# Where we are

Reps of $S_n$. For $\forall \lambda$ - partition of $n$ irred $V_\lambda$

Young tableau $\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$ $T$

- <u>Def</u> $P_T \subset S_n$ — preserve rows

  $Q \subset S_n$ preserve columns

$P_T \simeq \prod S_{\lambda_i}$ $Q_T \simeq \prod S_{\lambda_j'}$

$a_\lambda = \frac{1}{|P_\lambda|} \sum_{g \in P_\lambda} g$ $b_\lambda = \frac{1}{|Q_\lambda|} \sum_{g \in Q_\lambda} \operatorname{sgn} g \; g$ $c_\lambda = a_\lambda b_\lambda$

$V_\lambda := k S_n c_\lambda$

<u>Lemma 1</u> $c_\lambda x c_\lambda = \ell(x) c_\lambda$ $\quad \ell : kS_n \to k$ lin. function

<u>Def $\lambda > \mu$</u> if $\exists i$ s.t. $\lambda_1 = \mu_1, \lambda_2 = \mu_2, \ldots, \lambda_{i-1} = \mu_{i-1}$

Lexicographic order

$5 > (4,1) > (3,2) > (3,1,1) > (2,2,1) > (2,1,1,1) > (1,1,1,1,1)$

- <u>Lemma 2</u> $\lambda > \mu \implies a_\lambda \mathbb{C}[S_n] b_\mu = 0, \quad b_\mu \mathbb{C}[S_n] a_\lambda = 0$

<u>Pf</u> $T, T'$ tableaux for $\lambda, \mu$ $\quad \forall g \in S_n$

Want $\exists t \in P_T, \quad g^{-1} t g \in Q_{T'}$

$\exists i, j$ in the same box for $T$ same column for $T'$.

$b_\mu g a_\lambda = b_\mu t g a_\lambda = a_\lambda g (g^{-1} t g) b_\mu = -a_\lambda g b_\mu$

$a_\lambda g b_\mu$ – similar $\square$

- **Lemma 3** $c_\lambda^2 = \varkappa_\lambda c_\lambda$ where $\varkappa_\lambda = \frac{n!}{|P_T||Q_T| \dim V_\lambda}$ $c_\lambda$

Pf By Lemma 1 $c_\lambda^2 \sim c_\lambda$. It remains to find coefficient $\varkappa_\lambda$.
Consider operator of right multip. on $c_\lambda$ $r_{c_\lambda} : x \mapsto x c_\lambda$

Compute $\operatorname{Tr} r_{c_\lambda}$.
On one hand $r_{c_\lambda}$ projects on $kS_n c_\lambda = V_\lambda$ on this subspace $r_{c_\lambda}$ is equal to $\varkappa_\lambda$. Hence $\operatorname{Tr} r_{c_\lambda} = \varkappa_\lambda \dim V_\lambda$

On the other hand $\operatorname{Tr} r_g = |G| \delta_{g,e}$ (as character of regular representation). Clearly $P_T \cap Q_T = e \Rightarrow$ coeff. of $e$ in $c_\lambda$ is equal to $1/|P_T||Q_T|$ Hence $\operatorname{Tr} r_{c_\lambda} = |G|/|P_T||Q_T|$

Therefore $\varkappa_\lambda = \frac{n!}{|P_T||Q_T| \dim V_\lambda}$ $\square$

Pf of thm Compute $\dim \mathrm{Hom}_{S_n}(V_\lambda, V_\mu)$

$V_\lambda$ generated by $c_\lambda \Rightarrow \phi \in \mathrm{Hom}_{S_n}(V_\lambda, V_\mu)$ is determined by $\phi(c_\lambda)$. Since $c_\lambda^2 = \varkappa_\lambda c_\lambda \Rightarrow c_\lambda \phi(c_\lambda) = \varkappa \phi(c_\lambda) \Rightarrow \phi(c_\lambda) \in c_\lambda V_\mu$

Hence $\dim \mathrm{Hom}_{S_n}(V_\lambda, V_\mu) \leq \dim(c_\lambda V_\mu) = \dim(c_\lambda k S_n c_\mu)$

If $\lambda = \mu$ we have $c_\lambda k S_n c_\lambda = k c_\lambda \Rightarrow \dim \mathrm{End}_{S_n}(V_\lambda) \leq 1 \Rightarrow$
$V_\lambda$ - irreducible

If $\lambda > \mu$ we have $c_\lambda k S_n c_\mu = 0 \Rightarrow \dim \mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = 0$
$\Rightarrow V_\lambda, V_\mu$ are not isomorphic

$\square$

# Some constructions & reps of $S_n$

- **Restriction** $H \subset G$ subgroup $\rho: G \to GL(V)$ rep of $G$

  Then $\rho|_H : H \to GL(V)$ rep of $H$ $\quad \mathrm{Res}^G_H \rho$

Example

$S_2$

| | $e$ | $(12)$ |
|---|---|---|
| $(2)$ | 1 | 1 |
| $(1,1)$ | 1 | $-1$ |

$S_3$

| | $e$ | $(12)$ | $(123)$ |
|---|---|---|---|
| $(3)$ | 1 | 1 | 1 |
| $(2,1)$ | 2 | 0 | $-1$ |
| $(1,1,1)$ | 1 | $-1$ | 1 |

$S_4$

| | $e$ | $(12)$ | $(123)$ | $(12)(34)$ | $(1234)$ |
|---|---|---|---|---|---|
| $(4)$ | 1 | 1 | 1 | 1 | 1 |
| $(3,1)$ | 3 | 1 | 0 | $-1$ | $-1$ |
| $(2,2)$ | 2 | 0 | $-1$ | 2 | 0 |
| $(2,1,1)$ | 3 | $-1$ | 0 | $-1$ | 1 |
| $(1,1,1,1)$ | 1 | $-1$ | 1 | 1 | $-1$ |

$$\mathrm{Res}^{S_3}_{S_2} V_{(3)} = V_{(2)}$$

$$\mathrm{Res}^{S_3}_{S_2} V_{(2,1)} = V_{(2)} \oplus V_{(1,1)}$$

$$\mathrm{Res}^{S_3}_{S_2} V_{(1,1,1)} = V_{(1,1)}$$

$$\mathrm{Res}^{S_4}_{S_3} V_{\ydiagram{3,1}} = V_{\ydiagram{2,1}} \oplus V_{\ydiagram{3}}$$

$$\mathrm{Res}^{S_4}_{S_3} V_{\ydiagram{2,2}} = V_{\ydiagram{2,1}}$$

$$\mathrm{Res}^{S_4}_{S_3} V_{\ydiagram{2,1,1}} = V_{\ydiagram{2,1}} \oplus V_{\ydiagram{1,1,1}}$$

In general

$$\mathrm{Res}^{S_n}_{S_{n-1}} V_\lambda = \bigoplus_{\mu,\ \lambda = \mu + \square} V_\mu$$

- Opposite. Let $H \subset G$, $\rho : H \to GL(U)$ rep

Let $e = g_1, \dots, g_k$ representatives in $G/H$

$G = \bigsqcup_{i=1}^{k} g_i H$. For any $g, i$ let $g g_i = g_{s(g,i)} h(g,i)$

$s_{g,i} \in \{1, \dots, k\}$, $h(g,i) \in H$

<u>Def</u> $\mathrm{Ind}_H^G U = \mathbb{C}^{G/H} \otimes U = \bigoplus e_i \otimes U$

$g(e_i \otimes u) = e_{s(g,i)} \otimes h(g,i) u$

We have $\left.\begin{array}{l} s(g'g,i) = s(g', s(g,i)) \\ h(g'g,i) = h(g', s(g,i))\, h(g,i) \end{array}\right\} \Rightarrow$ definition is correct

$\dim \mathrm{Ind}_H^G U = |G/H| \cdot \dim U$

Let $f_U : U \to \mathrm{Ind}_U^G U$ $\quad v \mapsto e_1 \otimes v$ $\quad$ hom of $H$ reps

- Let $U$ - be $H$ rep $\quad V$ - be $G$ rep

Let $f : U \to V$ hom of $H$ reps. Then $\exists!\, \tilde{f}$ : s.t.

$$\tilde{f} \circ f_U = f$$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & V \\ \mathrm{Ind}_H^G U \arrow[ur, dotted, "\tilde{f}"'] & \end{tikzcd}$$

- Construction: $e_i \otimes v \mapsto g_i v$
- Check

$$\begin{tikzcd} (e_i \otimes v) \arrow[r, mapsto] \arrow[d, mapsto] & g_i v \arrow[d, mapsto] \\ e_{s(g,i)} \otimes h(g,i) v \arrow[r, mapsto] & g g_i v \end{tikzcd}$$

$$g g_i v = g_{s(g,i)} h(g,i) v$$

**Thm** (Frobenius reciprocity) There is a natural isomorphism between $\mathrm{Hom}_H(U, \mathrm{Res}^G_H V) = \mathrm{Hom}(\mathrm{Ind}^G_H U, \mathrm{Res}^G_H V)$

left adjoint functor to Res.

- Example ① $H = e \subset G$. $\mathrm{Ind}^G_H \mathbb{C}$ — regular rep

$\mathrm{Ind}^G_H \mathbb{C} = \langle e_1, \dots, e_N \rangle$ $\qquad G = \{1 = g_1, \dots, g_N\}$

$g e_i = g_{s(g,i)}$ $\qquad g g_i = g_{s(g,i)}$

In terms of characters $\chi_{\mathrm{Ind}^G_H \mathbb{C}}(g) = \begin{cases} |G| & g = e \\ 0 & \text{otherwise} \end{cases}$

② $H \subset G$ any subgroup, $\mathbb{C}$ - trivial rep

$\mathrm{Ind}^G_H \mathbb{C}$ - permutation rep $\mathbb{C}^{G/H}$

③ By Frobenius reciprocity $\mathrm{Ind}^{S_n}_{S_{n-1}} V_\mu = \bigoplus_{\lambda,\, \lambda = \mu + \square} V_\lambda$