INTERNATIONAL SCHOOL FOR ADVANCED STUDIES 1995
ENTRANCE EXAMINATION: CONDENSED MATTER

Solve at least one of the problems below (the order is irrelevant). Write out solution clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problems sheet, but use extra envelope. A single well-solved problem is better than many half-solved ones.

Problem 1
One-Dimensional Electron Motion in a Magnetic Field

Consider a circular ring of circumference L. Place on this ring a number 2N of electrons, where the density $2N/L$ is finite, and N is either (a) even, or (b) odd. Electron motion is strictly one-dimensional, on the ring. The electrons are fictitiously supposed to be non-interacting, and free of any external scalar potential. There is however an external constant magnetic field B, orthogonal to the ring.

1. Neglecting Zeeman coupling of the spin with the magnetic field, quantize the motion of a single electron on the ring, starting from the classical hamiltonian of a charged particle in a magnetic field. Find the one-electron energy levels, and discuss their behaviour as a function of B.
2. Fill up the one-electron levels found in (1) with the 2N electrons available. Determine the Fermi energy $E_F(B)$. Calculate then the total electron energy $E_T(B)$, and plot it out schematically as a function of the flux $\phi(B)$ through the ring, in cases (a) and (b).
3. Discuss in particular the behaviour of $E_T(B)$ for small B, and establish whether the system is paramagnetic or diamagnetic, again in cases (a) and (b) (neglect spin paramagnetism).
4. (Optional) Calculate and plot schematically the current $J(B)$ on the ring, as a funcion of flux $\phi(B)$, and electron density.

Problem 2
Two particles on a sphere

Consider a system of two particles of mass m and charge $+|e|$, constrained on the surface of a sphere of radius R. Suppose first that $e = 0$, so that the particles do not interact with each other.

1. Write down the hamiltonian of the system and discuss its symmetry properties.

2. Describe the spectrum of the hamiltonian derived at point 1. and write down the expression of its eigenvalues and eigenfunctions, indicating their quantum numbers and degeneracy.

3. Write down the general expression of the hamiltonian eigenfunctions which have the same quantum numbers as the ground state.

Suppose now that $e \neq 0$ and that the two particles interact with each other through the Coulomb interaction.

4. Calculate the value of the ground-state energy of the system by treating the interaction in second-order perturbation theory. Hints: the following two relations could be useful:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi \sum_{l=0}^{\infty} \frac{1}{2l+1} \frac{\min(r,r')^l}{\max(r,r')^{l+1}} \sum_{m=-l}^{l} Y^*_{lm}(\hat{\mathbf{r}'}) Y_{lm}(\hat{\mathbf{r}}),$$

$$\sum_{l=1}^{\infty} \frac{1}{l(l+1)(2l+1)} \approx 0.22741$$

Problem 3
Equilibrium shape of two-dimensional crystal

Consider a finite two-dimensional crystal, containing N atoms arranged in a square lattice with spacing a at $T = 0$. The total crystal area is fixed to be $A = Na^2$, but the geometrical **shape** of the whole crystal is free to vary. The presence of "surfaces" (which are called edges) in this two-dimensional crystal implies an additional excess energy per unit length (*surface energy*), due to the fact that some atomic bonds have been broken to form the edge. The surface energy of an edge depends upon its crystallographic orientation, relative to the square lattice.
Assuming a surface energy γ_1 for all the $\langle 10 \rangle$ orientations, a surface energy $\gamma_2 = \alpha\gamma_1$ for all the $\langle 11 \rangle$ orientations (with $\alpha \geq 1$), and neglecting the possibility of other orientations to occur,

1. calculate the equilibrium shape of the crystal (that is, the shape which has the full symmetry of the lattice and which minimizes total energy keeping A constant) as a function of α;

2. find the conditions for the equilibrium shape to be a square (that is, when are $\langle 11 \rangle$ edges completely absent?);

3. examine in detail the particular case of a square lattice with first-nearest-neighbor interactions, and an energy J associated to each bond. Find γ_1 and α as functions of J in this case, and determine the crystal shape as J varies.

Problem 4
States in a one-dimensional well

Consider a particle moving in a generic one dimensional potential well with reflection symmetry with respect to the origin:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x), \qquad V(x) = V(-x)$$

Suppose the well has N bound states: $\varepsilon_1 \leq \varepsilon_2 \leq ... \leq \varepsilon_N$, with wavefunctions $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_N(x)$.
a) discuss the simmetry and the degeneracy of the levels.
Consider a new well obtained by adding an impenetrable barrier at the origin to the old one.

$$V'(x) = \begin{cases} V(x) & \text{if } x > 0 \\ +\infty & \text{if } x \leq 0 \end{cases}$$

b) How many bound states does this well possess for even N? And for odd N?
c) Which are their energies? And the wavefunctions?
Consider now a generic (*i.e.* non symmetric) one dimensional well $V_2(x)$ with $2M$ bound states.
d) Where shall one put an impenetrable barrier in order to obtain a well V_2' with its M-th bound-state being degenerate with the last ($2M$-th) one of the original well V_2?
e) What is the best upper bound that you can give for the ground state of the new well V_2', without additional information on its shape?
f) What is the maximum number of bound states that the new well can have?

Problem 5
Ising spins in a one-dimensional lattice

On the N sites of a one dimensional ring, Ising spins $\sigma_x = \pm 1$ $(x = 1, 2, \ldots, N)$ interact with a Hamiltonian

$$H = -\sum_{x=1}^{N} \sum_{|r|=1}^{N/2} \sigma_x \sigma_{x+|r|} J(|r|) \tag{1}$$

where the integer N is assumed to be even.
1. Determine the free energy per site in the $N \to \infty$ limit at any temperature T for the case $J(r) = 0$ when $r > 1$, and $J(1) = J$.
Sketch the way to solve the case $J(2)$, $J(1)$ different from 0 and $J(r) = 0$ when $r > 2$.
2. Determine the ground state energies and the corresponding spin configurations as a function of $J(2)/J(1)$ for the case $J(r) = 0$ when $r > 2$. (Hint: rewrite the Hamiltonian as $H = -\sum_x [J(1)(\sigma_x \sigma_{x+1} + \sigma_{x+1}\sigma_{x+2})/2 + J(2)\sigma_x \sigma_{x+2})]$.)
3. Consider the case of the long-range interaction $J(r) = J/r^\alpha$ $(J > 0)$ and evaluate the energy difference of the following two spin configurations:

1. $\sigma_x = 1 \quad x = 1, 2, \ldots, N;$
2. $\sigma_x = 1 \quad x = 1, 2, \ldots, N/2, \qquad \sigma_x = -1 \quad x = N/2 + 1, \ldots, N.$

Neglect boundary effects and use the approximation:

$$\sum_{r=1}^{N/2} r^\beta \sim \int_1^{N/2} dr r^\beta \tag{2}$$

Given that an estimate of the entropy difference between configurations of type 2 and 1 is $\Delta S \sim \log N$, discuss the stability of ferromagnetic ordering as a function of α.

Problem 6
Fermions with spin-dependent interaction

Consider a system constituted by two fermions of spin 1/2 and mass m, interacting through the potential:

$$V(r_{12}) = v(r_{12}) + v_\sigma(r_{12})\sigma_1 \cdot \sigma_2, \tag{3}$$

where $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ is the interparticle distance, and σ_i are the Pauli matrices. The functions $v(r_{12})$ and $v_\sigma(r_{12})$ have the following expressions:

$$v(r_{12}) = -E_0 + 3Kr_{12}^2, \tag{4}$$

and

$$v_\sigma(r_{12}) = E_0 + Kr_{12}^2. \tag{5}$$

Find the eigenvalues of the energy for the two cases: (1) the two particles are in the spin singlet state ($S = 0$); (2) the two particles are in the spin triplet state ($S = 1$).

Problem 7
Spin-1/2 in a magnetic field

a) A spin-1/2 is immersed in a uniform magnetic field in the z-direction $\mathbf{B}_o = B_o\hat{\mathbf{z}}$. At time $t = 0$ the spin is pointing in the x-direction, i.e., $\langle S_x \rangle(t = 0) = 1/2$ (with $\hbar = 1$). Calculate the state $|\Psi(t)\rangle$ of the system, and the expectation value of the spin operator $\mathbf{S}$ at time t.

b) An additional magnetic field

$$\mathbf{B}_1(t) = B_1[\cos(\omega t)\hat{\mathbf{x}} - \sin(\omega t)\hat{\mathbf{y}}] \,.$$

is applied, so that the spin now moves in the field $\mathbf{B}(t) = \mathbf{B}_o + \mathbf{B}_1(t)$. Assume that the spin is pointing along $+\hat{\mathbf{z}}$ at time $t = 0$. Solve the time-dependent Schrödinger equation to find the state of the system $|\Psi(t)\rangle$ for $t > 0$. What is the probability that the spin will have flipped to $-\hat{\mathbf{z}}$ at time t ?
Hints: i) Neglecting constants, take as hamiltonian of a spin-1/2 in a magnetic field $H = -\mathbf{S} \cdot \mathbf{B}$. ii) If $\vec{\sigma}$ are the three Pauli matrices and $\vec{n}$ is a unit vector, then the following property could be useful

$$e^{i\phi\vec{n}\cdot\vec{\sigma}} = \cos(\phi)\mathbf{1} + i\sin(\phi)\vec{n}\cdot\vec{\sigma}$$