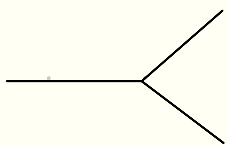
Quantum groups

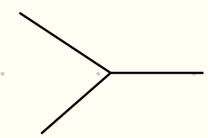
- A -algebra Rep_A, have $V_1 \oplus V_2$ what about $V_1 \otimes V_2$?

Symmetry of $V_1 \otimes V_2$ is $A \otimes A$

<u>Def</u> Coproduct $\Delta : A \to A \otimes A$ algebra homomorphism

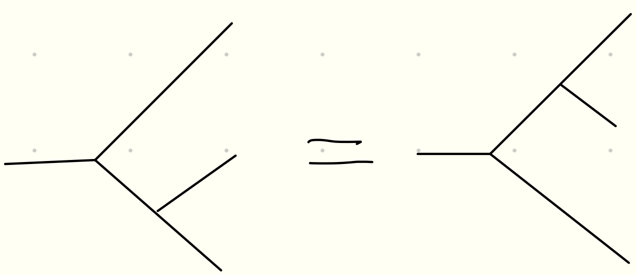
Pictures $\Delta : A \to A \otimes A$ c.f. product $m : A \otimes A \to A$





$(V_1 \otimes V_2) \otimes V_3 = V_1 \otimes V_2 \otimes V_3 = V_1 \otimes (V_2 \otimes V_3)$ c.f

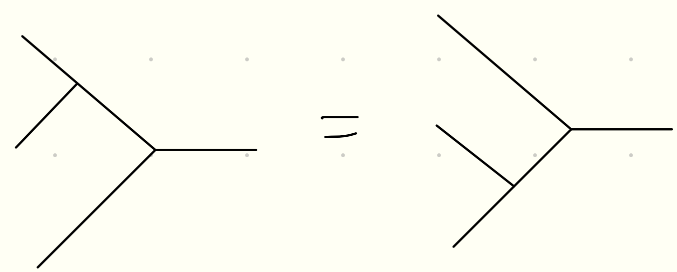
Coassociativity $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ associativity



$\Delta a = \sum a_{(1)} \otimes a_{(2)}$

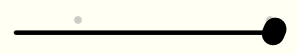
$\sum \Delta(a_{(1)}) \otimes a_{(2)} =$

$= \sum a_{(1)} \otimes \Delta(a_{(2)})$



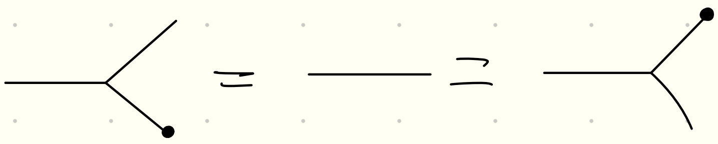
- trivial rep K_A c.f

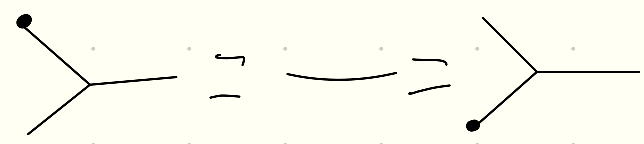
counit $\varepsilon : A \to \mathbb{C}$



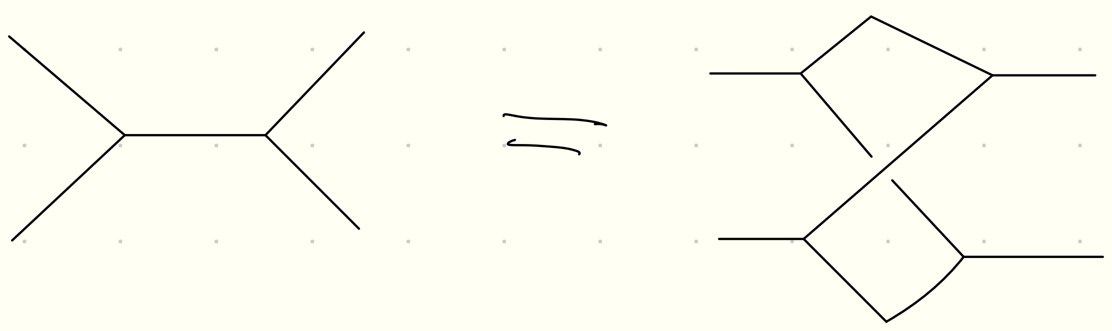
unit $i : \mathbb{C} \to A$

$K_A \otimes V = V = V \otimes K_A$





Δ is algebra homomorphism $\Delta(ab) = \Delta(a)\Delta(b)$

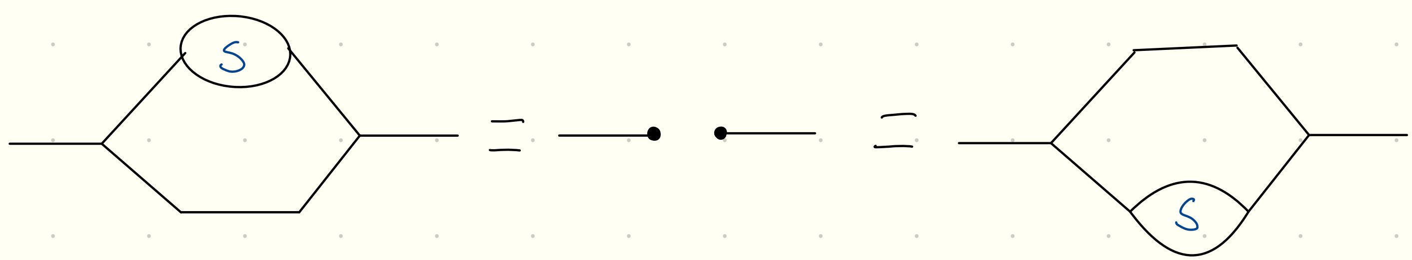


- Dual rep V^* Symmetry A^{op} (
$x \mapsto x^t$
antiautomorph)

Antipode $S: A \to A^{op}$ alg. hom. (invertable)

Action on V^* given by $(\rho_{V^*}(a)\varphi)(v) := \varphi(\rho_V(S(a))v)$
$\forall a \in A \quad \varphi \in V^*, v \in V$

Relation on S $\quad V^* \otimes V \to K_A, \quad K_A \to V \otimes V^*$



$\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \qquad \sum S(a_{(1)})a_{(2)} = i\,\varepsilon(a) = \sum a_{(1)} S(a_{(2)})$

(If $S^2 \neq id$ $\exists$ ${}^*V$ w/ action defined by S^{-1})

<u>Def</u> Hopf algebra $(A, m, \iota, \Delta, \varepsilon, S)$ w/ prop. above

Rem Δ, m determine ι, ε, S

Rem If A f.d. Hopf algebra $\Rightarrow$ A^* f.d. Hopf algebra

- Examples $\quad k = \mathbb{C}$

① G – finite or affine algebraic group
(e.g. $G = GL_n, SL_n$)

$A = \mathbb{C}[G]$ – algebra of functions on G

$$\Delta: \mathbb{C}[G] \longrightarrow \mathbb{C}[G] \times \mathbb{C}[G] = \mathbb{C}[G \times G]$$

$$G \longleftarrow G \times G$$

$$F \longmapsto \Delta F, \quad \Delta F(g_1, g_2) = F(g_1 g_2)$$

$$\varepsilon: \mathbb{C}[G] \to \mathbb{C} \qquad\qquad S: \mathbb{C}[G] \to \mathbb{C}[G]$$

$$\varepsilon(F) = F(e) \qquad\qquad S(F)(g) = F(g^{-1})$$

② G – finite group. kG – group algebra
$\mathrm{Rep}_{kG} = \mathrm{Rep}_{kG}$ has tensor product

$$\Delta(g) = g \otimes g \qquad \varepsilon(g) = 1 \qquad S(g) = g^{-1} \qquad \forall g \in G$$

③ G – Lie group $\mathfrak{g} = \mathrm{Lie}\,\mathfrak{g}$ $\mathcal{U}(\mathfrak{g})$ – universal enveloping algebra

$\forall x \in \mathfrak{g}$ $\Delta(e^{tx}) = e^{tx} \otimes e^{tx} \rightsquigarrow \frac{d}{dt}\Big|_{t=0}$ $\Delta(x) = x \otimes 1 + 1 \otimes x$

$\varepsilon(x) = 0$, $S(x) = -x$ $\forall x \in \mathfrak{g}$

Remark Examples ② & ③ not commutative but cocommutative i.e $\mathrm{Im}\,\Delta \in S^2 A$

Example ① is commutative but not cocommutative

Example ① is Hopf algebra dual to examples ② & ③

- Quantum group

Quantization of G $\rightsquigarrow$ noncommutative deformation $\mathbb{C}[G] \rightsquigarrow \mathbb{C}_q[G]$ in the class of Hopf algebras

Dually: deformation of $\mathcal{U}(\mathfrak{g})$ to $\mathcal{U}_q(\mathfrak{g})$ in class of Hopf algebras

- Main example

• Recall $\mathcal{U}(\mathfrak{sl}_2)$ generated by e, f, h

Relations $[h,e]=2e$ $\quad [h,f]=2f$ $\quad [e,f]=h$

Coproduct $\Delta x = x\otimes 1 + 1\otimes x$ $\quad \forall x = e, h, f$

• $\mathcal{U}_q(\mathfrak{sl}_2)$ generated by $E, F, K^{\pm 1}$

Relations $KE = q^2 EK$, $\quad KF = q^{-2} FK$ $\quad [E,F] = \dfrac{K - K^{-1}}{e^{\hbar} - e^{-\hbar}}$

Coproduct $\Delta(E) = E\otimes K + 1\otimes E$, $\Delta K = K\otimes K$, $\Delta F = F\otimes 1 + K^{-1}\otimes F$

q - complex number (or formal parameter)
$K^{\pm 1} = q^{\pm H}$ ~ group-like element (c.f. Harish-Chandra pairs)

Formal version over ring $\mathbb{C}[[\hbar]] = \{ \sum_{i=0}^{\infty} a_i \hbar^i,\ a_i \in \mathbb{C} \}$

$\mathbb{C}[[\hbar]]$-algebra $U_\hbar(sl_2)$ w/ generators E, F, H

Relations $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{\hbar}}$ — defined in $U_\hbar(sl_2)$
Coproduct $\Delta(E) = E\otimes e^{\hbar H} + 1\otimes E$, $\Delta F = F\otimes 1 + e^{-\hbar H}\otimes F$, $\Delta H = H\otimes 1 + 1\otimes H$

The quotient $U_\hbar(sl_2)/(\hbar) \simeq U(sl_2)$ as a Hopf algebra
If we set $q = e^{\hbar}$ in $U_q(sl_2)$ we get $U_\hbar(sl_2)$

Th $U_\hbar(sl_2) \simeq U(sl_2) \otimes \mathbb{C}[[\hbar]]$ as ass. algebra.

Hence $\mathrm{Rep}_{U_\hbar(sl_2)}$ close to $\mathrm{Rep}_{U(sl_2)}$

But what gives something new is tensor product.