Topics in Representation Theory, 2025

Assignment 3.

The ground field is $\mathbb{C}$, unless stated otherwise.

Problem 1. Consider tensor product $M_\lambda \otimes L_1$ of Verma module and two-dimensional representations of $\mathfrak{sl}_2$.

(a) Find spectral type (eigenvalues and Jordan blocks) for the Casimir element $C = 2fe + h^2/2 + h \in U(\mathfrak{sl}_2)$ on the $M_\lambda \otimes L_1$

(b) Show that for $\lambda \neq -1$ $M_\lambda \otimes L_1 \simeq M_{\lambda+1} \oplus M_{\lambda-1}$ and for $\lambda = -1$ $M_\lambda \otimes L_1$ is indecomposable module. For the latter find filtration by with irreducible quotients.

Hint: C commutes with $\mathfrak{sl}_2$ so suffices to compute its action on some vectors. For $\lambda \neq -1$ C acts with two different eigenvalues, while for $\lambda = -1$ C has Jordan block.

Problem 2. (a) Using action of $SL(3)$ on $\mathbb{P}^2$ realize Lie algebra $\mathfrak{sl}_3$ by differential operators in 2 variables.

(b) Using action of $SL(3)$ on the space of complete flags $\mathbb{FL}$ realize Lie algebra $\mathfrak{sl}_3$ by differential operators in 3 variables.

Problem 3. Consider principal series representation P_s^+ of $\mathfrak{sl}_2$ with $s \in 2\mathbb{Z}+1$. Find filtration with irreducible quotients.

Problem 4. Show isomorphism between discrete series representation of $SL(2,\mathbb{R})$ M_{-n}^+ and representation in the space of $n/2$ forms on the upper half plane.

Hint: First identify forms in upper half plane with forms in the unit disc. Then identify vectors w_n with functions z^n

Problem 5. Consider quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$.

(a) Show that coproduct defined by

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K$$

is an algebra homomorphism.

(b) Find counit ϵ and antipode S.

Problem 6. Let $V = L_1^+$ be standard two-dimensional representation of $U_q(\mathfrak{sl}_2)$.

(a) Show that $V^* \simeq V$.

(b) Define interwining operators

$$\mathrm{ev}\colon V^* \otimes V \to \mathbb{C}, \quad \mathrm{ev}^*\colon V \otimes V^* \to \mathbb{C}, \quad \mathrm{coev}\colon \mathbb{C} \to V^* \otimes V, \quad \mathrm{coev}^*\colon \mathbb{C} \to V \otimes V^*$$

such that Rademeister I relation holds.

(c) Compute Resehtikhin-Turaev invariant for the Hopf link.