Reps of $SL(2,\mathbb{R})$

$$SU_2, \quad SL_2(\mathbb{R}) \subset SL_2(\mathbb{C})$$

$$SU_2 = \{g \mid \det g = 1,\ gg^* = 1\} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \,\middle|\, |a|^2 + |b|^2 = 1 \right\}$$

Corresponding Lie algebras

$$sl_2(\mathbb{C}) = \Big\langle \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{=e}, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{=h}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{=f} \Big\rangle_{\mathbb{C}}$$

$$sl_2(\mathbb{R}) = \langle e, f, h \rangle_{\mathbb{R}} = \langle \sigma_1, i\sigma_2, \sigma_3 \rangle$$

$$su_2 = \{A \mid \operatorname{Tr} A = 0,\ A + A^* = 0\} = \Big\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \Big\rangle = \langle i\sigma_1, i\sigma_2, i\sigma_3 \rangle$$

Complexification $\quad sl_2(\mathbb{C}) = sl_2(\mathbb{R}) \otimes \mathbb{C} = su_2 \otimes \mathbb{C}$

- Topologically $SU_2 = S^3 \subset \mathbb{R}^4$ conn, 1-conn
  $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad t \quad 0$

<u>Lemma</u> [Iwasawa decomposition ~ QR decomposition)

(a) $G = SL_2(\mathbb{C})$, $K = SU_2$, $A = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ $t \in \mathbb{R}$ $N = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$

(b) $G = SL_2(\mathbb{R})$, $K = SO_2$, $A = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ $t \in \mathbb{R}$ $N = \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix}$

Then $G = K \cdot A \cdot N$

<u>Remark</u> (i) Explicitly $\forall g \in G$ $\exists!\, k \in K, a \in A, n \in N$ s.t $g = k \cdot a \cdot n$

(ii) Holds for any real reductive real group.

<u>Pf</u> g defines some basis

Gram-Schmidt action of N makes orthogonal.

action of A makes orthonormal $\square$

Corollary $SL_2(\mathbb{C}) \simeq$ as top space $S^3 \times \mathbb{R}^3$ — connected
1-connected

$SL_2(\mathbb{R}) \simeq$ as top space $S^1 \times \mathbb{R}^2$ — connected
not 1-connected

(Weyl unitarity trick)

Prop V - f.d. vector space $/\mathbb{C}$. There is $1 \leftrightarrow 1$ correspondance between following rep structures on V (preserving submods, morphisms)

(a) Smooth rep of $SL_2(\mathbb{R})$
(b) Smooth rep of SU_2
(c) Holomorphic rep of $SL_2(\mathbb{C})$
(d) rep of $\mathfrak{sl}_2(\mathbb{R})$
(e) rep of $\mathfrak{su}_2$
(f) $\mathbb{C}$-linear rep of $\mathfrak{sl}_2(\mathbb{C})$

Pf (c) $\xRightarrow[\text{restriction}]{}$ (a) (b) $\xRightarrow[\text{differentiate}]{}$ (d) (e) $\xRightarrow[\text{complexification}]{}$ (f) $\xRightarrow[SL_2(\mathbb{C})\text{-1 connected}]{}$ (c) $\square$

Corollary Irred f.d reps of $SL_2(\mathbb{R})$, SU_2 - are L_n

- Def Rep $\rho : G \to End(V)$ is unitary if $\exists$ positive hermitian form $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$ s.t

$\langle v_1 , v_2 \rangle = \langle \rho(g) v_1 , \rho(g) v_2 \rangle$ $\forall g \in G$, $v_1, v_2 \in V$

Equivalently $Im \rho \in U(V, \langle \cdot , \cdot \rangle)$

For some basis of V, $Im \rho \subset U_{\dim V}$

Prop (a) L_n is unitary rep of SU_2 $\forall n \in \mathbb{Z}_{\geq 0}$

(b) L_n is not unitary rep of $SL_2(\mathbb{R})$ $\forall n \in \mathbb{Z}_{\geq 0}$

$\underline{\text{Pf}}$ (a) $L_1 = \mathbb{C}^2$ unitary by def.

$L_n = L_{\boxed{\,|\,|\,|\,}} = S^n L_1$ - has induced $\langle \cdot, \cdot \rangle$

(Another pf of a) - take any $\langle \cdot, \cdot \rangle$ and then average using measure on SU_2

(b) Consider L_n. By (a) $\rho_{L_n}(i\sigma_2)$ hermitian antisymm

$\Rightarrow$ eigenvalues of $\rho_{L_n}(i\sigma_2)$ - purely imaginary

$\Rightarrow$ eigenvalues of $\rho_{L_n}(\sigma_1), \rho_{L_n}(\sigma_3)$ - real $\Rightarrow \rho_{L_n}(\sigma_1), \rho_{L_n}(\sigma_3)$ hermitian antisymm only if $\rho_{L_n}(\sigma_1) = \rho_{L_n}(\sigma_3) = 0 \Rightarrow \rho_{L_n}(\sigma_2) = 0$

(Another pf of b) U_n cmpt $SL_2(\mathbb{R})$ not $\Rightarrow$ $\rho: SL_2(\mathbb{R}) \to U_n$ should have kernel. The only normal subgroups in $SL_2(\mathbb{R})$ are $\{e\}, \{\pm e\}, SL_2(\mathbb{R})$ but first two are not co-compact

$\square$

- <u>Goal</u> Classify unitary reps of $SL_2(\mathbb{R})$

<u>Motivation</u> (i) Physics ~ QM, QFT

(ii) Number theory $SL_2(\mathbb{R}) \curvearrowright H = \{z \mid \operatorname{Im} z > 0\}$

modular forms

Some geometry.

- $\mathbb{R}P^1 = SL_2(\mathbb{R})/B$ where $B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \subset SL_2(\mathbb{R})$
- $SL_2(\mathbb{R})/SO_2 = H = \{z \mid \operatorname{Im} z > 0\}$

<u>Pf</u> $SL_2(\mathbb{R})$ acts on H transitively. $z \longmapsto \dfrac{az+b}{cz+d}$

Stabilizer of $i = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \dfrac{ai+b}{ci+d} = i\right\} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a=d,\ b=-c,\ a^2+b^2=1\right\}$

$$= \left\{\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \,\middle|\, \varphi \in [0, 2\pi)\right\} = SO_2$$

Geometrically: hyperbolic rotations w/ center at i

• One more real form

$SU_{1,1} = \{g \mid g\begin{pmatrix}1&0\\0&-1\end{pmatrix}g^* = \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \det g = 1\} = \{g \mid g \text{ preserves } |z_1|^2-|z_2|^2,\ \det g = 1\}$

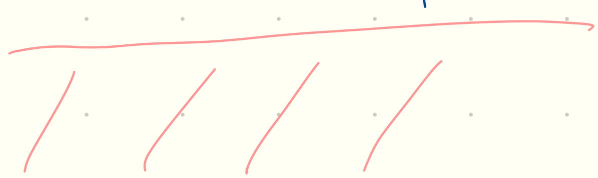
$= \left\{\begin{pmatrix}\alpha&\beta\\ \bar\beta&\bar\alpha\end{pmatrix} \,\middle|\, |\alpha|^2-|\beta|^2=1\right\}$

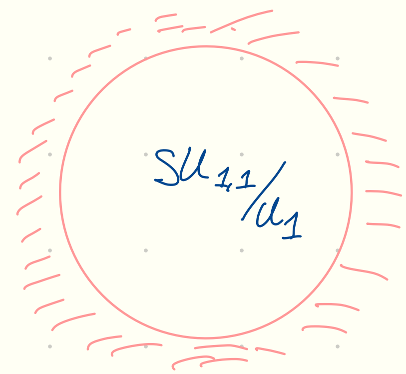
Lie algebra $su_{1,1} = \langle \sigma_1, \sigma_2, i\sigma_3\rangle = \langle e+f, i(e-f), ih\rangle$

c.f. $sl_2(\mathbb{R})$

Geometrically: $SU_{1,1}$ preserves unit circle $|z|^2 = 1$

$SL_2(\mathbb{R})/SO_2$





<u>Lemma</u> $SL_2(\mathbb{R})$ and $SU_{1,1}$ are conjugated in $SL_2(\mathbb{R})$

This conjugation relates SO_2 and U_1

E.g. one can take conj. by $c = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-i\\1&i\end{pmatrix}$

- Below V – top. vector space. Examples

(a) V — Hilbert space. E.g. $V = L^2(X,\mu)$

(b) V – normed space, i.e. $\exists \, \|\cdot\| : V \to \mathbb{R}$, s.t
(i) $\|\lambda x\| = |\lambda| \, \|x\|$ (ii) $\|x+y\| \le \|x\| + \|y\|$ (iii) $\|x\| = 0 \iff x = 0$
topology induced by $d(x,y) = \|x-y\|$

Ex $V = L^p(X,\mu)$ $\|F\| = \left(\int |F|^p d\mu\right)^{1/p}$ $L^\infty(X,\mu)$

(c) V has countable set of seminorms $\|\cdot\|_j$
(now $\|F\|_j = 0$ does not imply $F = 0$)
topology induced by $d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|x-y\|_j}{1+\|x-y\|_j}$

Ex $X = \bigcup_j K_j$ – cmpt subsets $C(X)$ $\|F\|_j = \sup_{x \in K_j} |F(x)|$
$C^\infty(X)$ $\|F\|_{j,k} = \sup_{x \in K_j} |F^{(k)}(x)|$

Def A <u>continuous rep.</u> of G is a top. vector space V w/ continuous linear action $a: G \times V \to V$

Ex $G = \mathbb{R}$, acts on $L^p(\mathbb{R})$ by translations

Def A <u>subrepresentation</u> of a cont. rep. is closed G-invariant subspace

Ex (a) $G = \mathbb{R}$ acts on $V = L^2(\mathbb{R})$ by translations
Do Fourier transform: act by multiplication e^{ix}
Any $U_K = \{F \mid \operatorname{supp} F \subset K\}$ K-cmpt is subrep.

(b) $U(1)$ acts on $L^2(S^1)$ Fourier $L^2(S^1) = \oplus R_n$ where $R_n = \langle e^{2\pi i n\theta} \rangle$ 1-dim rep of $U(1)$

(c) $SU_{1,1}$ acts on $L^2(S^1)$ now irreducible