<u>Recall</u> $\mathrm{Res}: \mathrm{Rep}\,G \to \mathrm{Rep}\,H$

$\mathrm{Ind}: \mathrm{Rep}\,H \longrightarrow \mathrm{Rep}\,G$

$U \longmapsto K^{G/H} \otimes_{KH} U = \mathrm{Ind}_H^G V$

More elementary $G = \bigsqcup_{i=1}^{|G/H|} g_i H$ $\quad g\, g_i = g_{s(g,i)} \cdot h_{g,i}$

$\mathrm{Ind}_H^G V = \langle e_i \otimes U \mid i = 1, \ldots, G/H \rangle$

$g\,(e_i \otimes v) = e_{s(g,i)} \otimes h(g,i)\, v$

Matrix for g - block perm. corresp. to $i \mapsto s(g,i)$ and block are given by $h(g,i) = g_{s(g,i)}^{-1}\, g\, g_i$

<u>Th</u> (Frobenius character formula)

$$\chi_{\mathrm{Ind}_H^G U}(g) = \frac{1}{|H|} \sum_{\substack{p \in G \\ p^{-1} g p \in H}} \chi_U(p^{-1} g p) \quad \Big(= \sum_{i,\ s(g,i)=i} \chi_U(g_i^{-1} g\, g_i) \Big)$$

Remark $\mathrm{CoInd}_H^G(U) = \{ f : G \to U \mid f(hx) = hf(x) \ \forall x \in G, h \in H \}$
and action $(gf)(x) = f(xg) \quad \forall g \in G$

Geometrically H sections of bundle on G = sections of bundle on $H \backslash G$

For finite groups $\mathrm{Ind}_H^G(U) \simeq \mathrm{Coind}_H^G(U)$.

- **Recall** λ, T, P_T - row-stab Q_T - column stab

$a_\lambda \qquad b_\lambda \qquad c_\lambda = a_\lambda b_\lambda$

- **Question** Characters of V_λ - ?

Our approach: use induced reps

Let $U_\lambda = \mathbb{C}[S_n]a_\lambda$. We have $U_\lambda \simeq \mathrm{Ind}_{P_\lambda}^{S_n} \mathbb{C}$

(Indeed, let $S_n = \bigsqcup_{i=1}^{k} g_i P_\lambda$, then U_λ has basis $g_1 a_\lambda, \dots, g_k a_\lambda$
and action $g\, g_i a_\lambda = g_{s(g,i)}\, p(g,i) a_\lambda = g_{s(g,i)} a_\lambda$)

<u>Remark</u> Similarly $\mathbb{C}[S_n] b_\lambda = \mathrm{Ind}_{Q_\lambda}^{S_n} \mathrm{sgn}$.

<u>Prop</u> $U_\lambda = \sum_{\mu \geq \lambda} K_{\lambda,\mu} V_\mu$ and $K_{\lambda,\lambda} = 1$

<u>Remark</u> ① $K_{\lambda,\mu}$ — Kostka numbers

② χ_{U_λ} - form another basis in space of characters, connected to χ_{V_λ} by <u>triangular</u> matrix

<u>Pf</u> $K_{\lambda,\mu} = \dim \mathrm{Hom}_{S_n}(U_\lambda, V_\mu) = \dim \mathrm{Hom}_{S_n}(\mathrm{Ind}_{P_\lambda}^{S_n} \mathbb{C}, V_\mu) \overset{\text{Frobenious reciprocity}}{=} \dim \mathrm{Hom}_{P_\lambda}(k, V_\mu)$

$$= \dim(a_\lambda \mathbb{C}[S_n] b_\mu a_\mu) = \begin{cases} 0 & \text{for } \lambda > \mu \text{ by Lemma} \\ 1 & \text{for } \lambda = \mu \text{ by Lemma} \\ ? & \text{for } \lambda < \mu \end{cases}$$

(Recall $a_\lambda x b_\mu = 0$ $\lambda > \mu$.)

- Notations $p_m(x) = \sum_{j=1}^{N} x_j^m$

Partition $\mu = (\mu_1, \dots) = (1^{i_1}, 2^{i_2}, \dots)$ i_m parts of length m. Let C_i be corresp conjugacy class with i_m cycles of length m, $\forall m$

$$g \sim (a_1), \dots, (a_{i_1})(b_1, b_1'), \dots, (b_{i_2}, b_{i_2}')(c_1, c_1', c_1''), \dots, (c_{i_3}, c_{i_3}', c_{i_3}''), \dots$$

$$p_\mu = \prod p_{\mu_i} = \prod_m p_m^{i_m}$$

<u>Th1</u> Let $N > \ell(\lambda)$ Then $\chi_{U_\lambda}(C_i)$ is equal to the coefficient of $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N}$ in p_μ

<u>Pf</u> Use Frobenious character formula. Take $g \in C_i$

$$\chi_{U_\lambda}(g) = \frac{1}{|P_T|} \sum_{p \in G,\, p^{-1}gp \in P_T} \chi_{triv}(p^{-1}gp) = \frac{1}{|P_T|} |\{p \mid p^{-1}gp \in P_T\}| =$$

$$= \frac{1}{|P_T|} |\{(x, p) \mid x \in C_i \cap P_T,\, x = p^{-1}gp\}| = \frac{|Z_g|}{|P_T|} |C_i \cap P_T|$$

Since $p_1^{-1}gp_1 = p_2^{-1}gp_2 \iff p_2 p_1^{-1} \cdot g = g\, p_2 p_1^{-1} \iff p_2 p_1^{-1} \in Z_g$ - centalizer of g

- We have $P_T = \prod S_{\lambda_i} \Rightarrow |P_T| = \prod \lambda_i!$
- $p \in Z_g \Rightarrow \forall m$ p can permute i_m cycles of length m and cyclically shift each of them $\Rightarrow |Z_g| = \prod_m i_m! \cdot m^{i_m}$
- $|C_i \cap P_\lambda| = \sum_\Gamma \prod_{j \geq 1} \frac{\lambda_j!}{\prod m^{r_m} r_{jm}!}$. here summation goes over collections $r_{j,m}$ s.t

$$\sum_m m\, r_{j,m} = \lambda_j \qquad \sum_j r_{j,m} = i_m$$

$\left(C_{\bar{i}} \subset S_n \Rightarrow C_{\bar{r}_1} \times C_{\bar{r}_2} \times \dots \times C_{\bar{r}_{\ell(\lambda)}} \subset S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_{\ell(\lambda)}}, \quad |C_{\bar{r}_j}| = \frac{|S_{\lambda_j}|}{|Z_{C_{\bar{r}_j}}|} = \frac{\lambda_i!}{\prod_m r_{j,m}!\, m^{r_{j,m}}} \right)$

Hence $\chi_{U_\lambda}(C_i) = \sum_\Gamma \prod_m \frac{i_m!}{\prod_j r_{jm}!}$

On the other hand we want to obtain $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_p^{\lambda_p}$

$x_1^{\lambda_1}$ from $p_1^{r_{11}} p_2^{r_{12}} p_3^{r_{13}} \cdots$

$x_2^{\lambda_2}$ from $p_1^{r_{21}} p_2^{r_{22}} \cdot p_3^{r_{23}} \cdots$

$- - - -$

Example S_3

$e \sim i_1 = 3,\ i_2 = i_3 = \dots = 0$ $\qquad (\Sigma x_i)^3 = \Sigma x_i^3 + 3\Sigma x_i^2 x_j + 6\Sigma x_i x_j x_k$

$(1,2) \sim i_1 = 1,\ i_2 = 1,\ i_3 = i_4 = \dots = 0$ $\qquad (\Sigma x_i)(\Sigma x_i^2) = \Sigma x_i^3 + \Sigma x_i^2 x_j$

$(1,2,3) \sim i_1 = i_2 = 0,\ i_3 = 1,\ i_4 = \dots = 0$ $\qquad \Sigma x_i^3 = \Sigma x_i^3$

Hence

		e	(12)	(123)
irreps V_λ	$V_{(3)}$	1	1	1
	$V_{(2,1)}$	2	0	-1
	$V_{(1,1,1)}$	1	-1	1
induced reps U_λ	$U_{(3)}$	1	1	1
	$U_{(2,1)}$	3	1	0
	$U_{(1,1,1)}$	6	0	0

Hence

$$U_{(3)} = V_{(3)}$$
$$U_{(2,1)} = V_{(3)} + V_{(2,1)}$$
$$U_{(1,1,1)} = V_{(3)} + V_{(2,1)} + V_{(1,1,1)}$$

Three bases in the space of characters

$$\begin{tikzcd} \{\delta_{C_\mu}\} \arrow[r, leftrightarrow, "\text{Th 1}"'] \arrow[rr, leftrightarrow, bend right, "\text{Th 2}"'] & \{\chi_{U_\lambda}\} \arrow[r, leftrightarrow, "\text{triangular}"'] & \{\chi_{V_\lambda}\} \end{tikzcd}$$

• Th2 (Frobenius character formula) Let $N \geq \ell(\lambda)$ then $\chi_{V_\lambda}(C_{\bar{i}})$ is a coefficient of $x_1^{\lambda_1+N-1} x_2^{\lambda_2+N-2} \dots x_N^{\lambda_N}$, $p=\ell(\lambda)$ in the polynomial $\prod_{i<j}(x_i - x_j) \prod_m H_m^{i_m}$

Remark Equivalently, $\chi_{V_\lambda}(C_{\bar{i}})$ is the coefficient of $x^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N}$ in $\prod_{i<j}(1 - x_i^{-1} x_j) p_\mu$

Plan of proof Let θ_λ be functions defined in theorem

① Show that $\theta_\lambda = \sum_{\mu > \lambda} L_{\lambda\mu} \chi_{U_\mu}$ $L_{\lambda\mu} \in \mathbb{Z}$, $L_{\lambda\lambda} = 1$

Indeed $\theta_\lambda = \sum_\sigma (-1)^\sigma \chi_{U_{\lambda+\rho-\sigma(\rho)}}$ – (terms w/ negative parts are zero

Reorder parts in $\lambda+\rho-\sigma(\rho)$ if necessary)

$\rho = (N-1, N-2, \dots, 0)$

② Show orthogonality $\langle \theta_\lambda, \theta_\lambda \rangle = 1$ ~ Cauchy identity

- In terms of symmetric functions

<u>Def</u> Monomial functions $m_\lambda = x_1^{\lambda_1} \cdots x_N^{\lambda_N} +$ permutations

$$\prod_{i<j} (x_i - x_j) \prod p_m^{i_m} = \sum_\lambda \chi_\lambda(C_\mu) \, \mathrm{Asym}(x_1^{l_1} \cdots x_N^{l_N})$$

here and below $l_i = \lambda_i + N - i$

<u>Def</u> Schur polynomial $S_\lambda(x) = \dfrac{\mathrm{Asym}(x_1^{l_1} \cdots x_N^{l_N})}{\prod_{i<j} (x_i - x_j)}$

<u>Th 1</u> $p_\mu = \sum_\lambda \chi_{U_\lambda}(C_\mu) \, m_\lambda$

<u>Th 2</u> $p_\mu = \sum_\lambda \chi_{V_\lambda}(C_\mu) \, S_\lambda$

<u>Prop</u> $S_\lambda = \sum K_{\lambda\mu} \, m_\mu$