Reps of $S_n$

- $\lambda \rightsquigarrow V_\lambda$ Frobenious formula $\sim 1900$

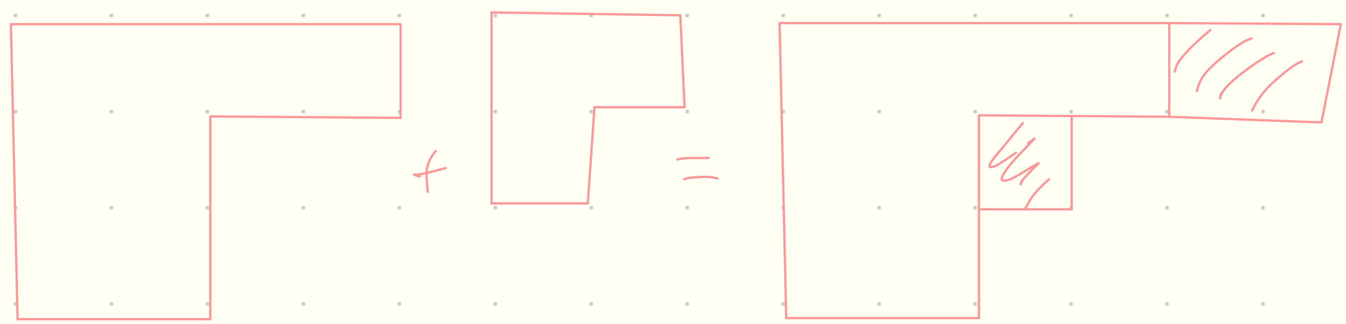
$$\chi_{V_\lambda}(C_\mu) = \text{coeff } x_1^{\ell_1}\cdots x_N^{\ell_N} \text{ in } \prod_{i<j}(x_i-x_j)\, p_\mu$$

where $\ell_i = \lambda_i + N - i$ $\quad p_\mu = \prod p_{\mu_i} = \prod p_m^{i_m}$ $\quad p_n = \sum x_i^n$

$+\ \ \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$

$N-1 > N-2 > N-3 > \ldots$

$\ell_1 > \ell_2 > \ell_3 >$



<u>Question</u> Is this formula useful?

- **Corollary** $\dim V_\lambda = \chi_{V_\lambda}(1) = \frac{n!}{\prod l_j!} \prod_{i<j} (l_i - l_j) \quad \forall N \geq l(\lambda)$

<u>Pf</u> $\dim V_\lambda$ – coeff of $x_1^{l_1} \cdots x_N^{l_N}$ in

$$\prod_{i<j}(x_i - x_j)\left(\sum x_i\right)^n = \det \begin{vmatrix} x_1^{N-1} & & x_N^{N-1} \\ \vdots & & \vdots \\ x_1 & & \\ 1 & & 1 \end{vmatrix} \sum \frac{n!}{k_1! \cdots k_n!} x_1^{k_1} \cdots x_N^{k_N} =$$

$$= \sum_\sigma (-1)^\sigma x_1^{N-\sigma(1)} \cdots x_N^{N-\sigma(N)} \sum \frac{n!}{\prod k_i!} \prod x_i^{k_i}$$

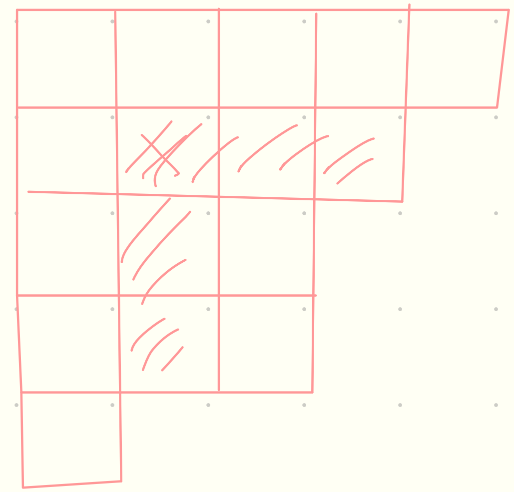
$$\dim V_\lambda = \sum_\sigma (-1)^\sigma \frac{n!}{\prod (l_i - N + \sigma(i))!} = \frac{n!}{\prod l_i!} \sum (-1)^\sigma l_i^{\downarrow N-\sigma(i)} =$$

$$= \frac{n!}{\prod l_i} \det \begin{vmatrix} l_1^{\downarrow N-1} & & l_N^{\downarrow N-1} \\ \vdots & & \\ l_1 & & l_N \\ 1 & \cdots & 1 \end{vmatrix} = \frac{n!}{\prod l_i} \prod (l_i - l_j)$$

$$a^{\downarrow k} = \underbrace{a(a-1)\cdots(a-k+1)}_{k}$$

**Remark** In this formula: many cancelations

**Def** hook of box $s \in \lambda$ $s=(i,j)$ $(i \le \lambda_j)$

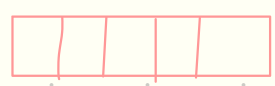


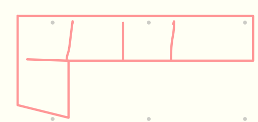

$\{(i,j') \in \lambda \mid j' \ge j\} \cup \{(i',j) \in \lambda \mid i' \ge i\}$

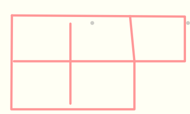
$h(i,j)$ - length $\lambda_j - i + \lambda'_i - j + 1$

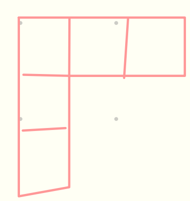
**Th** $\dim V_\lambda = \dfrac{n!}{\prod_{s \in \lambda} h(s)}$ does not depend on $N$

**Example** $n=5$

 $1 = \dfrac{5!}{5!}$

 $4 = \dfrac{5!}{4 \cdot 3!}$

 $5 = \dfrac{5!}{4 \cdot 3 \cdot 2}$

 $= \dfrac{5!}{5 \cdot 2 \cdot 2} = 6$

1 1 4 4 5 5 6

$\dim V_\lambda \mid n!$

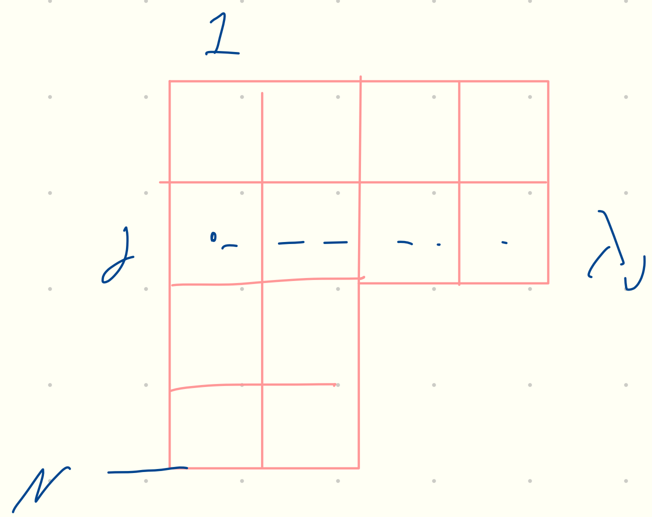
Pf Want to prove $\dfrac{\prod \ell_i!}{\prod_{i<j}(\ell_i-\ell_j)} = \prod_{s\in\lambda} h(s)$

Induction by $\ell(\lambda)$

If we add zero row: $N \to N+1$, $\lambda_{N+1}=0$, $\ell_i \mapsto \ell_i+1$

$\ell_{N+1}=0$

L.S. $\rightsquigarrow$ LS $\cdot \dfrac{\prod(\ell_i+1)}{\prod(\ell_i+1)} =$ LS




Can assume $N=\ell(\lambda)$, $\lambda_N\neq 0$

$s=(1,j)$ – in 1st column

$h_s = \lambda_j + N - j = \ell_j$

$\tilde\lambda_j = \lambda_j - 1$ $\quad$ $\tilde\ell_j = \ell_j - 1$

$$\frac{\prod \ell_j!}{\prod(\ell_i-\ell_j)} = \prod \ell_j \cdot \frac{\prod(\ell_j-1)!}{\prod(\ell_i-\ell_j)} = \prod \ell_j \prod_{s\in\tilde\lambda} h(s) = \prod_{s\in\lambda} h(s)$$

- Ex $\mathrm{Res}^{S_n}_{S_{n-1}} V_\lambda = \bigoplus_{\mu,\ \lambda=\mu+\square} V_\mu$.

- Another approach "new" - Vershik-Okounkov ~90s

Idea - $S_1 \subset S_2 \subset \dots \subset S_n \subset \dots$ Reconstruct rep. th of symmetric groups altogether.

$S_k$ - permutes $\{1 \dots k\}$

- General setting

$H \subset G$. $kH = \bigoplus_{U \in \mathrm{Irr}(H)} \mathrm{End}\, U$ $\quad kG = \bigoplus_{V \in \mathrm{Irr}\, G} \mathrm{End}\, V$

$\mathrm{Res}^G_H V = \bigoplus_U U \otimes M_{V,U}$ $\quad M_{V,U} = \mathrm{Hom}(U, \mathrm{Res}^G_H V)$

multiplicity spaces - reps of another algebra

**Def** Let $\sigma: B \to A$ algebra homomorphism

$Z_B(A) = \{a \in A \mid ab = ba \ \forall b \in B\}$ centralizer of $B$ in $A$.

In particular $Z_A(A) = Z(A)$ - center of $A$

We need $Z_{kH}(kG)$

**Lemma** $Z_{kH}(kG) = \bigoplus_{\substack{V \in Irr(G) \\ U \in Irr(H)}} End(M_{V,U})$ — meaning: By density thm $End(V)$ come from $kG$, $End\, M_{V,U}$ come from $Z_{kH} kG$.

**Pf** $a = (a_V)_{V \in Irr\, G} \in \bigoplus_{V \in Irr\, G} End(V) = kG$

$a \in Z_{kH} kG \iff \forall V \ a_V \in Z_{kH} End\, V$

But $Z_{kH} End\, V = \{\varphi \in End\, V \mid h\varphi = \varphi h \ \forall h\} = End_H V = \bigoplus_U End(M_{V,U})$

Hence $Z_{kH} kG = \bigoplus_V Z_{kH} End\, V = \bigoplus_{V,U} End(M_{V,U})$ $\square$

**Corollary** $Z_{KH} KG$ - commutative $\iff \dim M_{V,U} \leq 1 \ \forall V, U$

$$Z(kG) = \sum a_g g, \quad a_g \in k \quad \text{s.t} \quad a_{xgx^{-1}} = a_g \quad \forall g, x$$

**Lemma** $Z_{KH}(kG) = \sum a_g g, \ a_g \in k$ s.t $a_{hgh^{-1}} = a_g \ \forall g \in G, h \in H$

**Notation** $Z_m(n) = Z_{kS_m}(kS_n)$

**Example** $S_{n-1} \subset S_n \qquad J_n = \sum_{i=1}^{n-1} (i, n) \qquad J_n \in Z_{n-1}(n)$

Jucys-Murphy element

More generally $Z_m(n)$ contains

(a) $Z(kS_m)$

$$\underbrace{1 \ \dots \ m}_{S_m} \ m+1 \ \dots \ n$$

(b) $kS_{[m+1, n]}$ — permutations of $m+1, \dots, n$

(c) $J_{m+1}, \dots, J_n$

<u>Th</u> $Z_m(n)$ is generated as algebra by (a), (b), (c)

<u>Pf</u> See refs $\square$

<u>Corollary</u> (i) $Z_{n-1}(n)$ – commutative

(ii) For $V \in \operatorname{Irr} S_n$, $\operatorname{Res}^{S_n}_{S_{n-1}} V = \bigoplus U_i$ – pairwise non-isom.

(iii) $\forall i$ $J_n|_{U_i}$ = scalar

<u>Rem</u> We will show later that $J_n|_{U_i} \neq J_n|_{U_j}$ for $i \neq j$

i.e $S_{n-1}$ submodules are determined by $J_n$

<u>Example</u> $S_n$: $\mathrm{refl}_n = \{(x_1, \ldots, x_n) \mid \sum x_i = 0\}$ reflection rep

$(i,j)$ - reflection w.r.t. plane $x_i = x_j$

$$\mathrm{Res}^{S_n}_{S_{n-1}} \mathrm{refl}_n = \underbrace{\{(x_1, \ldots, x_{n-1}, 0) \mid \sum x_i = 0\}}_{= \mathrm{refl}_{n-1}} + \underbrace{\{(x, \ldots, x, (1-n)x)\}}_{= \mathrm{triv}_{n-1}}$$

Action of $\mathcal{J}_n$: $(x_1, \ldots, x_n) \mapsto (x_n, x_2, \ldots, x_{n-1}, x_1) + (x_1, x_n, \ldots, x_{n-1}, x_2) + \ldots$

$$= (x_n + (n-2)x_1, x_n + (n-2)x_2, \ldots, x_n + (n-2)x_{n-1}, -x_n).$$

Hence $\mathcal{J}_n\big|_{\mathrm{refl}_{n-1}} = (n-2)\,\mathrm{Id}$ $\qquad$ $\mathcal{J}_n\big|_{\mathrm{triv}_{n-1}} = -\mathrm{Id}$