Unitary reps of $SL_2(\mathbb{R})$

<u>Example</u> $U(1)$ acts on $L^2(S^1)$, coordinate $z=e^{i\theta}$
Can we define action of Lie algebra?
Problem $\frac{d}{d\theta}$ does not act on L^2
Acts on $C^\infty(S^1)$ - dense space of smooth functions

Smaller subset $\langle e^{in\theta} | n \in \mathbb{Z} \rangle$ - of trig. polynomials

Each $e^{in\theta}$ generate 1-dim rep of $U(1)$

<u>Def</u> Vector $v \in V$ is <u>smooth</u> if map $G \to V$ given by $g \mapsto \pi(g)v$ is smooth.
The space of smooth vectors denoted by V^∞

V^∞ is G invariant subspace (but not closed)

Prop Let $v \in V^\infty$. The we have a linear map
$\mathfrak{g} \to V^\infty \quad \pi_X(\xi) = \frac{d}{dt}(\pi(e^{t\xi}v))\Big|_{t=0}$

This defines a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V^\infty)$

- **Prop** V^∞ dense in V

Idea of proof $\forall v \in V \quad v = e \cdot v = \lim_{n\to\infty} \pi(\phi_n v$

here ϕ_n - seq. of smooth measures on G that converges to δ_e — δ function measure

$\pi(\phi_n)v = \int \pi(g)v \, d\phi_n \qquad \pi(\phi_n)v \in V^\infty$ since ϕ_n-smooth

$\square$

• **Def** $v \in V$ is K-finite if it is contained in f.d rep of K. $V^{K\text{-fin}}$ – space of K-fin vectors

Example $K = U(1)$, $V = L^2(S^1)$ $V^{K\text{-fin}}$ – trig polynomials.

Below K – max cmpt subgroup of G

E.g. $G = U(1) \Rightarrow K = U(1)$, $G = SL_2(\mathbb{R}) \Rightarrow K = SO_2(\mathbb{R})$

Def (π, V) – rep G, $K \subset G$. V – is K-admissible if for $\forall$ f.d rep ρ of K $\dim \mathrm{Hom}_K(\rho, V) < \infty$.

[I.e. K is suff big, for $K = e$, $\dim V = \infty \Rightarrow V$ – not admissible]

Example $G = K = U(1)$, $V = L^2(S^1)$

$\mathrm{Irr}\, K = \{ R_n \mid n \in \mathbb{Z} \}$ $R_n : e^{i\varphi} \mapsto e^{in\varphi}$

$\dim \mathrm{Hom}_K(R_n, V) = 1$, $R_n \subset V$ gen by $e^{in\theta}$

Example More generally $G=K$ - cmpt group

Peter-Weyl thm $\Rightarrow L^2(K)=\bigoplus_{\rho\in \mathrm{Irr}\,K} \rho\boxtimes\rho^* = \bigoplus_{\rho\in \mathrm{Irr}\,K} \mathrm{End}(\rho)$

$$L^2(K)^{K\text{-fin}} = \bigoplus_{\rho} \mathrm{End}\,\rho = \bigoplus_{\rho} \rho^{\dim\rho}$$

Example $G=SL_2(\mathbb{R})$, $K=SO(2)$, $B=\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$

$G \curvearrowright G/B = \mathbb{R}P^1$ — hence $V=L^2(\mathbb{R}P^1)$ rep of G $\quad$ x-coord

Better to consider half-densities $f(x)|dx|^{1/2}$

Change of coordinates $y=\varphi(x)$ $\quad f\mapsto f\,|\det\varphi'|^{1/2}$

Action $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$: $f(x)\mapsto \left|\frac{1}{cx+d}\right| f\left(\frac{ax+b}{cx+d}\right)$

Pairing $\langle f,g\rangle = \int \bar{f}(x)g(x)dx$

Recall $G = K\cdot A\cdot N$ $\quad A = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ Hence $G/B = K/\pm 1$

$$L^2(\mathbb{RP}^1) = L^2(S^1)^{\{\pm 1\}} = \overline{\bigoplus_{n\in 2\mathbb{Z}} R_n}$$

Conclusion: $L^2(\mathbb{RP}^1)$ – unitary, admissable rep of G

K-fin vectors $\bigoplus_{n\in 2\mathbb{Z}} R_n$.

Generalization Densities of the form $f(x)\,|dx|^{1/2+is}$ $s\in\mathbb{R}$

Action $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$: $f(x) \mapsto \left|\frac{1}{cx+d}\right|^{1+2is} f\left(\frac{ax+b}{cx+d}\right)$

$$\langle f(x)|dx|^{1/2+is}, g(x)\,|dx|^{1/2+is}\rangle = \int \bar{f}(x)\,g(x)\,dx$$

well defined herm. form $\Rightarrow$ Principal unitary series P_s^+

K-finite subspace – the same $\bigoplus_{n\in 2\mathbb{Z}} R_n$ since $\exists$ K-invariant 1-form – Haar form (e.g $\frac{1}{x^2+1}dx$)

(Appearence of $\mathbb{P}^1$ c.f. Beilinson-Bernstein localization)

Prop V - K-admissible $\Rightarrow V^{K\text{-fin}} \subset V^\infty$ and $\mathfrak{g}$ inv.

Hence $V^{K\text{-fin}}$ is $(\mathfrak{g}, K)$ module

Def $(\mathfrak{g}, K)$ module is vector space M w/ actions of $\mathfrak{g}$ and K s.t.

(i) M - is direct sum of smooth K-modules

(ii) Two action of $\mathfrak{k} = \mathrm{Lie}\, K$ (one from K and one from $\mathfrak{g}$) coincide

M is admissible if $\forall \rho \in \mathrm{Irr}\, K$, $\dim_{\mathbb{C}} \mathrm{Hom}(\rho, M) < \infty$

Admissible $(\mathfrak{g}, K)$ module f.g. over $\mathfrak{g}$ is called Harish-Chandra module.

Th (Harish-Chandra)

(i) $\forall$ unitary irred of G is K-admissible

(ii) $V \mapsto V^{K\text{-fin}}$ defines exact faithful functor $\mathrm{Rep}\,G \to HC_G$, irreps to irreps

(iii) $\forall$ unitory irred HC module uniquely integrates to irred. unitary rep of G.

- M - irred $(\mathfrak{g}, K)$ module $M = \bigoplus_{k \in \mathbb{Z}} R_k \otimes M_k$

multiplicity space

Here $\mathbb{Z} = \mathrm{Irr}\, SO_2$

$$C = 2fe + \frac{h^2}{2} + h$$

Lemma $C|_M$ – const

Pf M-irred. C commutes w/ $\mathcal{U}(\mathfrak{g}) \Rightarrow C$ acts on M by isom. Furthermore if $C|_M \neq const$ $\forall$ polynomial $P \in \mathbb{C}[x]$ $P(C)|_M$ is isom $\Rightarrow$ $\mathbb{C}(x)$ acts on. M $R(x) \longrightarrow R(C)$

$\mathcal{U}(\mathfrak{g})$ -countable dim $\Rightarrow$ M – countable (or finite) dim

But $\mathbb{C}(x)$ continuum dimensional $\{\frac{1}{x-a} | a \in \mathbb{C}\}$ – linear indep.
Hence $\sum \frac{\beta_i}{C-a_i} v = 0 \Rightarrow$ contradiction $\square$

Lemma $\mathcal{U}(\mathfrak{sl}_2) = (\langle e^k | k \in \mathbb{Z}_{>0}\rangle \oplus \langle 1 \rangle \oplus \langle f^k | k \in \mathbb{Z}_{>0}\rangle) \cdot \mathbb{C}[h, C]$
Pf Follows from PBW $\square$

Hence M generated by $v, e^k v, f^k v \mid k \in \mathbb{Z}_{>0}$
Hence $\dim M_k \leq 1$ $\forall k$

Let $P(M)=\{k \mid M_k \neq 0\}$ = set of weights of M.

① $P(M)$ finite $\Rightarrow$ M is f.d $M \simeq L_n$ $n \in \mathbb{Z}_{\geq 0}$

②

2.1 $P(M)$ - infinite bounded from above

$n=\max P(M)$ - highest weight. Then $M \simeq M_n = M_n^+$ - Verma module

Irreducibility $\Rightarrow$ $n \in \mathbb{Z}_{<0}$

2.2 $P(M)$ - infinite bounded from below

$n=\min P(M)$ - lowest weight. Then $M \simeq M_n^-$ lowest weight Verma module

Irreducibility $\Rightarrow$ $n \in \mathbb{Z}_{>0}$

③ $P(M)$ - unbounded c - value of C

3.1 $P(M)=2\mathbb{Z}$, basis v_k

$h v_k = k v_k$ $f v_k = v_{k-2}$ $e v_k = \lambda_k v_{k+2}$ $\lambda_k \neq 0$

$\Lambda_k v_k = fe\, v_k = \left(\frac{1}{2}c - \frac{1}{4}h^2 - \frac{h}{2}\right) v_k = \left(\frac{s^2}{4} - \frac{1}{4} - \frac{k}{2} - \frac{k^2}{4}\right) v_k$

where $c = s^2/2 - 1/2$ Rescalling $v_k \rightsquigarrow w_k$

$h w_k = k w_k$ $f w_k = \frac{1}{2}(s-1+k) w_{k-2}$ $e w_k = \frac{1}{2}(s-1-k) w_{k+2}$

Denote $P_+(s)$ $s \notin 2\mathbb{Z}+1$ (since $\Lambda_k \neq 0$)

(3.2) Similarly for $P(M) = 2\mathbb{Z}+1$. $P_-(s)$

$P_\pm(s)$ – <u>principal series</u>

<u>Th</u> Simple $(\mathfrak{g}, K)$-mod: L_n, $n \in \mathbb{Z}_{\geq 0}$

M^+_{-n}, M^+_n $n \in \mathbb{Z}_{\geq 1}$

$P_+(s)$ $s \notin 2\mathbb{Z}+1$

$P_-(s)$ $s \notin 2\mathbb{Z}$.

- Realization of principal series: forms on $\mathbb{RP}^1$

Coordinate $z = e^{i\theta}$ - (unit circle ~ $SU_{1,1}$ realization)

$P_+(s)$: forms on $\mathbb{RP}^1$ $\quad \phi(e^{i\theta})\, d\theta^{\frac{1-s}{2}}$

$P_-(s)$: forms on $\mathbb{RP}^1$ $\quad e^{i\theta/2}\phi(e^{i\theta})\, d\theta^{\frac{1-s}{2}}$

Group action $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f(z) = |cz+d|^{s-1} f\left(\frac{az+b}{cz+d}\right)$

Lie algebra action $h = 2z\partial_z \quad f = \partial_z \quad e = -z^2\partial_z$

Basis $\quad w_k = e^{ik\theta/2}\, d\theta^{\frac{1-s}{2}} = z^{k/2}\left(\frac{dz}{iz}\right)^{\frac{1-s}{2}}$

Action of f, h, e by formulas above

Geometrically, sections of the bundle $K^{\frac{1-s}{2}}$ on $\mathbb{RP}^1 = G/B$

Coindused representations. In the fiber of B fixed point B acts w/ character $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto |a|^{s-1}$

- Unitarity. Work in realization $su_{1,1}$:

Basis $\{ih, e+f, i(e-f)\}$. unitarity $x^* = -x \ \forall x \in su_2$

Hence $e^* = -f$, $f^* = -e$, $h^* = h$

$$\langle w_k, w_\ell\rangle = \frac{1}{k}\langle hw_k, w_\ell\rangle = \frac{1}{k}\langle w_k, hw_\ell\rangle = \frac{\ell}{k}\langle w_k, w_\ell\rangle \Rightarrow$$

Form is diagonal in w_k basis

Recurrence $\langle fw_k, fw_k\rangle = -\langle efw_k, w_k\rangle = \left(\frac{(k+1)^2 - s^2}{4}\right)\langle w_k, w_k\rangle$

should be positive $\forall k$.

For M_n, M_{-n} – no restriction

Th (Gelfand-Naimark, Bargmann) The irred. unitary reps of $SL_2(\mathbb{R})$ are Hilbert space completions of

- Triv rep $\mathbb{C} = L_0$
- Discrete series M_{-n}^+, M_n^- $\quad n \in \mathbb{Z}_{\geq 1}$
- Unitary principal series $P_\pm(s)$ $\quad s \in i\mathbb{R}$
- The complementary series $P_+(s)$ $\quad s \in \mathbb{R}$ $\quad 0 < |s| < 1$

- Realization For $P_\pm(s)$ - already know (Unitarity clear only for $s \in i\mathbb{R}$)

Discrete series M_{-n}^+ $\quad H = SL_2(\mathbb{R})/SO_2 = \{z \mid \operatorname{Im} z > 0\}$

$\{f$ analytic on $H \mid \|f\|^2 = \int |f(z)|^2 y^{n-2}\, dx\, dy < \infty\}$.

action $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f(z) = (cz+d)^{-n} f\left(\frac{az+b}{cz+d}\right)$

(In other words $\omega = f(z)\,dz^{n/2}$ - holomorphic $n/2$ forms)