d(y)$

is close to the directions $d(x) = \{d_1, d_2\}$. By blowing up we obtain that

$$\begin{aligned}\lim_{\rho \to 0} \frac{1}{\rho^{n-1}} \int_\Omega \phi\left(\frac{x-y}{\rho}\right) \mathrm{div}\, d(y) =& -\lim_{\rho\to 0} \frac{1}{\rho^n} \int_\Omega \nabla\phi\left(\frac{x-y}{\rho}\right) \cdot d(y) dy \\ =& -\lim_{\rho\to 0} \int_{x+(\Omega-x)/\rho} \nabla\phi(x-y)\cdot d(x+(y-x)/\rho)dy \\ =& -\int_{y\cdot n>0} \nabla\phi \cdot d_1 dy - \int_{y \cdot n<0} \nabla \phi \cdot d_2 dy = (d_1 - d_2)\cdot n.\end{aligned}$$

We thus have the following theorem:

Theorem 6.5. *The set $E = \cup_m J_2^m$ is rectifiable, and the measure* div $d|_E$ *can be written as*

$$\mathrm{div}\, d|_E = (d_2 - d_1)\cdot n(x) d\mathcal{H}^{n-1}\Big|_E, \tag{6.13}$$

where $n(x)$ is the measure theoretical normal defined $\mathcal{H}^{n-1}$-a.e. with the orientation $(d_2 - d_1)\cdot n(x) > 0$. Moreover div $d(A) = 0$ *for all sets $A \subset \Omega \setminus E$ with finite $\mathcal{H}^{n-1}$-measure.*

Proof. It remains to show that outside E the Radon-Nychodim derivative of div d w.r.t. $\mathcal{H}^{n-1}$ is 0. For almost any ball $B(x,r)$ we can write

$$|\mathrm{div}\, d(B(x,r))| = \left|\int_{\partial B} d(y)\cdot y/|y| dy\right| \leq \sup_{y\in\partial B(x,r)} |d(y) - d(x)||\partial B|_{\mathcal{H}^{n-1}}.$$

It thus follows that if A is a compact set contained in $\Omega \setminus E$ with finite $\mathcal{H}^{n-1}$ measure, then for δ sufficiently small there are balls $B(x_i, r_i)$, with $r_i \leq \delta$, such that $A \subset \cup_i B(x_i, r_i)$ and

$$\begin{aligned}\mathcal{H}^{n-1}(A) \geq& \sum_{i\in I} |\partial B(x_i,r_i)| - \epsilon \geq \left(\sup_{y\in A, |y-x|\leq 2r} |d(y) - d(x)|\right)^{-1} \sum_i |\mathrm{div}\, d(B(x_i,r_i))| \\ \geq& \left(\sup_{y\in A,|y-x|\leq 2r} |d(y)-d(x)|\right)^{-1} \Big(|(\mathrm{div}\, d)_s(A)| - \mathcal{O}(1)\sum_i |B(x_i,r_i)|\Big).\end{aligned}$$

Since A is a compact set, so that $\mathcal{D}$ is uniformly continuous, it follows that $(\mathrm{div}\, d)_s(A) = 0$. The general case follows by approximations with compact sets. □

It is an open question whether div d has a positive Cantor part and whether its support is contained in $J \setminus E$.

7. The Euler-Lagrange equation in the strictly convex case

In this section we prove that when D^* is strictly convex, the solution to the Euler-Lagrange equation can be reduced to an ODE along the segments $x + td(x)$ in S. The line of the proof follows closely the analysis of Section 5. Finally we apply these results to the conjecture stated in [5].

Let $Z(s)$, $C(h)$ be the sets considered in (5.9), (5.11). If r is sufficiently small, then $d_1(x) = d(x)\cdot e_1 \geq 1/2$ for all $x \in B(\bar{x}, 2r)$. From the weak formulation of the Euler-Lagrange equation and the fact that p is positive and $g'(u) > 0$ almost everywhere in Ω, it follows that

$$\int_{B(x,r)} p(y)dy > 0$$

for all $B(x,r) \subset \Omega$. For $x \in B(\bar{x}, r)$, $\delta \leq r$, define thus the vector field

$$\tilde{d}_\delta = \left(\int \rho_\delta(y)p(x-y)d(x-y)dy\right)\left(\int \rho_\delta(y)p(x-y)dy\right)^{-1}, \tag{7.1}$$

where $\rho_\delta = \delta^{-n}\rho(x/\delta)$ is a convolution kernel.

From the definition of $C(h)$ it follows that the first component of the above vector field satisfies $\tilde{d}_{1,\delta}(x) = e_1 \cdot \tilde{d}_1(x) \geq 1/2$ for $x \in C(h) + B(0, 2\delta)$. By taking ρ_δ as test function for the Euler-Lagrange equation we can write

$$\mathrm{div}(p_\delta \tilde{d}_\delta) = g'_\delta(x) = \int_\Omega g'(u(y))\rho_\delta(x-y)dy.$$

If we integrate the above equation in a cylinder $\tilde{C}_\delta(h) \subset B(\bar{x}, r)$, defined as in the proof of Lemma 5.5 by means of the flow of the vector field $\tilde{d}_\delta/(\tilde{d}_\delta)_1$, we obtain

$$\int_{\tilde{Z}_\delta(s)} p_\delta(x)\tilde{d}_{1,\delta}(x)d\mathcal{H}^{n-1}(x) - \int_Z p_\delta(x)\tilde{d}_{1,\delta}(x)d\mathcal{H}^{n-1}(x) = \int_{\tilde{C}_\delta(s)} g'_\delta(x)dx.$$

Due to the uniform continuity of d in $B(\bar{x}, 2r)$, we have that $\tilde{d}_\delta$ converges uniformly to d, and since p_δ converges for a.e. s in $L^1(Z'(s))$, we obtain that

$$\begin{aligned}\int_Z p\bigg(x + s\frac{d(x)}{d_1(x)}\bigg)d_1(x)\alpha(s,x)d\mathcal{H}^{n-1}(x) &= \int_{Z(s)} p(x)d_1(x)d\mathcal{H}^{n-1}(x)\\ &= \int_Z p(x)d_1(x)d\mathcal{H}^{n-1}(x) + \int_{C(s)} g'(u(x))dx\\ &= \int_Z p(x)d_1(x)d\mathcal{H}^{n-1}(x) + \int_0^s \int_Z g'\bigg(u\bigg(x + t\frac{d(x)}{d_1(x)}\bigg)\bigg)\alpha(t,x)d\mathcal{H}^{n-1}(x)dt. \qquad (7.2)\end{aligned}$$

By passing to the limit $|Z|_{\mathcal{H}^{n-1}} \to 0$, we obtain that for a.e. $x \in Z$

$$p\bigg(x + s\frac{d(x)}{d_1(x)}\bigg)\alpha(s,x) = p(0,x) + \int_0^s g'\bigg(u\bigg(x + t\frac{d(x)}{d_1(x)}\bigg)\bigg)dt,$$

and by (5.30) this is equivalent to the ODE

$$\frac{d}{dt}p(x + td(x)) + p(x + td(x))(\mathrm{div}\ d)_{\mathrm{a.c.}}(x + td(x)) = g'(u(x + td(x))).$$

The above limits holds for almost every segment $x + td(x)$ in S. Since S is of full Lebesgue measure in Ω, as in Theorem 5.7 we have the following result:

Theorem 7.1. *The solution $p(x)$ to the Euler-Lagrange* (4.1) *is absolutely continuous on almost every segment $x + td(x)$ and satisfies*

$$\frac{d}{dt}p(x + td(x)) + p(x + td(x))(\mathrm{div}\ d)_{\mathrm{a.c.}}(x + td(x)) = g'(u(x + td(x))). \qquad (7.3)$$

Conversely, if p is an $L^\infty_{\mathrm{loc}}(\Omega)$ solution of the above equation with initial data $p(a(x)) = 0$, then it is a weak solution of the Euler-Lagrange equation.

As it follows from the proof, it is sufficient that $p(a(x)) = 0$ if $a(x) \in \Omega$. Moreover, since $\alpha(s,x)$ and $(\mathrm{div}\ d)_{\mathrm{a.c.}}$ are locally bounded on the segment $(a(x), b(x))$, the function $p(x + sd(x))$ is locally Lipschitz (but in general not bounded as x approaches to boundary $\partial\Omega$).

Proof. Consider the ODE

$$\frac{d}{dt}p(x + td(x)) + p(x + td(x))(\mathrm{div}\ d)_{\mathrm{a.c.}} = g(u(x + td(x)), \quad p(a(x)) = 0.$$

The above ODE is meaningful for almost every $x \in \Omega$, because $\mathrm{div}\ d \in L^\infty_{\mathrm{loc}}$ for a.e. $x + td(x)$, and $p \in L^\infty_{\mathrm{loc}}(\Omega)$. By using (5.36), we have that for all test functions $\phi \in C^\infty_0(\Omega')$

$$\begin{aligned}\int_\Omega p(x)d(x)\cdot\nabla\phi(x)dx &= \sum_k \int_{Z_k}\bigg(\int_{a(x)}^{b(x)} p(x,t)\frac{d\phi(x,t)}{dt}\alpha(t,x)dt\bigg)d\mathcal{H}^{n-1}(x)\\ &= \sum_k \int_{Z_k}\bigg(\int_{a(x)}^{b(x)} \phi(t,x)\bigg(\frac{dp(x,t)}{dt}\alpha(t,x) + p(t,x)\frac{d\alpha(t,x)}{dt}\bigg)dt\bigg)d\mathcal{H}^{n-1}(x).\end{aligned}$$

Using the fact that

$$\frac{dp}{dt}\alpha + p\frac{d\alpha}{dt} = \big(g'(u)\alpha - p(\mathrm{div}\ d)_{\mathrm{a.c.}}\alpha\big) + p(\mathrm{div}\ d)_{\mathrm{a.c.}}\alpha = g'(u)\alpha,$$

it follows that

$$\sum_k \int_{Z_k} \bigg(\int_{a(x)}^{b(x)} \phi(t,x) \Big(\frac{dp(x,t)}{dt} \alpha(t,x) + p(t,x) \frac{d\alpha(t,x)}{dt} \Big) dt \bigg) d\mathcal{H}^{n-1}(x)$$

$$= \sum_k \int_{Z_k} \bigg(\int_{a(x)}^{b(x)} \phi(t,x) g'(u(t,x)) \alpha(t,x) dt \bigg) d\mathcal{H}^{n-1}(x) = \int_\Omega \phi(x) g'(u(x)) dx,$$

so that the Euler-Equation holds. □

As a corollary we obtain that p is different form 0 a.e. in Ω.

Corollary 7.2. *The function p is different from 0 a.e., and we have the uniform estimate on $a(x)+td(x)$*

$$p(a(x)+td(x)) \geq \frac{1}{(t+s^-)^\varrho} \int_0^{t+s^-} s^\varrho g'\Big(u\big(a(x)+(t+s^-)d(x)\big)\Big) ds > 0, \tag{7.4}$$

where $x - s^- d(x) = a(x)$.

Proof. In fact, for $x \in S$, consider the segment $(a(x), b(x))$, with

$$a(x) = x - s^- d(x), \quad b(x) = x + s^+ d(x).$$

From the fact that S is open and remark 5.2, it easily follows that we have a uniform estimate on the divergence of $d(x)$,

$$-\frac{\varrho}{s^+ - t} \leq \operatorname{div} d(x+td(x)) \leq \frac{\varrho}{t+s^-}. \tag{7.5}$$

We can thus estimate (7.3) by

$$\frac{d}{dt} p(x+td(x)) \geq g'(u(t)) - \frac{p\varrho}{t+s^-}.$$

Since $p \geq 0$, we recover the estimate

$$p(x+td(x)) \geq \frac{1}{(t+s^-)^\varrho} \int_0^{t+s^-} s^\varrho g'\Big(u\big(a(x)+(t+s^-)d(x)\big)\Big) ds.$$

□

The complementary inequality is

$$p(x+td(x)) \leq \frac{1}{(s^+-t)^\varrho} \int_0^{t+s^-} (s^+-t)^\varrho g'\Big(u\big(a(x)+(t+s^-)d(x)\big)\Big) ds. \tag{7.6}$$

It is easy to see that the above estimates are sharp.

Note that the solution is in general not unique, because the initial data of p on $a(x)$ can be arbitrary. This typically happens when $a(x) \in \Omega$ and u remains smooth near $a(x)$. In Theorem 7.1 a particular solution $\bar{p}$ is selected by the initial data $\bar{p}(a(x)) = 0$ for a.e. $x \in S$.

Let $p(x)$ be a solution to the Euler-Lagrange equation. Since p satisfies the ODE (7.3), then it follows that repeating the computations (in inverse direction) to prove the second part of Theorem 7.1 we have that for all $\phi \in C_c(\Omega)$

$$\sum_k \int_{A_k} \alpha(a(x)) p(\alpha(x)) \phi(a(x)) d\mathcal{H}^{n-1}(x) = 0. \tag{7.7}$$

This implies that $\alpha(a(x))p(a(x)) = 0$ for almost every $x \in A_k$. Since we have that for almost every segment $x + td(x)$

$$\frac{d}{dt}\big(p(x+td(x))\alpha(t+td(x))\big) = g'(u(x+td(x)))\alpha(x+td(x)), \tag{7.8}$$

we conclude that for almost every $x \in \Omega$

$$p(a(x)+td(x)) = \frac{1}{\alpha(a(x)+td(x))} \int_0^t g'(u(a(x)+td(x)))\alpha(a(x)+sd(x)) ds. \tag{7.9}$$

In particular it follows that p is unique if $a(x) \in \Omega$ for all $x \in \Omega$, i.e. no segment $x + td(x)$ has both end points on the boundary.

Proposition 7.3. *If $a(x) \in \Omega$ for all $x \in \Omega$, then the solution p is unique.*

Remark 7.4. If the end point $a(x)$ of the line is a point on E, then a rescaling argument shows that the initial data have to be 0. In fact, since E is rectifiable, for $\mathcal{H}^{n-1}$-a.e. $x \in E$ there exists a unit normal n and by continuity d has a jump on the two faces. Using the test function $\phi_\delta = \delta^{-n}\phi(x/\delta)$, we have that near the point $x = 0$

$$\int p(x)d(x) \cdot \nabla\phi_\delta(x)dx = \frac{1}{\delta}\int p(\delta x)d(\delta x) \cdot \nabla\phi(x)dx = \int g'(u(\delta x)\phi(x)dx.$$

Taking any weak limit of $p(\delta x)$ as $\delta \to 0$, and using the strong convergence of d and $g'(u)$, we obtain that any weak limit satisfies

$$\int_{x_1<0} p(x)d^- \cdot \nabla\phi dx + \int_{x_1>0} p(x)d^+ \cdot \nabla\phi dx = 0.$$

Thus any weak limit is the local solution of a linear equation

$$\mathrm{div}(p\bar{d}) = 0,$$

with $\bar{d}$ discontinuous on a plane surface, and monotone. Since p is positive, it turns out that $p = 0$. Thus any weak limit of $p(\delta x)$ is 0. The initial data is thus 0 on the discontinuity set J.

7.1. **A conjecture of Bertone-Cellina.** In this section we consider the following conjecture. Let Ω be an open bounded set in $\mathbb{R}^n$, and D a convex closed bounded set in $\mathbb{R}^n$. Let $u \in W^{1,\infty}(\Omega)$ such that $\nabla u \in D$ a.e.. The conjecture stated in [5] is the following:

(1) either there exists a function $\eta \in W_0^{1,\infty}(\Omega)$, $\eta \neq 0$, such that $\nabla\eta + \nabla u \in D$;
(2) or there exists a divergence free vector $\pi \in (L^1_{\mathrm{loc}}(\Omega))^n$ such that $\pi \neq 0$ and

$$\pi(x) \cdot \nabla u(x) = \max_{k \in D}\{\pi \cdot k\} \tag{7.10}$$

a.e. in Ω.

We first give a proof that if (2) holds, then the only variation admissible in (1) is $\eta = 0$.

Proposition 7.5. *If there exists π satisfying point (2), then $\eta = 0$.*

Proof. First of all, if $\eta \in W_0^{1,\infty}(\Omega)$ is a variation, then also

$$\eta' = \max\{0, \eta\},$$

is a variation. Similarly we can consider

$$\eta'' = \min\{0, \eta\}.$$

By assumptions, at least one of these functions is different from 0: let us assume that $\eta'(x) > 0$ for some $x \in \Omega$. If $\delta > 0$ is sufficiently small, then the function

$$\eta''' = \max\{u, u + \eta - \delta\} - u$$

is a variation different from 0 and with compact support in Ω. We thus assume that the variation η is positive and with compact support in Ω.

Since $\nabla\eta + \nabla u \in D$, it follows that a.e. in Ω

$$\pi(x) \cdot \nabla\eta(x) \leq \max_{k \in D}\{\pi(x) \cdot k\} - \pi(x) \cdot \nabla u(x) \leq 0.$$

Since $\mathrm{div}_x\pi = 0$, then

$$0 = \int_\Omega \pi(x) \cdot \nabla\eta(x)dx \leq 0.$$

Thus $\pi \cdot \nabla\eta = 0$ a.e..

Next consider the test function ηu. Clearly this function belongs to $W_0^{1,\infty}(\Omega)$, so that from the assumption on 0 divergence and $\pi \cdot \nabla\eta = 0$ it follows

$$0 = \int_\Omega u\pi \cdot \nabla\eta dx + \int_\Omega \eta\pi \cdot \nabla u dx = \int_\Omega \eta\pi \cdot \nabla u dx.$$

If D contains a ball and π satisfies (7.10), then $\pi \cdot \nabla u \geq \delta$, with $\delta > 0$. Hence we have a contradition, unless $\eta \equiv 0$. □

In the strictly convex case the other implication (i.e. if no variations η exists apart $\eta = 0$ then there exists a divergence free vector field π satisfying (2)) is a consequence of Corollary 7.2. Consider in fact the two minimization problem

$$\inf_{\bar{u}+W_0^{1,\infty}(\Omega)} \int_\Omega (\mathbb{I}_D(\nabla u) - u)dx, \qquad \inf_{\bar{u}+W_0^{1,\infty}(\Omega)} \int_\Omega (\mathbb{I}_D(\nabla u) + u)dx. \tag{7.11}$$

Since we assume that there are no variations, then the two solutions coincide with u, so in particular there are two positive functions $p^-(x)$, $p^+(x)$ belonging to $L^\infty_{\mathrm{loc}}(\Omega)$ and satisfying the Euler-Lagrange equation

$$\mathrm{div}(p^-(x)d(x)) = -1, \quad \mathrm{div}(p^+(x)d(x)) = 1. \tag{7.12}$$

For the second minimization problem, we have to reverse the directions on $x + td(x)$, by setting

$$a(x) = \{x + td(x), t \in \mathbb{R}\} \cap \partial\Omega, \quad b(x) = \{x - td(x), t \in \mathbb{R}\} \cap \partial\Omega.$$

Clearly, it is equivalent to consider the minimization problem

$$\inf_{-\bar{u}+W_0^{1,\infty}(\Omega)} \int_\Omega (\mathbb{I}_{-D}(\nabla v) - v)dx,$$

and setting $u = -v$.

By adding the two equations in (7.12), it follows that $\pi(x) = p^+(x)d(x) + p^-(x)d(x)$ satisfies point (2), if we can prove that it is different from 0: this follows from Corollary 7.2.

Remark 7.6. We observe that a direct proof of Theorem 7.1 is much simpler if we assume that there are no variations. In fact, it follows that for all $x \in \Omega$, $a(x) \in \partial\Omega$, so that the set $S^{\epsilon,h+\epsilon'}$ have non empty interior, and the divergence formula can be deduced easily, see Remark 5.2.

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