- Consider $G = SL_2(\mathbb{F}_p)$ p-odd

Subgroups $H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right\}$, $|H| = p-1$

$U = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right\}$ $|U| = p$ $B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \right\}$ $|B| = p(p-1)$

$G/B = \mathbb{P}^1$ $|\mathbb{P}^1| = |\mathbb{A}^1| + |\mathbb{A}^0| = p+1$ $|G| = p^3 - p$

$H \simeq \mathbb{F}_p^{\times}$ - cyclic group $\mathbb{Z}/(p-1)\mathbb{Z}$ ε - generator $\varepsilon^{p-1} = 1$

conj classes

Case 1 g has eigenvalues in $\mathbb{F}_p$

$$g \sim \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \varepsilon^{\ell} & 0 \\ 0 & \varepsilon^{-\ell} \end{pmatrix} \quad \ell = 1, \dots \frac{p-3}{2}.$$

$C_g = |G| / |Z_g(G)|$ $Z_g(G) = H$ $\Rightarrow |C_g| = p(p+1)$

Two exceptions $g = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $|C_g| = 1$

Case 2 Eigenvalues not in F_p $\Rightarrow$ eigenvalues in $\mathbb{F}_{p^2}$

$\mathbb{F}_{p^2} = \mathbb{F}_p[\sqrt{\xi}] = \{\alpha + \beta\sqrt{\xi} \mid \alpha, \beta \in \mathbb{F}_p\}$ ξ - is not square in $\mathbb{F}_p$

$\xi = \varepsilon^{2\ell+1} \Rightarrow \sqrt{\xi} = \varepsilon^{\ell}\sqrt{\varepsilon} \Rightarrow$ w.l.o.g. $\mathbb{F}_{p^2} = \mathbb{F}_p[\sqrt{\varepsilon}]$

In basis $1, \sqrt{\varepsilon}$ of $\mathbb{F}_{p^2}$ mult by $\alpha + \beta\sqrt{\varepsilon}$ has the form

$$\begin{pmatrix} \alpha & \varepsilon\beta \\ \beta & \alpha \end{pmatrix}$$

Let $K = \left\{ \begin{pmatrix} \alpha & \varepsilon\beta \\ \beta & \alpha \end{pmatrix} \middle| \alpha^2 - \varepsilon\beta^2 = 1 \right\}$ det = 1 condition

$(\alpha + \beta\sqrt{\varepsilon})(\alpha - \beta\sqrt{\varepsilon}) = 1 \Rightarrow \alpha + \beta\sqrt{\varepsilon}$ solves eq. $x^2 - 2\alpha x + 1 = 0$

eqs $= p$ # eqs w/ sols in $\mathbb{F}_p$ — $\frac{p-3}{2} + 2 = \frac{p+1}{2}$ # other eqs $\frac{p-1}{2}$

$|K| = 2\,\frac{p-1}{2} + 2 = p+1$ $K \simeq \mathbb{Z}/(p+1)\mathbb{Z}$ Let $\varkappa$ generator of K

$\pm 1 \in K$

(Another pf $\alpha - \beta\sqrt{\varepsilon} = Fr(\alpha + \beta\sqrt{\varepsilon}) = (\alpha + \beta\sqrt{\varepsilon})^p$ Hence

$\alpha^2 - \beta^2\xi = 1 \Leftrightarrow (\alpha + \beta\sqrt{\varepsilon})^{p+1} = 1$ $K = \sqrt[p+1]{1} \subset \mathbb{F}_{p^2}$)

$$Z_{\varkappa^k} = K \Rightarrow |Z_{\varkappa^k}| = p+1 \Rightarrow |C_{\varkappa^k}| = p(p-1)$$

Case 3 Unipotent matrices $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} \in GL_2(\overline{\mathbb{F}_p})$

$\det = 1 \Rightarrow \lambda = \pm 1$

Conjugation in $GL_2(\overline{\mathbb{F}_p})$ $\quad Mg_1M^{-1} = g_2 \Leftrightarrow$ linear eqs $Mg_1 = g_2M \Rightarrow$ conjugation in $GL_2(\mathbb{F}_p)$

But not necessary in $SL_2(\mathbb{F}_p)$ since $\det g$ could be not-square

Namely, conjugate $\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}$ to $\begin{pmatrix}1 & \lambda'\\ 0 & 1\end{pmatrix}$, $\varkappa, \varkappa' \neq 0$

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & \lambda'\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

$$\begin{pmatrix}\alpha & \alpha\lambda+\beta\\ \gamma & \gamma\lambda+\delta\end{pmatrix} \qquad \begin{pmatrix}\alpha+\gamma\lambda' & \beta+\delta\lambda'\\ \gamma & \delta\end{pmatrix}$$

$$\Rightarrow \begin{cases}\gamma = 0\\ \alpha\lambda = \delta\lambda'\\ \alpha\delta = 1\end{cases} \Rightarrow \lambda/\lambda' \text{ - square in } \mathbb{F}_p^\times$$

2 conj. classes $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & \varepsilon\\ 0 & 1\end{pmatrix}$. Similarly $\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$, $\begin{pmatrix}-1 & \varepsilon\\ 0 & -1\end{pmatrix}$

From computation above $Z_{\begin{pmatrix}1&1\\0&1\end{pmatrix}} = \left\{ \begin{pmatrix}\pm1 & \beta\\ 0 & \pm1\end{pmatrix} \right\} \Rightarrow \left| Z_{\begin{pmatrix}1&1\\0&1\end{pmatrix}} \right| = 2p \Rightarrow$

$\left| C_{\begin{pmatrix}1&1\\0&1\end{pmatrix}} \right| = \frac{p^2-1}{2}$. Similarly $\left| C_{\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}} \right| = \left| C_{\begin{pmatrix}-1&1\\0&-1\end{pmatrix}} \right| = \left| C_{\begin{pmatrix}-1&\varepsilon\\0&-1\end{pmatrix}} \right| = \frac{p^2-1}{2}$

Summarize

			$\ell = 1, \dots, \frac{p-3}{2}$	$k = 1, \dots, \frac{p-1}{2}$				
$\lvert C \rvert$	1	1	$p(p+1)$	$p(p-1)$	$\frac{p^2-1}{2}$	$\frac{p^2-1}{2}$	$\frac{p^2-1}{2}$	$\frac{p^2-1}{2}$
Conj. clas	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}\varepsilon^{\ell}&0\\0&\varepsilon^{-\ell}\end{pmatrix}$	x^k	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&1\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&\varepsilon\\0&-1\end{pmatrix}$

\# conj classes $p+4$

\# elements $1 + 1 + \frac{p-3}{2} p(p+1) + \frac{p-1}{2} p(p-1) + 4 \cdot \frac{p^2-1}{2} = \dots = p^3 - p$

- Reps. Trivial $\mathbb{1}_G$.

Let $P_s = \mathrm{Ind}_B^G R_s$, R_s - 1 dim rep of B

R_s trivial on U, $H = \mathbb{Z}/(p-1)\mathbb{Z}$ reps $\varepsilon^e \mapsto \omega^{se}$, where $s \in \mathbb{Z}/(p-1)\mathbb{Z}$ $\omega = e^{\frac{2\pi i}{p-1}}$

Frobenious formula $\chi_{\mathrm{Ind}_B^G R_s}(g) = \frac{1}{|B|} \sum\limits_{\substack{h \in G \\ h^{-1}gh \in B}} \chi_{R_s}(hgh^{-1}) = \sum\limits_{\substack{hB \in G/B, \\ h^{-1}gh \in B}} \chi_{R_s}(hgh^{-1})$

Case 1 $\chi_{P_s}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \dim P_s = \frac{|G|}{|B|} = p+1$ $\chi_{P_s}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) = (-1)^s (p+1)$

$h^{-1}\begin{pmatrix} \varepsilon^e & 0 \\ 0 & \varepsilon^{-e} \end{pmatrix} h \in B \iff h \in B$ or $h \in \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B \implies$

$\chi_{P_s}\left(\begin{pmatrix} \varepsilon^e & 0 \\ 0 & \varepsilon^{-e} \end{pmatrix}\right) = \omega^{se} + \omega^{-se}$

Case 2 $C_{x^k} \cap B = \emptyset$ since $\forall$ element of B has eigenvalues in $F_p \implies \chi_{P_s}(x^k) = 0$

Case 3 $h^{-1}\begin{pmatrix}1&1\\0&1\end{pmatrix}h \in B \iff h \in B \Rightarrow \chi_{P_s}\begin{pmatrix}1&1\\0&1\end{pmatrix} = \chi_{R_s}\begin{pmatrix}1&1\\0&1\end{pmatrix} = 1$

Character in the table below. Note $P_s = P_{p-1-s}$

<u>Lemma</u> $\langle \chi_{P_s}, \chi_{P_s} \rangle = \begin{cases} 2 & s=0 \text{ or } s=\frac{p-1}{2} \\ 1 & \text{otherwise} \end{cases}$

<u>Pf</u> $\langle \chi_{P_s}, \chi_{P_s} \rangle = \frac{1}{p^3-p}\left(2(p+1)^2 + 4\frac{p^2-1}{2} + p(p+1)\sum_{\ell=1}^{(p-3)/2}(\omega^{\ell s}+\omega^{-\ell s})(\omega^{\ell s}+\omega^{-\ell s})\right) =$

$= \frac{1}{(p-1)}\left(4 + \sum_{\ell=1}^{(p-3)/2}(\omega^{2\ell s} + \omega^{-2\ell s} + 2)\right)$ | Note $\sum_{\ell=1}^{(p-3)/2}\omega^{2\ell s} = \begin{cases} (p-3)/2 & \text{if } \omega^{2s}=1 \\ \frac{\varepsilon^{\frac{p-1}{2}2s} - \varepsilon^2}{\varepsilon^{2s}-1} = -1 & \text{if } \omega^{2s} \neq 1 \end{cases}$

$= \begin{cases} \frac{1}{p-1}(4 + (p-3) + (p-3)) & \varepsilon^{2s}=1 \\ \frac{1}{p-1}(4 + (p-3) - 2) & \varepsilon^{2s} \neq 1 \end{cases} = \begin{cases} 2 & \varepsilon^{2s}=1 \\ 1 & \varepsilon^{2s} \neq 1 \end{cases}$ $\square$

Hence for $1 \le s \le \frac{p-3}{2}$ P_s – irred

For $s=0$ $P_0 = \mathbb{1}_G + St$, St – q dim rep (Steinberg rep)

For $s=\frac{p-1}{2}$ $P_{(p-1)/2} = P^+_{(p-1)/2} \oplus P^-_{(p-1)/2}$

- Consider now $\mathrm{Ind}_K^G R_n$. Here R_n rep of $K \in \mathbb{Z}/(p+1)\mathbb{Z} \Rightarrow n \in \mathbb{Z}/(p+1)\mathbb{Z}$ $\quad R_n(\varkappa^k) = \sigma^{kn}$, $\sigma = e^{2\pi i/(p+1)}$

Compute character using Frobenius formula

Note that conj. classes of $\begin{pmatrix} \varepsilon^e & 0 \\ 0 & \varepsilon^{-e} \end{pmatrix}$, $\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$, $\begin{pmatrix} \pm 1 & \varepsilon \\ 0 & \pm 1 \end{pmatrix}$ do not intersect K.

$h\varkappa^k h^{-1} \in K \iff h \in K,\ h\varkappa^k h^{-1} = \varkappa^k$ or $h \in \left\{ \begin{pmatrix} \gamma & -\delta\varepsilon \\ \delta & -\gamma \end{pmatrix} \middle| \gamma^2 - \delta\varepsilon^2 = -1 \right\}$ and $h\varkappa^k h^{-1} = \varkappa^{-k}$

Hence:

Conj. clas	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \varepsilon^e & 0 \\ 0 & \varepsilon^{-e} \end{pmatrix}$	$\varkappa^k$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & \varepsilon \\ 0 & -1 \end{pmatrix}$
$\mathrm{Ind}_K^G R_n$	$p(p-1)$	$(-1)^n p(p-1)$	0	$\sigma^{kn} + \sigma^{-kn}$	0	0	0	0

Note $\mathrm{Ind}_K^G R_n = \mathrm{Ind}_K^G R_{p+1-n}$

$\mathrm{Ind}_K^G R_n$ too big to be irred. Consider $M_n = P_n \cdot St - P_n - \mathrm{Ind}_K^G R_n$

Character of this (virtual) rep is in table below.

<u>Lemma</u> $\langle \chi_{M_n}, \chi_{M_n} \rangle = \begin{cases} 2 & n = 0, \frac{p+1}{2} \\ 1 & \text{otherwise} \end{cases}$

Pf $\langle \chi_{M_n}, \chi_{M_n} \rangle = \frac{1}{p^3-p}\left(2(p-1)^2 + 4\frac{p^2-1}{2} + p(p-1)\sum_{k=1}^{(p-1)/2}(\sigma^{kn}+\sigma^{-kn})(\sigma^{kn}+\sigma^{-kn})\right)$

$= \frac{1}{p+1}\left(4 + \sum_{k=1}^{(p-1)/2}(\sigma^{2kn}+\sigma^{-2kn}+2)\right) = \Big|$ Similarly to before $\Big|$

$= \begin{cases} \frac{1}{p+1}(4+(p-1)+(p-1)) & \text{if } \sigma^{2n}=1 \\ \frac{1}{p+1}(4+(p-1)-2) & \text{if } \sigma^{2n}\neq 1 \end{cases} = \begin{cases} 2 & \text{if } \sigma^{2n}=1 \\ 1 & \text{if } \sigma^{2n}\neq 1 \end{cases}$ $\square$

Hence M_n, $n=1,\dots,\frac{p-1}{2}$ are irreps. And $M_{\frac{p+1}{2}} = M^+_{\frac{p+1}{2}} \oplus M^-_{\frac{p+1}{2}}$

(Note also $M_0 = St - 1\!\!1_G$ - not new rep)

Overall we found $1+\frac{p-3}{2}+1+2+\frac{p-1}{2}+2 = p+4$ reps

$= \#$ conj classes

$\sum \dim^2 = 1 + \frac{p-3}{2}(p+1)^2 + p^2 + 2\left(\frac{p+1}{2}\right)^2 + \frac{p-1}{2}(p-1)^2 + 2\left(\frac{p-1}{2}\right)^2 = p^3-p = |G|$

Hence we found full list

			$\ell=1,\dots,\frac{p-3}{2}$	$k=1,\dots,\frac{p-1}{2}$				
$\lvert C \rvert$	1	1	$p(p+1)$	$p(p-1)$	$\frac{p^2-1}{2}$	$\frac{p^2-1}{2}$	$\frac{p^2-1}{2}$	$\frac{p^2-1}{2}$
Conj. class	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}\varepsilon^{\ell}&0\\0&\varepsilon^{-\ell}\end{pmatrix}$	$\varkappa^{k}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&1\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&\varepsilon\\0&-1\end{pmatrix}$
1_G	1	1	1	1	1	1	1	1
P_s, $s=1,\dots,\frac{p-3}{2}$	$(p+1)$	$(-1)^s(p+1)$	$\omega^{\ell s}+\omega^{-\ell s}$	0	1	1	$(-1)^s$	$(-1)^s$
St	p	p	1	-1	0	0	0	0
$P^+_{\frac{p-1}{2}}$	$\frac{p+1}{2}$	$(-1)^{\frac{p-1}{2}}\frac{p+1}{2}$	$(-1)^{\ell}$	0	$\frac{1}{2}\left(1+\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{1}{2}\left(1-\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{(-1)^{\frac{p-1}{2}}}{2}\left(1+\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{(-1)^{\frac{p-1}{2}}}{2}\left(1+\sqrt{(-1)^{(p-1)/2}p}\right)$
$P^-_{\frac{p-1}{2}}$	$\frac{p+1}{2}$	$(-1)^{\frac{p-1}{2}}\frac{p+1}{2}$	$(-1)^{\ell}$	0	$\frac{1}{2}\left(1-\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{1}{2}\left(1+\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{(-1)^{\frac{p-1}{2}}}{2}\left(1+\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{-1)^{\frac{p-1}{2}}}{2}\left(1+\sqrt{(-1)^{(p-1)/2}p}\right)$
M_n, $n=1,\dots,\frac{p-1}{2}$	$(p-1)$	$(-1)^n(p-1)$	0	$-\sigma^{kn}-\sigma^{-kn}$	-1	-1	$(-1)^{n+1}$	$(-1)^{n+1}$
$M^+_{\frac{p+1}{2}}$	$\frac{p-1}{2}$	$(-1)^{\frac{p+1}{2}}\frac{p-1}{2}$	0	$(-1)^{k+1}$	$\frac{1}{2}\left(-1+\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{1}{2}\left(-1-\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{(-1)^{\frac{p+1}{2}}}{2}\left(-1+\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{(-1)^{\frac{p+1}{2}}}{2}\left(-1+\sqrt{(-1)^{(p-1)/2}p}\right)$
$M^-_{\frac{p+1}{2}}$	$\frac{p-1}{2}$	$(-1)^{\frac{p+1}{2}}\frac{p-1}{2}$	0	$(-1)^{k+1}$	$\frac{1}{2}\left(-1-\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{1}{2}\left(-1+\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{(-1)^{\frac{p+1}{2}}}{2}\left(-1+\sqrt{(-1)^{(p-1)/2}p}\right)$	$\frac{(-1)^{\frac{p+1}{2}}}{2}\left(-1+\sqrt{(-1)^{(p-1)/2}p}\right)$