Reps of $U_q(sl_2)$

- $U_q(sl_2)$ generated by $E, F, K^{\pm 1}$

Relations $KE = q^2 EK$, $KF = q^{-2} FK$ $\quad [E,F] = \dfrac{K - K^{-1}}{q - q^{-1}}$

$\Delta(E) = E \otimes K + 1 \otimes E$, $\quad \Delta K = K \otimes K$, $\quad \Delta F = F \otimes 1 + K^{-1} \otimes F$

<u>Example</u> 1dim rep $[E,F] \to 0 \Longrightarrow \left[\begin{array}{l} K \mapsto 1 \\ K \mapsto -1 \end{array}\right. \Longrightarrow \left[\begin{array}{ll} E, F \mapsto 0, K \mapsto 1 & \varepsilon = \varepsilon^+ \\ E, F \mapsto 0, K \mapsto -1 & \varepsilon^- \end{array}\right.$

<u>Def</u> $V_{(\lambda)} = \{ v \in V \mid \exists m \text{ s.t } (K - \lambda)^m v = 0 \}$

<u>Lemma</u> $E V_{(\lambda)} \subset V_{(q^2 \lambda)}$ $\quad F V_\lambda = V_{(q^{-2} \lambda)}$

<u>Def</u> λ - h.w of V if $V_{(\lambda)} \neq 0$, $\quad V_{q^2 \lambda} = 0$

Assume q is <u>not</u> root of 1. Then $\forall$ f.d rep has h.w.

$U_q(\mathfrak{b})$ - subalgebra generated by $E, K^{\pm 1}$

$\mathbb{C}_u$ - 1dim rep of $U_q(\mathfrak{b})$ $\quad E \mapsto 0,\ K^{\pm 1} \mapsto u^{\pm 1}$

<u>Def</u> Verma module $M_\lambda = \mathrm{Ind}_{U_q(\mathfrak{b})}^{U_q(\mathfrak{g})} \mathbb{C}_u$

Basis (by PBW) $\{F^\ell v_\lambda \mid \ell \in \mathbb{Z}\}$

$$\begin{tikzcd} \cdots & F^4 v_u \arrow[r, bend left, "E"] & F^3 v_u \arrow[r, bend left, "E"] \arrow[l, bend left] & F^2 v_u \arrow[r, bend left, "E"] \arrow[l, bend left] & F v_u \arrow[r, bend left, "E"] \arrow[l, bend left, "F"] & v_u \arrow[l, bend left, "F"] \end{tikzcd}$$

$F \cdot F^\ell v_u = F^{\ell+1} v_u \qquad K F^\ell v_u = q^{-2\ell} u\, v_u \qquad E F^\ell v_u = \dfrac{u q^{1-\ell} - u^{-1} q^{\ell-1}}{q - q^{-1}} \, \dfrac{q^\ell - q^{-\ell}}{q - q^{-1}} F^{\ell-1} v_u$

let $u = q^\lambda$, $[x] = \dfrac{q^x - q^{-x}}{q - q^{-1}}$ - q-number

$E F^\ell v_u = [\lambda+1-\ell][\ell] F^{\ell-1} v_u$

Lemma (i) $u \neq \pm q^n$ $n \in \mathbb{Z}_{\geq 0}$ $\Rightarrow$ M_u irred

(ii) $u = \pm q^n$ $n \in \mathbb{Z}_{\geq 0}$ $\Rightarrow$ one proper submodule gen. by. $F^{n+1} v_u$

$L_n^{\pm} = M(\pm q^n) / \text{proper submod}$ $\qquad$ $\dim L_n^{\pm} = n+1$

Th Let q not root of 1.

(i) All f.d irred reps of $U_q(\mathfrak{sl}_2)$ are $L_n^{\pm}$

(ii) Any f.d. rep of $U_q(\mathfrak{sl}_2)$ is S/S

For (ii) Casimir $2fe + \frac{1}{2}h^2 + h \rightsquigarrow FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$

- Character of rep $\sim$ Tr K.

$\chi_{L_n^{\pm}} = \pm(q^n + q^{n-2} + \dots + q^{-n}) = \pm$ character of L_n-rep of $\mathfrak{sl}_2$

Since $\Delta K = K\otimes K$ we have $\chi_{V_1\otimes V_2} = \chi_{V_1}\chi_{V_2}$

Therefore (and by Th (ii)) tensor product the same as for $U(sl_2)$

$$L_n^+\otimes L_m^+ = \bigoplus_{k=|n-m|,\ k\equiv n+m \bmod 2}^{n+m} L_k^+ \qquad L_n^-\otimes L_m^+ = \bigoplus_{k=|n-m|,\ k\equiv n+m \bmod 2}^{n+m} L_k^- \qquad L_n^-\otimes L_m^- = \bigoplus_{k=|n-m|,\ k\equiv n+m \bmod 2}^{n+m} L_k^+$$

<u>Example</u> L_1^+ - has basise v_q, Fv_q. $L_1^+\otimes L_1^+$ has basis $v_q\otimes v_q,\ v_q\otimes Fv_q,\ Fv_q\otimes v_q,\ Fv_q\otimes Fv_q$

$$K\mapsto\begin{pmatrix} q^2 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & q^{-2}\end{pmatrix} \qquad E\mapsto\begin{pmatrix} 0 & 1 & q & 0\\ 0 & 0 & 0 & q^{-1}\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix} \qquad F\mapsto\begin{pmatrix} 0 & 0 & 0 & 0\\ q^{-1} & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & q & 0\end{pmatrix}$$

$L_1^+\otimes L_1^+ = L_2^+\oplus L_0^+$ $\qquad L_0^+ = \langle q\, v_q\otimes Fv_q - Fv_q\otimes v_q\rangle$

- Tensor product $V_1\otimes V_2$ vs $V_2\otimes V_1$

want $\widetilde{R}_{V_1,V_2}: V_1\otimes V_2 \longrightarrow V_2\otimes V_1$

In cocomm case $P: V_1\otimes V_2 \to V_2\otimes V_1$ isom. of reps

Properties

$$\begin{tikzcd}
V_1 \otimes V_2 \otimes V_3 \arrow[rr, "\tilde{R}_{V_1, V_2 \otimes V_3}"] \arrow[dr, "\tilde{R}_{V_1 V_2} \otimes id"'] & & V_2 \otimes V_3 \otimes V_1 \\
& V_2 \otimes V_1 \otimes V_3 \arrow[ur, "id \otimes \tilde{R}_{V_1 V_3}"'] &
\end{tikzcd}$$

Triangle

$$\begin{tikzcd}
V_1 \otimes V_2 \otimes V_3 \arrow[rr, "\tilde{R}_{V_1 \otimes V_2, V_3}"] \arrow[dr, "id \otimes \tilde{R}_{V_2 V_3}"'] & & V_3 \otimes V_1 \otimes V_2 \\
& V_1 \otimes V_3 \otimes V_2 \arrow[ur, "\tilde{R}_{V_1 V_3} \otimes id"'] &
\end{tikzcd}$$

Corollary

$$\begin{tikzcd}
& V_1 \otimes V_3 \otimes V_2 \arrow[r] & V_3 \otimes V_1 \otimes V_2 \arrow[dr] & \\
V_1 \otimes V_2 \otimes V_3 \arrow[ur] \arrow[dr] & & & V_3 \otimes V_2 \otimes V_1 \\
& V_2 \otimes V_1 \otimes V_3 \arrow[r] & V_2 \otimes V_3 \otimes V_1 \arrow[ur] &
\end{tikzcd}$$

Hexagon

$R : P\tilde{R}$ $\quad R_{V_1 \otimes V_2} : V_1 \otimes_\Delta V_2 \to V_1 \otimes_{\Delta^{op}} V_2$

<u>Def</u> Universal R-matrix $R \in A \otimes A$ s.t.

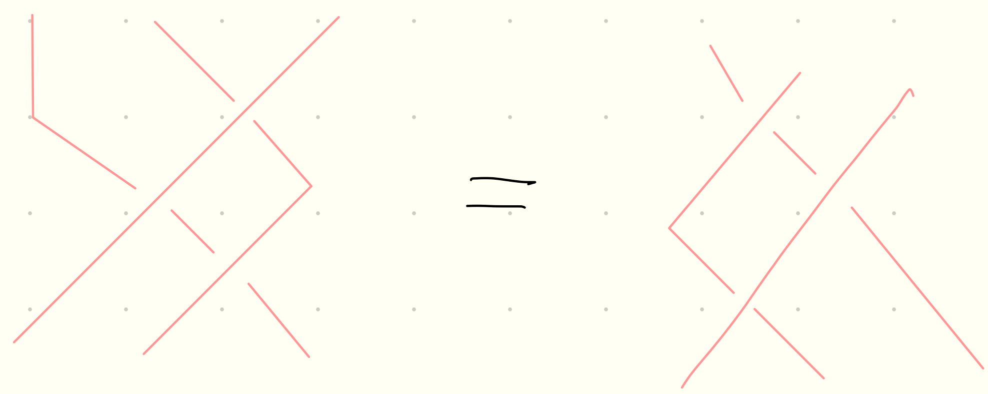
- $R\,\Delta(x) = \Delta^{op}(x)\,R \quad \forall x \in A$
- $(\Delta \otimes Id)(R) = R_{13} R_{23}, \qquad (Id \otimes \Delta) R = R_{13} R_{12}$

triangles

(Notations $R = \sum a_i \otimes b_i \qquad R_{13} = \sum a_i \otimes 1 \otimes b_i$

$R_{23} = \sum 1 \otimes a_i \otimes b_i \qquad (\Delta \otimes Id)(R) = \sum \Delta(a_i) \otimes b_i$)

hexagon – Yang-Baxter $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$



Th $$R = e^{\frac{1}{2}\hbar H\otimes H} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n \otimes F^n$$

Notations $q = e^{\hbar}$, $\binom{n}{2} = \frac{n(n-1)}{2}$, $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, $[n]_q! = [n]_q \cdot [n-1]_q \cdot \ldots \cdot [1]_q$

Ex $V_1 = V_2 = L_1^+$ $\quad H \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$R \mapsto q^{-1/2} \begin{pmatrix} q & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & q \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (q-q^{-1}) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

Def **Knot**: continuous embedding S^1 into $\mathbb{R}^3$

Link: continuous embedding $\underbrace{S^1 \sqcup \ldots \sqcup S^1}_{k}$ into $\mathbb{R}^3$

Braid: configuration of n strands in $\mathbb{R}^2 \times [0,1]$ connecting points $(i,0,0)$ to points $(j,0,1)$ (one to one) in some order s.t.

(i) each strand projects isom. to $[0,1]$

(ii) Strands do not intersect

Tangle: oriented link in $\mathbb{R}^2 \times [0,1]$ and oriented strands that connect some n fixed points on $\mathbb{R}^2 \times \{0\}$ and m fixed points on $\mathbb{R}^2 \times \{1\}$

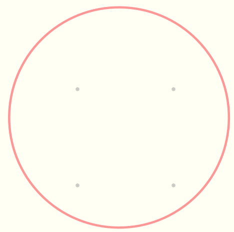
Points are connected 1 to 1 (hence $n+m$ - even)

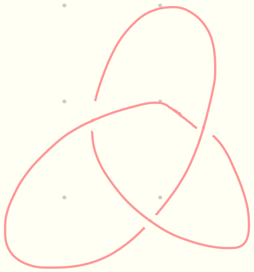
All objects up to isotopy.

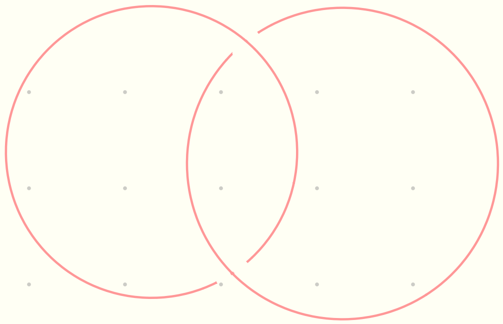
Tangles include links and braids.

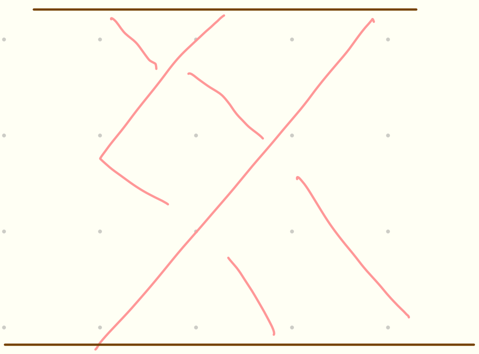
Drawn by projection $\mathbb{R}^3_{(x,y,t)} \to \mathbb{R}^2_{x,t}$

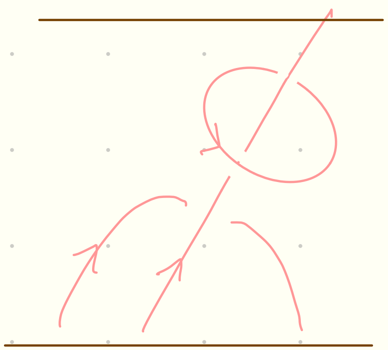
<u>Ex</u>

 unknot

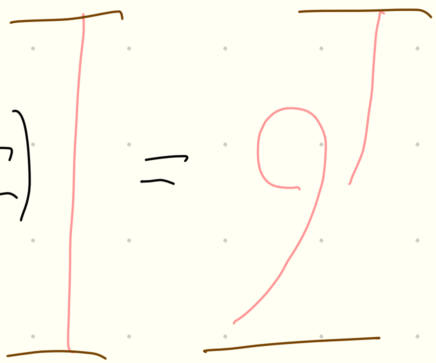
 threefoil

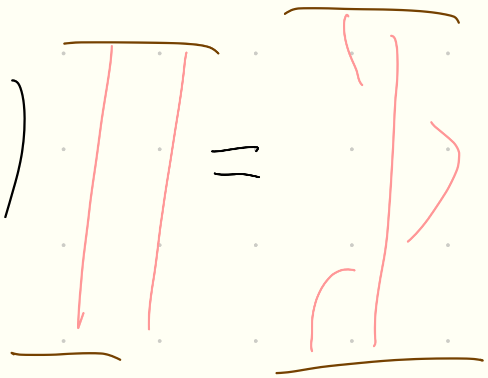
 - Hopf link

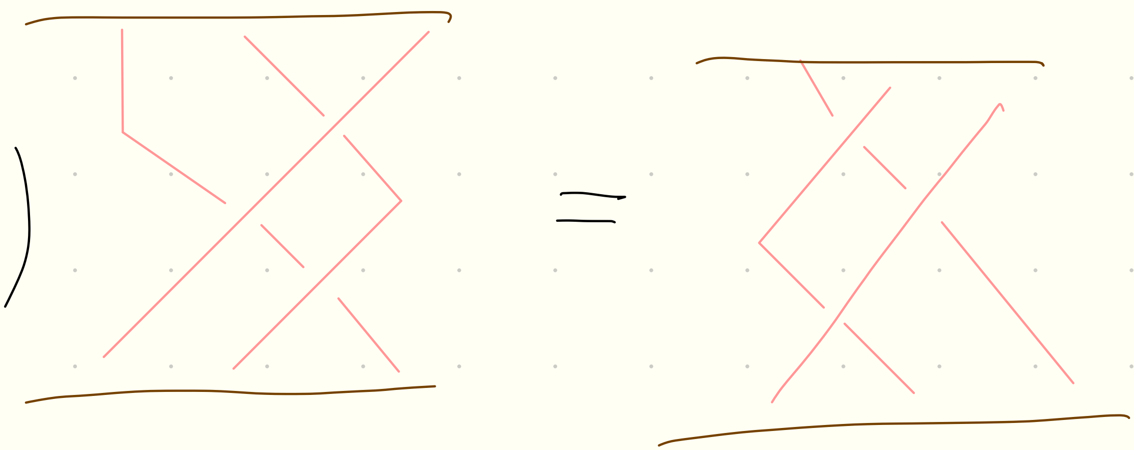
 Braid

 - tangle.

Equivalence by Rademeister moves

(RI)


(RII)


(RIII)


- Signed set — set w/ map to $\{\pm\}$

Tangle ~ signed set structure on $\{1,..,n\}$, $\{1,..,m\}$:
sources on $\mathbb{R}^2\times\{0\}$ and sinks on $\mathbb{R}^2\times\{1\}$ $\longrightarrow +$
Sinks on $\mathbb{R}^2\times\{0\}$ and sources on $\mathbb{R}^2\times\{1\}$ $\longrightarrow -$

For two signed sets N, M let $T(N,M)$ - set of isotopy classes of tangles w/ boundaries of type N, M

Partial composition $T(K,N)\times T(N,M) \to T(K,M)$

Tensor product $T(N_1,M_1)\times T(N_2,M_2) \to T(N_1\sqcup N_2, M_1\sqcup M_2)$

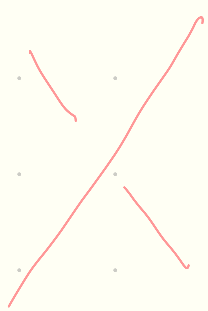
- Let $V\in \mathrm{Rep}_{U_q}$

Reshetikhin-Turaev

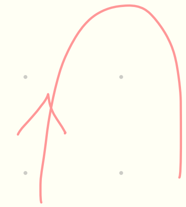
Functor $T\in T(N,M) \rightsquigarrow \varphi_T : V^{\otimes N} \to V^{\otimes M}$ s.t

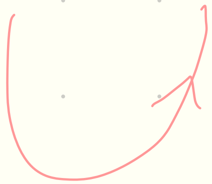
$\varphi_{T_1\circ T_2} = \varphi_{T_1}\circ\varphi_{T_2}$ $\quad$ $\varphi_{T_1\otimes T_2} = \varphi_{T_1}\otimes\varphi_{T_2}$

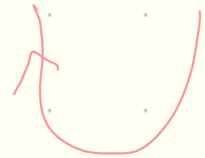
Strands  colored by V  colored by V^*

 $\sim q^2 R$  $q^{-2}R^{-1}$

 $\sim$ $ev: V^*\otimes V \to \mathbb{C}$

 $ev^*: V\otimes V^* \to \mathbb{C}$

 $coev: \mathbb{C} \to V^*\otimes V$

 $coev^*: \mathbb{C} \to V\otimes V^*$

These tangles generate all.

(RIII) – follows from Yang-Baxter

(RII) – trivial

(RI) – direct check