- Reps of G, $|G|<\infty$ $k=\mathbb{C}$

Any rep — $\oplus$ irreds $kG = \bigoplus_{i=1}^{r} \mathrm{End}(V_i) = \bigoplus \mathrm{Mat}_{d_i}(k)$

r = # irreds = # conj classes

$\sum d_i^2 = |G|$ $\forall$ $\chi_V : G \to \mathbb{C}$ character

$\chi_{V\oplus U} = \chi_V + \chi_U$, $\chi_{U\otimes V} = \chi_U \cdot \chi_V$ $\chi_{U^*} = \overline{\chi_U}$

$\langle \chi_U, \chi_V \rangle = \frac{1}{|G|} \sum \overline{\chi_U(g)} \chi_V(g)$ $\langle \chi_{V_i}, \chi_{V_j} \rangle = \delta_{ij}$

<u>Example</u> S_3

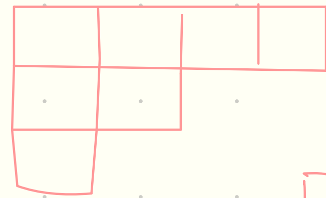
		e	(12)	(123)
trivial rep	χ_{triv}	1	1	1
sign rep	χ_{sgn}	1	-1	1
Recall	χ_{U_2}	2	0	-1

$\mathbb{C}^3 = U_2 + triv$

Conj. classes in S_n — product of cycles ~ partitions of n ~ Young diagr w/ n boxes

Def Partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0)$ $\quad \ell(\lambda) = p$ — length

$(4,2,1)$



Example $G = S_4$

		e	(12)	(123)	$(12)(34)$	(1234)
χ_1	χ_{tri}	1	1	1	1	1
	χ_{sign}	1	-1	1	1	-1
	$\chi_{\mathbb{C}^3-triv}$	3	1	0	-1	-1
	$sgn \otimes \ldots$	3	-1	0	-1	1
		2	0	-1	2	0

$24 = 1+1+9+9+4$

$\chi_1 + \chi_2 + 3\chi_3 + 3\chi_4 + 2\chi_5 = \chi_{reg}$

Lemma V - rep, $\langle \chi_V, \chi_V \rangle = 1 \Rightarrow V$ irred

Pf $V = \sum n_i V_i \Rightarrow \sum n_i^2 = 1 \Rightarrow V \simeq V_i$ for some i $\square$

Example $G = S_5$

$120 = 1+1+16+16+25+25+36$

	e	(12)	(123)	$(12)(34)$	(1234)	$(12)(345)$	(12345)
χ_{triv}	1	1	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1	-1	1
$\chi_{\mathbb{C}^5 - triv}$	4	2	1	0	0	-1	-1
$sgn \otimes \ldots$	4	-2	1	0	0	1	-1
	5						
	5						
	6						

$$\left.\begin{array}{l} 120 = 1+1+16+16+1+36+49 \\ 120 = 1+1+16+16+1+4+81 \end{array}\right\} \text{not work due to the following.}$$

Lemma The only two 1-dim reps of S_n are triv, sgn

Th 1-dim reps G $\longleftrightarrow$ irreds $G/[G,G]$
where $[G,G]$ generated by $aba^{-1}b^{-1}$

Pf $\rho(aba^{-1}b^{-1}) = \rho(a)\rho(b)\rho(a)^{-1}\rho(b)^{-1} = 1$ $\square$

Lemma $[S_n, S_n] = A_n$ – subgroup of even permutations

Sketch $(12)(23)(12)(23) = (132) \Rightarrow \forall$ cycle $(i,j,k) \in [S_n, S_n]$
Cycles of length 3 generate $A_n \Rightarrow A_n \subset [S_n, S_n]$
$aba^{-1}b^{-1}$ – even $\Rightarrow [S_n, S_n] \subset A_n$ $\square$

Then for $S_n/[S_n, S_n] = S_n/A_n \simeq C_2 \Rightarrow$
There are only 2 1dim irreds – triv + sgn

Th $|G| < \infty$, $k = \mathbb{C}$ V – irred $\Rightarrow \dim V \mid |G|$
(Pf requires some number theory)

- For general n $\forall \lambda$ $|\lambda|=n \rightsquigarrow V_\lambda$ irred $d_\lambda = \dim V_\lambda$

Strategy — find idempotents $c_\lambda \in kS_n$ that correspond to $E_{11} \in Mat_{d_\lambda} = \bigoplus Mat_{d_\lambda} = kS_n$
E_{11} is not canonical (unless $d_\lambda = 1$) $\sim$ we will need some choices

<u>Ex</u> triv rep $\frac{1}{|G|}\sum g = c_{triv}$

sgn rep $\frac{1}{|G|}\sum \operatorname{sgn} g \cdot g = c_{sgn}$

- <u>Def</u> Young tableau is numbering of boxes of λ by integer numbers

<u>Standard</u> — " — different numbers $1, \ldots, |\lambda|$ in increasing order in $\forall$ row and $\forall$ column

Young tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array} \qquad T$$

- <u>Def</u> $P_T \subset S_n$ consist of elements that preserve rows
  $Q \subset S_n$ consist of elements that preserve columns

$P_T \simeq \prod S_{\lambda_i} \qquad Q_T \simeq \prod S_{\lambda'_j}$

$$a_\lambda = \frac{1}{|P_\lambda|}\sum_{g \in P_\lambda} g \qquad b_\lambda = \frac{1}{|Q_\lambda|}\sum_{g \in Q_\lambda} \operatorname{sgn} g \; g \qquad c_\lambda = a_\lambda b_\lambda$$

<u>Ex</u> $\lambda = \begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline\end{array}$ $\qquad T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline\end{array}$

$P_\lambda = S_n, \quad Q_\lambda = e \qquad c_\lambda = \frac{1}{n!}\sum g$

$\lambda = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$ $T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}$ $P_\lambda = e$ $Q_\lambda = S_n$

$$c_\lambda = \frac{1}{n!} \sum \operatorname{sgn} g \, g$$

<u>Th</u> $V_\lambda := kS_n c_\lambda$ is irred. rep of S_n

Any irred. rep of S_n is V_λ for unique λ.

Specht module

- <u>Lemma</u> $\forall x \in \mathbb{C}[S_n]$ $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$, where $\ell_\lambda : \mathbb{C}[S_n] \to \mathbb{C}$ some linear function

<u>Pf</u> W.l.o.g. $x = g$

If $g = pq$, $p \in P_\lambda$, $q \in Q_\lambda$ $a_\lambda x b_\lambda = \operatorname{sgn} q \, a_\lambda b_\lambda = \operatorname{sgn} q \, c_\lambda$

Let $g \notin P_\lambda Q_\lambda$ Want to show that $a_\lambda x b_\lambda = 0$

Plan: show that $\exists\, t$ - transposition s.t

$t \in P_T$ $\quad g^{-1} t g \in Q_T$ $\quad (\Leftrightarrow t \in Q_{gT})$ Indeed

$$a_\lambda g b_\lambda = a_\lambda t g b_\lambda = a_\lambda g (g^{-1} t g) b_\lambda = -a_\lambda g b_\lambda$$

$t = (i, j)$

we need (i, j) — in one row of T / in one column of gT

Consider first row. If not $\exists\, p_1 \in P_T$, $q_1' \in Q_{gT}$

s.t $p_1 T$ and $q_1' g T$ have the same first row ...

$\exists\, p \in P_T$, $q' \in Q_{gT}$ s.t $\quad pT = q' g T \Rightarrow p = q' g$

$\Rightarrow g = p q$ where $\quad q = g^{-1} (q')^{-1} g \in Q_T$

$\square$